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THREE-DIMENSIONAL MANIFOLDS, SKEW-GORENSTEIN RINGS AND THEIR COHOMOLOGY

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Dedicated to Ralf Fröberg and Clas Löfwall at their 65th birthdays.

ABSTRACT. Graded skew-commutative rings occur often in practice. Here are two examples: 1) The cohomology ring of a compact three-dimensional manifold. 2) The cohomology ring of the complement of a hyperplane arrangement (the Orlik-Solomon algebra). We present some applications of the homological theory of these graded skew-commutative rings. In particular, we find compact oriented 3-manifolds without boundary for which the Hilbert series of the Yoneda Ext-algebra of the cohomology ring of the fundamental group is an explicit transcendental function. This is only possible for large first Betti numbers of the 3-manifold (bigger than, or maybe equal to, 11). We give also examples of 3-manifolds where the Ext-algebra of the cohomology ring of the fundamental group is not finitely generated.

0. Introduction. Let X be an oriented compact 3-dimensional manifold without boundary. The cohomology ring $H = H^*(X, \mathbf{Q})$ is a graded skew-commutative ring whose augmentation ideal $\overline{H}$ satisfies $\overline{H}^4 = 0$. The triple (cup) product $x \cup y \cup z = \mu(x,y,z).e$, where e is the orientation generator of H^3, defines a skew-symmetric trilinear form on H^1 with values in $\mathbf{Q}$, i.e. a *trivector*, and conversely, according to a theorem of Sullivan [**36**] any such form comes in this way from a 3-manifold X (not unique) whose cohomology algebra can be reconstructed from μ since by Poincaré duality $H^2 \simeq (H^1)^*$. In the more precise case when H^* is also a Poincaré duality algebra, i.e., the cup product $H^1 \times H^2 \to H^3$ is nondegenerate, it follows that H^* is a Gorenstein ring (cf. Section 1 below). Such Gorenstein rings will be studied here. Any 3-manifold M can be decomposed in a unique way

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as a connected sum of prime 3-manifolds:

$$M = P_1 \sharp P_2 \cdots \sharp P_k$$

of prime manifolds P_i (cf., e.g., Milnor [**28**, Theorem 1], and with the exception of S^3 and $S^2 \times S^1$ any prime manifold is also irreducible ([**28**] and for them $\pi_2(M) = 0$ ([**28**, Theorem 2]). If furthermore $\pi_1(M)$ is infinite then also the higher homotopy groups are 0 (M is said to be aspherical) so that M is the Eilenberg-Maclane space $K(\pi_1(M), 1)$ and the cohomology ring of M is isomorphic to the cohomology ring of the group $\pi_1(M)$ (this is of course also true for any 3-manifold X which is aspherical). (For most of the applications below we suppose that $H = H^*(X)$ is a Poincaré duality algebra and we suppose that the base field k is of characteristic 0, the preference being $\mathbf{Q}$.) The ring H has interesting homological properties which have not yet been fully studied, and we wish to continue such a study here. For small values of the first Betti number $b_1(X) = \dim_{\mathbf{Q}} H^1(X, \mathbf{Q})$ the ring H is a Koszul algebra (cf. Section 2 below) so that in particular the generating series

$$P_H(z) = \sum_{i \geq 0} |\mathrm{Tor}_i^H(k, k)| z^i = H(-z)^{-1} \tag{0.1}$$

where $H(z) = 1 + |H^1(X, \mathbf{Q})|z + |H^2(X, \mathbf{Q})|z^2 + z^3$, and where $|V|$ denotes the dimension of the vector space V. But for bigger Betti numbers many new phenomena occur. In particular we will see that for $b_1(X) = 12$ (and maybe even for $b_1(X) = 11$) there are a few examples where $P_H(z)$ is an explicit transcendental function (thus we are far away from the Koszul case of formula (0.1)!). However, maybe 11 is the best possible number here. On the other hand for bigger $b_1(X)$ the possible series $P_H(z)$ are rationally related to the family of series which occurred in connection with the Kaplansky-Serre questions a long time ago [**1**]. But even for smaller $b_1(X)$ other strange homological properties of H occur: we will give examples (probably best possible) where $b_1(X) = 11$ and the Yoneda Ext-algebra $\mathrm{Ext}_H^*(k, k)$ is not finitely generated. The implications of all this for 3-manifold groups have not been fully explored. Compare also [**35**]. Conversely, the connection with 3-manifolds makes it possible to go backwards and to deduce results in the homology theory of skew-commutative algebras (and there are also related results in the *commutative* case). Note that we are studying everything in characteristic 0. There seem to be some relations with Benson [**5**] but he works over finite fields. Finally, let us remark that even for the special 3-dimensional case when X is the boundary manifold of a line arrangement in $P^2(\mathbf{C})$ we can have, e.g., the same strange transcendental phenomenon as above but the price to pay for this is to accept even bigger $b_1(X)$. The three-manifolds that occur here are called graphic manifolds [**7**] and they are aspherical if the line arrangement is not a pencil of lines.

1. Graded skew-Gorenstein rings and classification of trivectors. Let us first recall that a local commutative Gorenstein ring was defined in [**4**] as a ring R which has a finite injective resolution as a module over itself. In particular if R is artinian this means that R is injective as a module over itself. It is also equivalent to saying that the socle of R, i.e., $\mathrm{Hom}_R(R/m, R)$ (m is the maximal ideal of R) is 1-dimensional over $k = R/m$. Things are more complicated in the noncommutative case [**10**], but if R is skew-commutative artinian the Gorenstein property is equivalent to R being injective as a module over itself (left or right–these two conditions are equivalent–and they are also equivalent to saying that the left, or right, socle of R is one-dimensional). In the special case when X is an oriented compact three-dimensional manifold without boundary, and when $H = H^*(X, \mathbf{Q})$ is the cohomology ring of X, we let $R = H = H^*(X, \mathbf{Q})$. From the preceding definition it follows that R is Gorenstein if and only if R is generated by H^1 and H is a Poincaré duality algebra (we assume that $|H^1| > 1$).

We now turn to the classification of such R's: when $|H^1(X, \mathbf{Q})| \leq 8$. We will use the classification of trivectors in H^1 described in section 35 of the book [**15**]. For a background we refer the reader to our introduction, to [**36**] and to Section 4 [Intermezzo: Classification of skew-symmetric forms] of [**35**]. Since we are only studying nondegenerate trivectors, their ranks are equal to the dimension of H^1.

For rank 5 there is only one trivector (denoted by III in [**15**, page 391]) and which can be denoted by (here e_1, e_2, e_3, e_4, e_5 is a basis for H^1 and the e^i's are the dual basis elements in $(H^1)^*$):

$$e^1 \wedge e^4 \wedge e^3 + e^2 \wedge e^5 \wedge e^3$$

and the corresponding Gorenstein ring is a Koszul algebra (the preceding trivector is the correct one as in [**15**]–in [**35**] there is a minor misprint in its form). It is only when the ranks are ≥ 6 that non-Koszul Gorenstein rings occur. Let us give the details in the first nontrivial case of [**15**], namely case IV of rank 6, page 391:

$$f = e^1 \wedge e^2 \wedge e^3 + e^3 \wedge e^4 \wedge e^5 + e^2 \wedge e^5 \wedge e^6.$$

Note that H^* is a quotient of the exterior algebra

$$E(e_1, e_2, e_3, e_4, e_5, e_6),$$

by an ideal that we want to determine. To begin with we want to determine all quadratic elements $g = \sum_{j<k} g_{j,k} e_j \wedge e_k$ that go to 0 in the quotient H^2. Since the pairing $H^1 \otimes H^2 \to H^3$ is nondegenerate this boils down to determine when $e_s \wedge g$ goes to zero for $s = 1, \dots, 6$. Using f we get the conditions $\sum_{j<k} g_{j,k} f(e_s, e_j, e_k) = 0$ for $s = 1, \dots, 6$. This gives, using the explicit form of f and the condition that f is skew-symmetric (note that if f were only the monomial $f_{\text{mon}} = e^1 \wedge e^2 \wedge e^3$, then $f_{\text{mon}}(e_{i1}, e_{i2}, e_{i3})$ is non zero $(= \pm 1)$ if and only if i_1, i_2, i_3 is a permutation of 1, 2, 3) the six conditions $s = 1, \dots, 6$:

$$g_{2,3} = 0,\ -g_{1,3} + g_{5,6} = 0,\ g_{1,2} + g_{4,5} = 0,\ g_{3,5} = 0,\ g_{3,4} - g_{2,6} = 0,\ g_{2,5} = 0.$$

The 9 solutions of this system of 6 linear equations lead to the relations $e_1 \wedge e_3 + e_5 \wedge e_6$, $e_1 \wedge e_2 - e_4 \wedge e_5$ and $e_3 \wedge e_4 + e_2 \wedge e_6$ and to 6 monomial relations, leading to the following ring with quadratic relations (we do not write wedge for multiplication):

$$R_{IV} = \frac{E(e_1, e_2, e_3, e_4, e_5, e_6)}{(e_1e_2 - e_4e_5, e_1e_3 + e_5e_6, e_2e_6 + e_3e_4, e_1e_4, e_1e_5, e_1e_6, e_2e_4, e_3e_6, e_4e_6)}.$$

But this quotient ring R_{IV} has Hilbert series $R_{IV}(z) = 1 + 6z + 6z^2 + 2z^3$. This means that we have to find a last cubic relation. One finds that the cube of the maximal ideal of R_{IV} is generated by $e_2e_5e_6$ and $e_2e_3e_5$. But $f(e_2, e_3, e_5) = 0$ and $f(e_2, e_5, e_6) = 1$ so that the corresponding Gorenstein ring is $G_{IV} = R_{IV}/e_2e_3e_5$ and we will see that this Gorenstein ring is not a Koszul algebra. For case V of loc. cit. the f is given by

$$f = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$$

leading in the same way to the ring with quadratic relations:

$$R_V = \frac{E(e_1, e_2, e_3, e_4, e_5, e_6)}{(e_1e_4, e_1e_5, e_1e_6, e_2e_4, e_2e_5, e_2e_6, e_3e_4, e_3e_5, e_3e_6)},$$

which also has Hilbert series $R_V(z) = 1 + 6z + 6z^2 + 2z^3$, but in this case we have to divide by $e_1e_2e_3 - e_4e_5e_6$ to get the Gorenstein ring $G_V = R_V/(e_1e_2e_3 - e_4e_5e_6)$ which is, as we will see below, not a Koszul algebra either.

In [**15**, pages 393-395] the classification of 3-forms of rank 7 is given as the 5 cases VI, VII, VIII, IX, X and those forms of rank 8 are described as the 13 cases XI, XII,..., XXIII. The classification of 3-forms of rank 9 are given in [**38**]. We will describe the homological behavior of the corresponding Gorenstein rings in Section 4.

2. Calculating the Koszul dual of the Gorenstein ring associated to a 3-form. Let R be any finitely presented ring (connected k-algebra) generated in degree 1 and having quadratic relations. It can be described as the quotient $T(V)/(W)$, where $T(V)$ is the tensor algebra on a (finite-dimensional) k-vector space V, placed in degree 1 and (W) is the ideal in $T(V)$, defined by a sub-vector space $W \subset V \otimes_k V$. The Yoneda Ext-algebra of R is defined by

$$\mathrm{Ext}^*_R(k,k) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_R(k,k)$$

where k is an R-module in the natural way and where the multiplication is the Yoneda product. The sub-algebra of $\mathrm{Ext}^*_R(k,k)$ generated by $\mathrm{Ext}^1_R(k,k)$ is called the Koszul dual of R, and it is denoted by $R^!$. It can be calculated as follows: consider the inclusion map $W \to V \otimes_k V$. Taking k-vector space duals (denoted by W^* and $(V \otimes_k V)^*$ we get the exact sequence ($W^\perp$ = those linear $f : V \otimes_k V \to k$ that are 0 on W):

$$0 \leftarrow W^* \leftarrow (V \otimes_k V)^* \leftarrow W^\perp \leftarrow 0.$$

Now $R^! = T(V^*)/(W^\perp)$ (we have used that $(V \otimes_k V)^* = V \otimes_k V^*$).

For all this, cf. [**24**]. Note that $(R^!)^!$ is isomorphic to R. Note also that if R also has cubic relations and/or higher relations then $\mathrm{Ext}^*_R(k,k)$ is still defined, and the subalgebra generated by $\mathrm{Ext}^1_R(k,k)$ is still given by the formula $T(V^*)/(W^\perp)$ where W is now only the "quadratic part" of the relations of R. In particular $(R^!)^!$ is now only isomorphic to $T(V)$ divided by the quadratic part of the relations. Note also that in general, if R is skewcommutative then $\mathrm{Ext}^*_R(k,k)$ is a cocommutative Hopf algebra which is the enveloping algebra of a graded Lie algebra, and $R^!$ is a sub Hopf algebra which is the enveloping algebra of a smaller graded Lie algebra. (All this is also true if R is commutative, but now the Lie algebras are super Lie algebras.) In [**24**, Corollary 1.3, pages 301–302] there is a recipe about how to calculate the Koszul dual of an algebra with quadratic relations. Applying this to the case of R_{IV} of the previous section we find that $R^!_{IV}$ is the algebra

$$\frac{k\langle X_1,X_2,X_3,X_4,X_5,X_6\rangle}{([X_1,X_2]+[X_4,X_5],[X_3,X_4]-[X_2,X_6],[X_1,X_3]-[X_5,X_6],[X_2,X_3],[X_2,X_5],[X_3,X_5])}$$

where $[X_i,X_j]=X_iX_j-X_jX_i$ is for $i<j$ the Lie commutator of X_i and X_j. In general if one starts with a (skew)-commutative algebra A it is often easy to calculate the Hilbert series of A. But calculating the Hilbert series of $A^!$ is often very difficult and this series can even be a transcendental function. But in this case it is rather easy: we get that $R^!_{IV}(z)=1/(1-6z+6z^2-2z^3)$ so that $R_{IV}(-z)R^!_{IV}(z)=1$ and we even have (the in general strictly stronger assertion ([**29, 33**])) that R_{IV} is a Koszul algebra. (For the definitions and equivalent characterizations of Koszul algebras we refer to [**24**, page 305, Theorem 1.2].) A similar result for R_V holds true, and in this case we can directly apply a result of Fröberg [**11**], since R_V has quadratic monomial relations. But neither the Gorenstein quotients G_{IV} nor G_V are Koszul algebras and in the next section we will see how to relate their Hilbert series and the corresponding Hilbert series of their Koszul duals to their two-variable Poincaré-Betti series

$$P_G(x,y)=\sum_{i,j}|\mathrm{Tor}^G_{i,j}(k,k)|x^iy^j.$$

But we will first deduce a few results about explicitly calculating the Gorenstein ring and its Koszul dual associated to a given 3-form.

An old result of Macaulay gives a nice correspondence between commutative artinian graded Gorenstein algebras having socle of degree j of the form $k[x_1,x_2,\dots,x_n]/I$ and homogeneous forms of degree j in the dual of $k[x_1, x_2, \dots, x_n]$ (cf., e.g., [**9**, Lemma 2.4]). Here is a skew-commutative version (here described only for socle degree 3), the proof of which follows from [**8**, Exercise 21.1, page 547], where we have to replace polynomial rings by exterior algebras (I thank Antony Iarrobino who suggested that such a result should be true):

Proposition 2.1 ("Skew"-Macaulay). *Let $T = E[y_1, \dots, y_n]$ be the exterior algebra in n variables of degree 1 over a field of characteristic 0. Consider a polynomial skew-differential operator D on T with constant coefficients of the form*

$$D = \sum_{i_1, \dots, i_n} a_{i_1}, \dots, i_n (\partial/\partial y_1)^{i_1} \cdots (\partial/\partial y_n)^{i_n}$$

(the i_j are 0 or 1). The symbol of D is the $\sum_{i_1, \dots, i_n} a_{i_1}, \dots, i_n x_1^{i_1} \cdots x_n^{i_n}$ which is obtained by replacing each $\partial/\partial y_i$ by x_i in a new exterior algebra $S = E[x_1, \dots, x_n]$. Let now $0 \neq f \in T$ be a homogeneous polynomial of degree 3. Let I be the set of symbols in S of differential operators D as above, such that $Df = 0$. Then S/I is a zero-dimensional Gorenstein ring and there is a converse assertion.

We will not use the preceding result, which is only included here for historical reasons.

We now go back to the reasoning that we used when we treated case IV in Section 1. Recall first that in ([**24**, Corollary 1.3, page 301]) it is proved in particular that if we have a quadratic algebra $R = E[x_1, \dots, x_n]/(f_1, \dots, f_r)$ where the

$$f_i = \sum_{j<k} b_{i,j,k} x_j \wedge x_k, \quad b_{i,j,k} \in k, \quad i = 1, \dots r,$$

then

$$R^! = k\langle X_1, \dots, X_n\rangle/(\phi_1, \dots, \phi_s),$$

where

$$\phi_i = \sum_{j<k} c_{i,j,k}[X_j, X_k], \quad c_{i,j,k} \in k, \quad i = 1, \dots, s,$$

where $[X_j, X_k] = X_j X_k - X_k X_j$ and $(c_{i,j,k})_{jk}$, $i = 1, \dots s$, is a basis of the solutions of system of linear equations:

$$\sum_{j<k} b_{i,j,k} X_{jk}, \quad i = 1, \dots, r \text{ (hence)}, \ s = \binom{n+1}{2} - r.$$

Now, given a skew-symmetric 3-form $(e^1, e^2, \ldots, e^n)$ of rank n, written as a linear combination of $e^{i_1} \wedge e^{i_2} \wedge e^{i_3}$, where $i_1 < i_2 < i_3$ (with coefficients that are $a_{i_1 i_2 i_3}$), we have that

$$f_i = \sum_{j<k} b_{ijk} e_j \wedge e_k, \quad i = 1, \ldots, r$$

are relations corresponding to Ψ (cf. the discussion of case IV in Section 1) if and only if

$$\sum_{j<k} b_{i,j,k} \Psi(e_i, e_j, e_k) = 0, \quad \text{for } i = 1, \ldots, n.$$

But, for fixed i, $\Psi(e_i, e_j, e_k)$ is a_{ijk} if and only if $i < j < k$ and $j < k < i$, and $-a_{ijk}$ if and only if $j < i < k$. This corresponds to taking the graded skew-derivative of Ψ with respect to e^i. Combining this with the Löfwall description of the calculation of $R^!$ given above, we finally arrive at the following useful Theorem-Recipe to calculate the Koszul dual $G^!$ of the Gorenstein ring associated to a skew 3-form:

Theorem-Recipe 2.1. *Let $\Psi(e^1, e^2, \ldots, e^n)$ be a skew-symmetric 3-form of rank n, X one of the 3-manifolds giving rise to Ψ and $G = H^*(X, Q)$. The Koszul dual $G^!$ of G is obtained as follows: Calculate the n skew-derivatives of Ψ with respect to the variables $e^1, e^2, \ldots, e^n$. Then*

$$G^! \simeq \frac{k\langle X_1, X_2, \ldots, X_n\rangle}{(q_1, q_2, \ldots, q_n)}$$

where $k\langle X_1, X_2, \ldots, X_n\rangle$ is the free associative algebra generated by the variables X_i that are dual to the e^i's in Ψ and where the q_i's are obtained by replacing each quadratic element $e^s \wedge e^t$ $(s < t)$ in the $\partial\Psi/\partial e^i$ by the commutator $[X_s, X_t] = X_s X_t - X_t X_s$ in $k\langle X_1, X_2, \ldots, X_n\rangle$, for $i = 1, \ldots, n$.

Note that if $G^!$ is given it is easy to calculate "backwards" the ring with quadratic relations $(G^!)^!$. After that it is easy to find the extra cubic relations we should divide with to get G. Using this theorem-recipe we finally find:

Theorem 2.2. a) *The double Koszul duals* $(G^!)^!$ *of the Gorenstein rings corresponding trivectors of* rank = 7 *are Koszul algebras in the cases* VI, VII, VIII, IX *and* X *and therefore* $G^!$ *is a Koszul algebra in cases* VI, VII, VIII, IX, X. (*But they are not isomorphic*). *The corresponding Gorenstein algebras* $G_{VI}, G_{VII}, G_{VIII}, G_{IX}, G_X$ *are also Koszul algebras.*

b) *When it comes to trivectors of rank* 8, *i.e., the cases* XI, XII,..., XXIII *the situation is more complicated already from the homological point of view*: *Indeed in case* XI *the double Koszul dual* $(G^!)^!$ *is already a Gorenstein ring with quadratic relations, thus equal to* G *and* $G^!$ *is not a Koszul algebra. But the case* XII *treated explicitly below is slightly different and not a Koszul algebra either. The cases* XIII, XIV *and* XV *are Koszul algebras. But case* XVI *is as above*: $(G^!)^!$ *is a Koszul algebra, but we have to divide out a cubic form to get* G_{XVI} *which is therefore not Koszul. Finally the algebras corresponding to the cases* XVII, XVIII, XIX, XX, XXI, XXII *and* XXIII *are Koszul algebras* (*but not isomorphic*).

Example 2.1. Here is a use of the Theorem-Recipe 2.1: Let us consider case XII of rank 8. Here the 3-form $f_{XII} = \Psi$ is

$$\begin{aligned}\Psi(e^1, e^2, \dots, e^8) = e^5 \wedge e^6 \wedge e^7 + e^1 \wedge e^5 \wedge e^4 \\ + e^2 \wedge e^6 \wedge e^4 + e^3 \wedge e^7 \wedge e^4 + e^3 \wedge e^6 \wedge e^8.\end{aligned}$$

We calculate the eight partial derivatives of this skew form (we do not write out the $\wedge$ sign):

$$\begin{gathered}\partial\Psi/\partial e^1 = e^5e^4, \quad \partial/\partial e^2 = e^6e^4, \quad \partial/\partial e^3 = e^7e^4 + e^6e^8, \\ \partial\Psi/\partial e^4 = e^1e^5 + e^2e^6 + e^3e^7, \quad \partial\Psi/\partial e^5 = e^6e^7 - e^1e^4, \\ \partial\Psi/\partial e^6 = -e^5e^7 - e^2e^4 - e^3e^8, \quad \partial/\partial e^7 = e^5e^6 - e^3e^4, \quad \partial/\partial e^8 = e^3e^6\end{gathered}$$

leading to $G^! = k\langle X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8\rangle$ divided by the ideal

$$\begin{gathered}([X_5, X_4], [X_6, X_4], [X_7, X_4] + [X_6, X_8], [X_1, X_5] + [X_2, X_6] + [X_3, X_7], \\ [X_6, X_7] - [X_1, X_4], [X_5, X_7] + [X_2, X_4] + [X_3, X_8], [X_5, X_6] \\ -[X_3, X_4], [X_3, X_6])\end{gathered}$$

and the Hilbert series $1/G^!(z) = 1 - 8z + 8z^2 - z^3 - z^4$. Furthermore one sees that $(G^!)^!$ has Hilbert series $1 + 8z + 8z^2 + z^3$ so that we do not have to divide by cubic elements to get G_{XII} which is non-Koszul. As a matter of fact we have a generalization of a formula of Löfwall (cf. next section), giving that

$$\frac{1}{P_{G_{XII}}(x,y)} = (1+1/x)/G^!_{XII}(xy) - G_{XII}(-xy)/x$$

so that

$$\frac{1}{P_{G_{XII}}(x,y)} = 1 - 8xy + 8x^2y^2 - x^3y^3 - x^3y^4 - x^4y^4,$$

leading to, e.g., $\mathrm{Tor}^{G_{XII}}_{3,4}(k,k)$ being 1-dimensional. The other cases I-XXIII are treated in a similar manner: sometimes $G^!(z)$ can be directly calculated since we have a finite Groebner basis for the non-commutative ideal and in all cases using the Backelin et al. programme BERGMAN [**3**].

3. A generalization of a formula of Löfwall to Gorenstein rings with $m^4 = 0$. In his thesis [**24**] Clas Lofwall proved in particular that if A is any graded connected algebra with $\overline{A}^3 = 0$ then the double Poincaré-Betti series is given by the formula

$$\frac{1}{P_A(x,y)} = (1+1/x)/A^!(xy) - A(-xy)/x. \tag{3.1}$$

In [**30**] we have an easy proof of this in the case when A is commutative (works also in the skew-commutative case [**31**]) using the fact that in these cases $A^!$ is a sub-Hopf algebra of the big Hopf algebra $\mathrm{Ext}^*_A(k,k)$ and according to a theorem of Milnor-Moore this big Hopf algebra is free as a module over $A^!$. We now show

Theorem 3.1. *The formula* (3.1) *is true when* (R,m) *is a graded (skew-) commutative Gorenstein ring with* $m^4 = 0$.

Proof. First we note that Avramov-Levin have proved [**2**] in the commutative case that the natural map

$$R \longrightarrow R/\mathrm{soc}\,(R)$$

is a Golod map (cf. Section 5 for the skew-commutative case that we use here). Now the socle of R is m^3 which is $s^{-3}k$. Therefore we have (3.2)

$$P_{R/\mathrm{soc}\,(R)}(x,y) = \frac{P_R(x,y)}{1 - x(P_R^{R/\mathrm{soc}\,(R)}(x,y)-1)} = \frac{P_R(x,y)}{1 - x^2y^3P_R(x,y)}.$$

But R/m^3 is a local ring where the cube of the maximal ideal is 0. Thus the formula of Löfwall can be applied, and we obtain using that $(R/m^3)^! = R^!$ the formula

$$\frac{1}{P_R/m^3(x,y)} = (1+1/x)/R^!(xy) - (R/m^3)(-xy)/x$$

which combined with (3.2) gives (3.1) since $R(z) = R/m^3(z) + z^3$.

Remark 3.1. Thus it is clear that we cannot obtain cases where the ring $H^*(X,Q)$ has "bad" homological properties if $b_1(X) \leq 8$. We therefore study the case when $b_1(X) = 9$. In this case there is a classification of the tri-vectors by Vinberg and Elashvili [**38**]. But even in this case it seems impossible to get "exotic" $H^*(X,\mathcal{Q})$. In fact we have, e.g., by the procedure above analyzed all cases in Table 6 of [**38**, pages 69–72] (those cases that have a $*$ in loc. cit. correspond to $b_1(X) < 9$ and they have already been treated).

Theorem 3.2. *For the* 3*-forms of rank* 9 *of Table* 6 [**38**, pages 69–72] *we have that the corresponding* $1/R^!(z) = 1 - 9z + 9z^2 - z^3$*, and the corresponding Gorenstein ring is a Koszul algebra in all cases except*

a) *the cases* 79*,* 81*,* 85 *where* $1/R^!(z) = 1 - 9z + 9z^2 - 3z^3$

b) *the case* 83 *where* $1/R^!(z) = 1 - 9z + 9z^2 - z^3 - 5z^4 + 4z^5 - z^6$*.*

Proof. The proof is by using Theorem-Recipe 2.1 above. For the Koszul cases we refer the reader to the Appendix. Here we only indicate what happens in the exceptional cases a) and b). In case 79 in a) the 3-form is (we do not write out the wedge for multiplication):

$$f_{79} = e^1e^2e^9 + e^1e^3e^8 + e^2e^3e^7 + e^4e^5e^6.$$

We take the 9 partial derivatives $\partial f_{79}/\partial e^i$ for $i = 1, \dots, 9$, and we obtain:

$$e^2e^9 + e^3e^8, \quad -e^1e^9 + e^3e^7, \quad -e^1e^8 - e^2e^7, \quad e^5e^6,$$
$$-e^4e^6, e^4e^5, e^2e^3, e^1e^3, e^1e^2$$

leading to

$$R^! = k\langle X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9\rangle/([X_2, X_9] + [X_3, X_8], - [X_1, X_9] + [X_3, X_7], [X_1, X_8] + [X_2, X_7], [X_5, X_6], [X_4, X_6], [X_4, X_5], [X_2, X_3], [X_1, X_3], [X_1, X_2]).$$

Now it turns out that the Gröbner basis of the ideal above is finite and in degree ≤ 2. This proves that $R^!$ is Koszul and has $R^!(z) = 1 - 9z + 9z^3 - 3z^3$, since the commutative ring $(R^!)^!$ has Hilbert series $1 + 9z + 9z^2 + 3z^3$. The ideal m^3 is generated by the three elements $(e_1e_2e_9, e_4e_5e_6, e_1e_2e_3)$ and

$$f_{79}(e_1, e_2, e_9) = 1, \quad f_{79}(e_4, e_5, e_6) = 1 \quad \text{and} \quad f_{79}(e_1, e_2, e_3) = 0.$$

It follows that if we divide out by the two extra elements $e_1e_2e_9 - e_4e_5e_6, e_1e_2e_3$ we do have the non-Koszul Gorenstein ring G_{79} we are looking for. The rings G_{81} and G_{85} in a) are treated in the same way. They correspond to:

$$f_{81} = e^1e^2e^9 + e^1e^3e^8 + e^1e^4e^6 + e^2e^3e^7 + e^2e^4e^5 + e^3e^5e^6$$

and

$$f_{85} = e^1e^2e^9 + e^1e^3e^5 + e^1e^4e^6 + e^2e^3e^7 + e^2e^4e^8.$$

In case 83 of b) the form

$$f_{83} = e^1e^2e^9 + e^1e^3e^5 + e^1e^4e^6 + e^2e^3e^7 + e^2e^4e^8 + e^3e^4e^9$$

and its 9 skew-derivatives lead to the quotient:

$$k\langle X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9\rangle$$

divided by the ideal

$$([X_2, X_9] + [X_3, X_5] + [X_4, X_6], -[X_1, X_9] + [X_3, X_7] + [X_4, X_8], -[X_1, X_5] - [X_2, X_7] + [X_4, X_9], [X_1, X_6] + [X_2, X_8] +[X_3, X_9], [X_1, X_3], [X_1, X_4], [X_2, X_3], [X_2, X_4], [X_1, X_2] + [X_3, X_4]).$$

In this case the Hilbert series of $R^!$ is $1/(1-9z+9z^2-z^3-5z^4+4z^5-z^6)$, and the ring $(R^!)^!$ has Hilbert series: $1+9z+9z^2+z^3$ so that $G_{83} = (R^!)^!$ which is a Gorenstein non-Koszul algebra and Theorem 3.2 is proved. □

Thus to obtain examples of X where the homological properties of the cohomology algebra $H^*(X, Q)$ are complicated we need to study the cases where the dimension of $H^1(X, \mathcal{Q})$ is ≥ 10. But the trivectors of rank 10 have not been classified, and even the cases of Sikora [**35**, Theorem 1] which uses for any simple Lie algebra g the trilinear skewsymmetric form $\Psi_g(x, y, z) = \kappa(x, [y, z])$ where κ is the Killing form of g, do not seem to give anything exotic from our point of view, at least when the dimension of the Lie algebra is 10. We therefore need a new way of constructing Gorenstein rings corresponding to trivectors of rank ≥ 10, but defined in another way. This will be done in the next section.

4. New skew-Gorenstein rings. Let (R, m) be any ring with $m^3 = 0$ which is the quotient of the exterior algebra $E(x_1, \ldots, x_n)$ by homogeneous forms of degrees ≥ 2 and let $I(k)$ be the injective envelope of the residue field $k = R/m$ of R. Then $G = R \propto I(k)$ ([**10**] is a skew-Gorenstein ring with maximal ideal $n = m \oplus I(k)$ satisfying $n^4 = 0$, which therefore corresponds to a trivector of rank $|m/m^2| + |I(k)/mI(k)| = |m/m^2| + |m^2|$ and according to the Sullivan theory it comes from a 3-manifold X with the dimension of $H^1(X, \mathcal{Q})$ being equal to $|m/m^2| + |m^2|$. It turns out that the homological properties of G are closely related to those of the smaller ring R. Indeed we have the following general result:

Theorem 4.1 (Gulliksen [**14**]). *Let R be (skew-)commutative and $R \propto M$ be the trivial extension of R with the R-module M. Then we have an exact sequence of Hopf algebras*:

$$k \longrightarrow T(s^{-1}\mathrm{Ext}^*_R(M, k)) \longrightarrow \mathrm{Ext}^*_{R\propto M}(k, k) \longrightarrow \mathrm{Ext}^*_R(k, k) \longrightarrow k$$

where $T(s^{-1}\mathrm{Ext}^*_R(M, k))$ is the free algebra on the graded vector space $s^{-1}\mathrm{Ext}^*_R(M, k)$ and where the arrow $\mathrm{Ext}^*_{R\propto M}(k, k) \to \mathrm{Ext}^*_R(k, k)$ is a split epi-morphism (there is a splitting ring map $R \propto M \to R$). In *particular we have the formula for Poincaré-Betti series*

$$P_{R\propto M}(x,y) = \frac{P_R(x,y)}{1 - xP_R^M(x,y)}.$$

Remark 4.0. Gulliksen's proof [**14**] is in the commutative setting and uses Massey operations. In [**23**], Clas Löfwall studies the more general case of a trivial extension $R \propto M$, where R is any ring (not necessarily commutative) and M is an R-bimodule. In [**23**, Section 6, pages 305-306] he derives the Gulliksen formula when R is commutative using an idea of mine (cf. [**23**, page 288]). In the same way one deduces the Gulliksen formula in the skew-commutative case.

Now turn to the artinian case and assume that M is finitely generated, and let $\widetilde{M} = \mathrm{Hom}_R(M, I(k))$ be the Matlis dual of M [**27**] (if M as an R-module is a vector space over k, then this is the ordinary vector space dual). In this case we have a well-known formula (cf., e.g., Lescot [**19**, Lemme 1.1])

$$\mathrm{Ext}_R^*(M,k) \simeq \mathrm{Ext}_R^*(k, \widetilde{M}).$$

In particular if $M = I(k)$ then $\widetilde{M} = \mathrm{Hom}_R(I(k), I(k)) = R$ [**27**] so that

$$\mathrm{Ext}_R^*(I(k),k) \simeq \mathrm{Ext}_R^*(k, R).$$

Thus, if we use the Theorem on $R \propto I(k)$ we are led to the study of the Bass series of R, i.e., the generating series in one or two variables of $\mathrm{Ext}_R^*(k,R)$. It turns out that in many (most) of the cases we study here, the Bass series of R divided by the Poincaré-Betti series P_R is a very nice explicit polynomial (for more details about this–the Bøgvad formula–we refer to Section 5 of this paper). Thus if we want strange homological properties of the Ext-algebra of the Gorenstein ring $R \propto I(k)$, i.e. the Ext-algebra of the cohomology ring of the corresponding fundamental group $\pi_1(X)$ of the 3-manifold X corresponding to the Gorenstein ring $R \propto I(k)$ we only have to find (R, m) with $m^3 = 0$ with strange properties.

Corollary 4.1. *In the case when R is the cohomology ring* (*over* Q) *of the complement of a line arrangement L in $P^2(\mathbf{C})$, i.e. the*

*Orlik-Solomon algebra of L, the 3-manifold X corresponding to the Gorenstein ring $R \propto I(R/m)$ can be chosen as the boundary manifold of a tubular neighborhood of L in $P^2(\mathbf{C})$ [**7**]. In this case we found in [**31**] an arrangement where the* Ext*-algebra of R was not finitely generated. Then the* Ext*-algebra of the cohomology ring of X that corresponds to $R \propto I(k)$ cannot be finitely generated either since by Theorem* 4.1 *it is mapped onto $\mathrm{Ext}^*_R(k,k)$. In this case the dimension of $H^1(X, \mathcal{Q})$ is* 12.

In [**31**] we found two cases of arrangements: the MacLane arrangement and the mleas arrangement where the corresponding $R^!(z)$ is a transcendental function, and this leads to two cases where $H^1(X, \mathcal{Q})$ is of dimension 20, respectively 21. In order to press down this dimension to 12 and maybe to 11, I have to use some of my earlier results. In our paper [**32**] describing the homological properties of quotients of exterior algebras in 5 variables by quadratic forms (there are 49 cases found), we have found 3 cases (cases 12, 15 and 20) where $R^!(z)$ is proved to be transcendental and is explicitly given, and 3 other cases (cases 21, 22 and 33) where we conjecture that $R^!(z)$ is transcendental, but no explicit formula can be given, even in case 33 where we have now calculated the series $R^!(z)$ up to degree 33 using Backelin's et al. programme BERGMAN (some details are given in [**31**], where the "educated guess" now has to be abandoned): Here is Case 20:

$$R_{20} = \frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_1x_4 + x_2x_3, x_1x_5 + x_2x_4, x_2x_5 + x_3x_4)}.$$

Here the Hilbert series is $R_{20}(z) = 1 + 5z + 7z^2$, and the Koszul dual $R^!_{20}$ is given (according to the recipe we have described above) by

(4.1)
$$\frac{k\langle X_1,X_2,X_3,X_4,X_5\rangle}{([X_1,X_2],[X_1,X_3],[X_3,X_5],[X_4,X_5],[X_1,X_4]-[X_2,X_3],[X_1,X_5]-[X_2,X_4],[X_2,X_5]-[X_3,X_4])}$$

where $k\langle X_1, X_2, X_3, X_4, X_5\rangle$ is the free associative algebra in the five variables X_i and $[X_i, X_j] = X_iX_j - X_jX_i$ is the commutator. The corresponding Hilbert series is:

$$\frac{1}{R^!_{20}(z)} = \prod_{n=1}^{\infty}(1 - z^{2n-1})^5(1 - z^{2n})^3.$$

The proof of this last statement is by an adaption of the proof given in [**26**] when the ring R is commutative (the proof is even easier in the skew-commutative case).

Here is Case 12:

$$R_{12} = \frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_1x_2, x_1x_3 + x_2x_4 + x_3x_5, x_4x_5)},$$

and the Hilbert series is still $R_{12}(z) = 1+5z+7z^2$, but the corresponding Hilbert series for the Koszul dual $R^!_{12}$ is given by

$$\frac{1}{R^!_{12}(z)} = (1-2z)^2 \prod_{n=1}^{\infty} (1-z^n).$$

This is proved in the same way as it was proved for the corresponding R in the commutative case [**25**]. Finally here is Case 15:

$$R_{15} = \frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_1x_4 + x_2x_3, x_1x_5, x_3x_4 + x_2x_5)}$$

which has $R_{15}(z) = 1 + 5z + 7z^2$, but

$$\frac{1}{R^!_{15}(z)} = (1-2z) \prod_{n=1}^{\infty} (1-z^{2n-1})^3(1-z^{2n})^2,$$

which is proved in a similar way.

Now if R is any of the 3 rings above, then the Gorenstein ring $G = R \propto I(k)$ has Hilbert series $G(z) = 1+12z+12z^2+z^3$ and by the Gulliksen formula $P_G(x,y) = P_R(x,y)/(1 - xy\mathrm{Ext}^*_R(k,R)(x,y))$, the Bøgvad formula $\mathrm{Ext}^*_R(k,R)(x,y)/P_R(x,y) = x^2y^2R(-1/xy)$ (cf. Section 5 below) and the Löfwall formula $1/(P_R(x,y)) = (1+1/x)/R^!(xy) - R(-xy)/x$ we finally find that the transcendental properties of $P_G(x,y)$ are rationally related to those of $R^!(z)$. For the validity of the Bøgvad formula, cf. Section 5.

We now present the 3 other cases of R with 4 quadratic relations (cases 21, 22 and 33) where the Hilbert series is $R(z) = 1 + 5z + 6z^2$ and which lead to possibly strange Gorenstein rings and 3-manifolds X with the dimension of $H^1(X, \mathcal{Q})$ equal to 11. They are extremely easy to describe:

$$R_{21} = R_{20}/(x_4x_5), \quad R_{22} = R_{20}/(x_3x_5) \text{ and } R_{33} = R_{15}/(x_4x_5),$$

but the corresponding series $R^!(z)$ are unknown. But, using the Backelin et al. programme BERGMAN [**3**] we have calculated these series up to degrees 25, 14, and 33 respectively. In the last case we found in characteristic 47 using a work-station with 48 GB of internal memory that

$$\begin{aligned}\frac{1}{(1-z)^2R^!_{33}(z)} \\ &= 1-3z-z^2+z^3+2z^4+3z^5+z^6+z^7 \\ &\quad -z^8-z^9-2z^{10}-z^{11}-3z^{12}-z^{13}-z^{14} \\ &\quad -z^{15}+z^{17}+z^{18}+2z^{19}+z^{20}+z^{21} \\ &\quad +3z^{22}+z^{23}+z^{25}+z^{26}-z^{29}-z^{30}-z^{31}-z^{32}-z^{33}\cdots.\end{aligned}$$

But to go from degree 31 to degree 32 we needed more than one week of calculations, and from degree 32 to degree 33 we needed three weeks of calculations, even with an optimal order of the variables. But we still think that the series might be transcendental here.

Remark 4.1. All these results are in case the characteristic of the base field is 0 (in cases 21, 22 and 33 characteristic 47 or higher). In Case 20 we have different $R^!_{20}(z)$ for all characteristics and the same remarks might be applicable to the cases 21, 22 and 33. What this gives for the corresponding fundamental groups of the corresponding 3-manifolds X has not been studied.

Remark 4.2. If the Yoneda Ext-algebra $\mathrm{Ext}^*_R(k,k)$ is not finitely generated as an algebra then $\mathrm{Ext}^*_{R\propto M}(k,k)$ is not so either, since the algebra $\mathrm{Ext}^*_R(k,k)$ is a quotient of $\mathrm{Ext}^*_{R\propto M}(k,k)$. One can use this for the Gorenstein ring $G_{33} = R_{33} \propto I(k)$ which has Hilbert series $G_{33}(z) = 1 + 11z + 11z^2 + z^3$. Now $\mathrm{Ext}^*_{R_{33}}(k,k)$ needs an infinite number of generators if and only if the $\mathrm{Tor}^{R^!_{33}}_{3,j}(k,k)$ is non-zero for an infinite number of $j : s$ (cf., e.g., [**31**, Theorem 3.1, (a)]). Indeed, by computer calculations using the ANICK command in the programme BERGMAN one obtains that the dimensions of $\mathrm{Tor}^{R^!_{33}}_{3,j}(k,k)$ are 1 for $j = 4$ and then 0, 3, 0, 2, 1, 2 for $j = 5, 6, 7, 8, 9, 10$ and again 0, 3, 0, 2, 1, 2 for $j = 11, 12, 13, 14, 15, 16$, etc.

5. A formula of Bøgvad and m^2-selfinjective rings. Recall that Bøgvad proved the following in [**6**]: Let (R, m) be a local commutative ring with $m^3 = 0$. Assume that R has some special properties ($\operatorname{soc}(R) = m^2$ and R being a "beast" [**6**]). Then we have the formula for the Bass series $\operatorname{Bass}_R(Z)$, i.e., the generating series of $\operatorname{Ext}^*_R(k, R)$, the Poincaré-Betti series $P_R(Z)$ (i.e., the generating series of $\operatorname{Ext}^*_R(k, k)$) and $R(Z) = 1 + |m/m^2|Z + |m^2|Z^2$ (the Hilbert series of R):

$$\operatorname{Bass}_R(Z)/P_R(Z) = Z^2 R\left(-\frac{1}{Z}\right).$$

In [**6**] this formula is broken up into two assertions, of which the first one is often valid (the proposition "$E(R/m^2)$") and the other is more special. Lescot has observed in [**19, 21**] that the proof of this in [**6**] boils down to prove that Condition 5.ii and Condition 5.iii below are valid under some conditions:

Condition 5.i. *The natural map* $\operatorname{Ext}^*_R(k, m) \to \operatorname{Ext}^*_R(k, R)$ *is an epimorphism.*

Condition 5.ii. *The natural map* $\operatorname{Ext}^*_R(k, m/m^2) \to \operatorname{Ext}^*_R(k, R/m^2)$ *is an epimorphism.*

Condition 5.iii. *The natural map* $\operatorname{Ext}^*_R(k, m^2) \to \operatorname{Ext}^*_R(k, R)$ *is an epimorphism (for* $* = 0$ *this means that the socle of* R *is* m^2*).*

Condition 5.i is true if (R, m) is nonregular. Condition 5.ii is also often true (cf. condition "$E(R/m^2)$" in [**6**] and Proposition 1.10 in [**19**] for $I = m^2$, as well as the assertion that $G \to G/\operatorname{soc}(G)$ is a Golod map for a Gorenstein ring G [**2**]). We have not yet proved all skew-commutative versions of the preceding results, but computer computations indicate that they are true up to "high degrees" and probably in all degrees.

We now present some results that should give the skew-versions of some of the preceding results. We hope to return to these problems rather soon.

Let R be any ring (with unit). Recall that in order to test that a left R-module M is injective it is sufficient to test that for any left R-ideal J, any R-module map $\phi : J \to M$ can be extended to a map $R \to M$. Since the last map is given as $r \to r \cdot m_\phi$ where m_ϕ is a suitable element of M, this means that $\phi(j) - jm_\phi = 0$ for any $j \in J$. In particular, R is self-injective to the left if and only if for any R-module map $\phi : J \to R$ there is an element $r_\phi \in R$ such that $\phi(j) - j \cdot r_\phi = 0$ for all $j \in J$. This explains condition b) in the Lemma that follows:

Lemma 5.1. *Let (R, m) be a local (skew)commutative local ring where $m^3 = 0$ and J an ideal in R. The following two conditions are equivalent:*

a) *The natural map*

$$\mathrm{Ext}^1_R(R/J, m^2) \longrightarrow \mathrm{Ext}^1_R(R/J, R)$$

is surjective.

b) *For any R-module map $\phi : J \to R$ there is an element $r_\phi \in R$ such that $\phi(j) - j \cdot r_\phi \in m^2$ for all $j \in J$.*

Proof. The short exact sequence $0 \to J \to R \to R/J \to 0$ and the natural map $m^2 \to R$ give rise to a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(R/J, m^2) & \longrightarrow & \mathrm{Hom}_R(R, m^2) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(R/J, R) & \longrightarrow & \mathrm{Hom}_R(R, R) & \xrightarrow{i} &
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Hom}_R(J, m^2) & \xrightarrow{\delta'} & \mathrm{Ext}^1_R(R/J, m^2) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \iota} & & \\
\mathrm{Hom}_R(J, R) & \xrightarrow{\delta} & \mathrm{Ext}^1_R(R/J, R) & \longrightarrow & 0
\end{array}$$

Let us first prove that a) $\Rightarrow$ b). Thus assume that ι is onto. We start with a $\phi \in \mathrm{Hom}_R(J, R)$ and put $\alpha = \delta(\phi)$. We can assume that $\alpha = \iota(\xi)$. But $\xi = \delta'(\tilde{\xi})$. Thus $\delta(\kappa(\tilde{\xi}) - \phi) = 0$, so that $\kappa(\tilde{\xi}) - \phi$ comes by i from a map $R \to R$ of the form $r \to r \cdot r_\phi$. Thus to any map $\phi : J \to R$ there is an r_ϕ in R such that for all $j \in J$, $\phi(j) - j \cdot r_\phi \in m^2$, i.e., we have proved b). The converse follows from the "reverse" reasoning.

Remark 5.1. If L is any R-module and $P(L) \to L$ is the projective envelope of L we have an exact sequence

$$0 \longrightarrow S(L) \longrightarrow P(L) \longrightarrow L \longrightarrow 0,$$

where the first syzygy $S(L)$ of L is included in $m.P(L)$, we can redo the same reasoning for $P(L)/S(L)$ as we did for R/J in Lemma 5.1. The result is that $\mathrm{Ext}^1_R(L, m^2) \to \mathrm{Ext}^1_R(L, R)$ is onto if and only if for any map $\phi : S(L) \to R$ there is a map $j\phi : P(L) \to R$ such that $\phi(s) - j_\phi(s) \in m^2$ for all $s \in S(L) \subset P(L)$. Note that $P(L)$ is free so that j_ϕ is given by a matrix of elements in R.

Remark 5.2. In the selfinjective case (0-selfinjective) it is of course sufficient to require b) for the maximal ideal $J = m$. In the "m^2-selfinjective" case it is not clear (for us) what the right definition should be. We hope to return to this later. Therefore the definition below is maybe too strong.

Definition 5.1. We say that R is m^2-selfinjective if the conditions of Remark 5.2 are valid for $L = k$ and all syzygies of k.

Using this definition we can formulate the following consequence of Lemma 5.1 and Remark 5.1:

Corollary 5.1. *The following conditions are equivalent*:

α) $\mathrm{Ext}^*_R(k, m^2) \to \mathrm{Ext}^*_R(k, R)$ *is onto.*

β) R *is* m^2*-selfinjective.*

Remark 5.3. In the commutative case, it seems that many rings (R, m) with $m^3 = 0$ of the form $R = k[x_1, x_2, \dots, x_n]/(f_1, f_2, \dots, f_t)$ where the f_i are homogeneous quadratic forms are m^2-selfinjective. For the case when $n = 4$, cf. Remark 5.4 below.

Observation 5.1. Let (R, m) be a local ring with $m^3 = 0$, residue field $k = R/m$ and $\mathrm{soc}\,(R) = m^2$.

The following two conditions are equivalent:

a) R is m^2-selfinjective and Condition 5.ii above is true.

b) The Bass series of R, i.e. $\mathrm{Bass}_R(Z) = \sum_{i\geq 0} |\mathrm{Ext}^i(k, R)| Z^i$ is related to the Poincaré-Betti series $P_R(Z) = \sum_{i\geq 0} |\mathrm{Ext}^i_R(k, k)| Z^i$ by the "Bøgvad formula" $\mathrm{Bass}_R(Z) = Z^2 R(-1/z) P_R(Z)$, where $R(Z) = 1 + |m/m^2| Z + |m^2| Z^2$ is the Hilbert series of R.

Proof. Consider the long exact sequence obtained when we apply $\mathrm{Ext}^*_R(k,-)$ to the short exact sequence $0 \to m^2 \to R \to R/m^2 \to 0$:

$$\begin{aligned}0 &\to \mathrm{Hom}_R(k,m^2) \to \mathrm{Hom}_R(k,R) \to \mathrm{Hom}_R(k,R/m^2) \to \mathrm{Ext}^1_R(k,m^2)\\ &\to \mathrm{Ext}^1_R(k,R) \to \mathrm{Ext}^1_R(k,R/m^2) \to \mathrm{Ext}^2_R(k,m^2) \to \mathrm{Ext}^2_R(k,R)\\ &\to \mathrm{Ext}^2_R(k,R/m^2) \to \mathrm{Ext}^3_R(k,m^2) \to\end{aligned}$$

Since $\mathrm{soc}\,(R) = m^2$ the natural monomorphism $\mathrm{Hom}_R(k,m^2) \to \mathrm{Hom}_R(k,R)$ is indeed an isomorphism. Now according to a) all the maps $\mathrm{Ext}^i_R(k,m^2) \to \mathrm{Ext}^i_R(k,R)$ are also epimorphisms for $i \geq 1$. It follows that we have an exact sequence:

$$0 \to s^{-1}\mathrm{Ext}^*_R(k,R/m^2) \to \mathrm{Ext}^*_R(k,m^2) \to \mathrm{Ext}^*_R(k,R) \to 0$$

so that $\mathrm{Bass}_R(Z) - |m^2|P_R(Z) + Z.\mathrm{Ext}^*_R(k,R/m^2)(Z) = 0$. But it is now easy to apply Condition 5.ii and this gives the result that a) $\Rightarrow$ b). The converse is easy.

Remark 5.4. In the commutative case most rings of embedding dimension 4, which are quotients of $k[x,y,z,u]$ by an ideal I generated by homogeneous quadratic forms and having $m^3 = 0$, are according to [**34**] given by (the numbers of the ideals comes from [**34**]):

$$\begin{aligned}I_{29} &= (x^2+xy, y^2+xu, z^2+xu, zu+u^2, yz)\\ I_{54} &= (x^2, xy, y^2, z2, yu+zu, u^2)\\ I_{55} &= (x^2+xy, xz+yu, xu, y^2, z^2, zu+u^2)\\ I_{56} &= (x^2+xz+u^2, xy, xu, x^2-y^2, z^2, zu)\\ I_{57} &= (x^2+yz+u^2, xu, x^2+xy, xz+yu, zu+u^2, y^2+z^2)\\ I_{71} &= (x^2, y^2, z^2, u^2, xy, zu, yz+xu)\\ I_{78} &= (x^2, xy, y^2, z^2, zu, u^2, xz+yu, yz-xu)\\ I_{81} &= (x^2, y^2, z^2, u^2, xy, xz, yz-xu, yu, zu).\end{aligned}$$

The Hilbert series $R(z)$ of the different cases $R = k[x,y,z,u]/I$ are

$$\begin{gathered}1+4z+5z^2 \text{ (case 29)}, \quad 1+4z+4z^2 \text{ (cases 54, 55, 56, 57)},\\ 1+4z+3z^2 \text{ (case 71)}, \quad 1+4z+2z^2 \text{ (case 78)}\\ \text{and } 1+4z+z^2 \text{ (case 81)}.\end{gathered}$$

But the Hilbert series $R^!(z)$ of the Koszul duals $R^!$ are respectively (all different):

$$\frac{(1+z)^4}{(1-z^2)^5}, \quad \frac{1}{1-4z+4z^2}, \quad \frac{(1-z+z^2)^2}{(1-z)^3(1-3z+3z^2-3z^3)},$$
$$\frac{1-z+z^2}{(1-z)^2(1-3z+2z^2-z^3)}, \ \frac{1}{(1-z)^2(1-2z-z^2)}, \ \frac{1}{1-4z+3z^2},$$
$$\frac{1}{1-4z+2z^2} \quad \text{and} \quad \frac{1}{1-4z+z^2},$$

and in all cases the "Löfwall formula"

$$\frac{1}{P_R(x,y)} = (1+1/x)/R^!(xy) - R(-xy)/x$$

holds true. In all these cases *except* I_{78} the Bøgvad formula also holds true.

Remark 5.5. The Bøgvad formula is indeed a two-variable formula:

$$\mathrm{Bass}_R(x,y)/P_R(x,y) = x^2y^2R\left(-\frac{1}{xy}\right),$$

which shows that the non-diagonal elements occur for the two-variable Bass series in "the same way" as they occur in the two-variable Poincaré-Betti series. In the case of I_{78} this is not true. Indeed I_{78} is a Koszul ideal but the corresponding Bass series has non-diagonal elements. More precisely we have in that case:

$$\mathrm{Bass}_R(x,y)/P_R(x,y) = x^2y^2R\left(-\frac{1}{xy}\right) + x^2y^2 + xy^2.$$

Remark 5.6. There are of course similar results to those of Remarks 5.4 and 5.5 in the skew-commutative case. What corresponds to the "bad" case I_{78} in the case of four commuting variables is the case of four skew-commuting variables: $E(x,y,z,u)/(xy, xz-yu, yz-xu, zu)$.

Remark 5.7. In general it is not true that $\mathrm{Bass}_R(x,y)/P_R(x,y)$ is a rational function. This was first noted by Lescot in his thesis [**21**]. Take, e.g., S a ring with transcendental Bass series. Form

$T = S \propto I(S/m)$ and let R be $T/(\operatorname{socle} T)$. Then R is a so-called Teter ring [**37**], where the maximal ideal is isomorphic to its Matlis dual (this even characterizes Teter rings [**16**]), and from this one sees that $\operatorname{Bass}_R(x, y)/P_R(x, y)$ is transcendental [**19**, Corollary 1.9].

APPENDIX

How to prove that an algebra is Koszul using non-commutative permuted Gröbner bases, or Macaulay2 [**13**].

It is well known that a quadratic algebra (generators in degree 1, relations in degree 2) is Koszul if it has a quadratic Gröbner basis [**12**] for some ordering of the variables. This is stronger than being a Koszul algebra, but most of the dual Koszul algebras of Theorem 3.2 above satisfy this stronger condition for a suitable permutation of the variables. Since there are 9 variables, there are 9! permutations. But several years ago Jörgen Backelin constructed at my request a programme `permutebetter.sl` written in PSL and running under BERGMAN that in this case goes through the permutations of the variables and indicates the length of Gröbner basis for each permutation of the variables. We will only indicate how this works for case 63 in Table 6 of Vinberg et al. Here the 3-form is in their notations 129 138 167 246 257 345 leading according to Th-Rec. 2.1 to the algebra

$$\frac{k\langle e1,e2,e3,e4,e5,e6,e7,e8,e9\rangle}{([e2,e9]+[e3,e8]+[e6,e7],[e1,e9]-[e4,e6]-[e5,e7],[e1,e8]-[e4,e5],[e2,e6]+[e3,e5]},$$
$$[e2, e7] - [e3, e4], [e1, e7] - [e2, e4], [e1, e6] + [e2, e5], [e1, e3], [e1, e2]).$$

We now construct the input file for BERGMAN (no variables mentioned –they will be permuted) `invinberg63`:

```
(setq embdim 9)
(setq maxdeg 6)
(noncommify)
(off GC)
(ALGFORMINPUT)
e2*e9-e9*e2+e3*e8-e8*e3+e6*e7-e7*e6,-e1*e9+e9*e1+e4*e6
-e6*e4+e5*e7-e7*e5,-e1*e8+e8*e1+e4*e5-e5*e4,e2*e6
-e6*e2+e3*e5-e5*e3,-e2*e7+e7*e2+e3*e4-e4*e3,
```

```
-e1*e7+e7*e1+e2*e4-e4*e2,e1*e6
-e6*e1+e2*e5-e5*e2,e1*e3-e3*e1,e1*e2-e2*e1;
```

We also need the `varsfile9` which numbers the variables e1, e2, e3, e4, e5, e6, e7, e8, e9:

$$((1\ .\ e1)\ (2\ .\ e2)\ (3\ .\ e3)\ (4\ .\ e4)\ (5\ .\ e5)\ (6\ .\ e6)(7\ .\ e7)(8\ .\ e8)(9\ .\ e9)).$$

We are now ready to go and we start BERGMAN and load in the programme needed to do the work: `permutebetter.sl` that can be found on `http://www.maths.lth.se/matematiklth/personal/ufn/bergman/permutebetter.sl`

We get a command `permutedgbases` that we use as seen below creating an outputfile `outvinberg63`:

```
Bergman 1.001, 29-Aug-20
1 lisp> (dskin ``permutebetter.sl'')
nil
permutedgbases
nextpermalist
*** Function `degreeenddisplay' has been redefined
degreeenddisplay
nil
nil
2 lisp> (permutedgbases ``varsfile9'' ``invinberg63''
``outvinberg63'')
```

The outputfile `outvinberg63` contains all the information we need. In particular we see using a PERL programme, that Torsten Ekedahl has written for me (I thank him for that), that the order of the variables: e5, e1, e2, e4, e6, e7, e3, e8, e9 gives a non-commutative quadratic Groebner basis and therefore case 63 is Koszul. More precisely: of the 362880 permutations there are 18618 that give a quadratic Groebner basis. Indeed:

```
fgrep -e '% 2' outvinberg63 | wc -l
```

gives 362880, but

```
fgrep -e '% 3' outvinberg63 | wc -l
```

gives 344262, and 362880-344262=18618.

However for case 77 in Table 6 of Vinberg et al: 129 136 138 147 234 256 both `wc -l` give 362880. In this case we can use Macaulay2 [**13**] on the skew-commutative Koszul dual and find that in this case we have a quadratic Groebner basis as follows (here we use `ei` as a notation for the dual variables too):

```
R:=QQ[e1,e2,e3,e4,e5,e6,e7,e8,e9,SkewCommutative => true]
I:=ideal(e1*e5,e1*e6,e2*e7,e2*e8,e3*e5,e3*e6,e3*e7,
e3*e9,e4*e5,e4*e6,e4*e8,e4*e9,e5*e7,
e5*e8,e5*e9,e6*e7,e6*e8,e6*e9,e7*e8,e7*e9,e8*e9,e1*e7
+e2*e3,e1*e8-e2*e4,
e1*e9+e3*e4,e1*e9+e5*e6,e2*e9-e3*e8,e3*e8-e4*e7)
G= gens gb I
```

The output file contains

```
i3 :  G= gens gb I
o3 = | e8e9 e7e9 e6e9 e5e9 e4e9 e3e9 e7e8 e6e8 e5e8 e4e8
e3e8-e2e9
e2e8 e6e7 e5e7 e4e7-e2e9 e3e7 e2e7 e5e6+e1e9 e4e6 e3e6
e1e6 e4e5
e3e5 e1e5 e3e4+e1e9 e2e4-e1e8 e2e3+e1e7 |
```

which shows that the (skew-commutative) Groebner basis is quadratic and thus case 77 also gives a Koszul algebra.

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