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SOLVABLE INFINITE FILIFORM LIE ALGEBRAS

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ABSTRACT. An infinite filiform Lie algebra L is residually nilpotent and it is graded associated with respect to the lower central series, has smallest possible dimension in each degree, but is still infinite. This means that $\mathrm{gr}\,(L)$ is of dimension two in degree one and of dimension one in all higher degrees. We prove that if L is solvable, then already $[L, L]$ is abelian. The isomorphism classes in this case are given in [**1**], but the proof is incomplete. We make the necessary additional computations and restate the result in [**1**] when the ground field is the complex numbers.

1. Introduction. Infinite filiform Lie algebras have been studied among others by Fialowski [**2**], Millionshchikov [**4**] and Shalev-Zelmanov [**5**]. They may be seen as projective limits of finite dimensional filiform Lie algebras introduced by Vergne [**6**] as nilpotent Lie algebras with maximal degree of nilpotency among all nilpotent Lie algebras of a certain dimension. One result is that there is only one infinite filiform naturally graded Lie algebra L, where naturally graded means that L is isomorphic to its graded associated with respect to the filtration defined by the lower central series ($L^1 = L,\ L^{i+1} = [L, L^i]$, $i \geq 1$). This Lie algebra, denoted M_0, has a basis $a, e_1, e_2, \ldots$ with $[a, e_i] = e_{i+1}$ and $[e_i, e_j] = 0$ for all i and j. We have that M_0 is generated by a, e_1 which is a basis for the component of degree 1, and e_i is a basis for the component of degree i for $i \geq 2$. A general infinite filiform Lie algebra L may be seen as a (filtered) deformation of M_0 such that $\mathrm{gr}\,(L) \cong M_0$. Thus we have the following definition:

Definition 1.1. An infinite filiform Lie algebra is a Lie algebra L with a basis $a, e_1, e_2, \ldots$ satisfying

$$[a, e_j] = e_{j+1} \quad \text{and} \quad [e_j, e_{j+1}] = \sum_{s=1}^{\infty} \lambda_{js} e_{2j+1+s} \quad \text{for all } j \geq 1$$

for some λ_{js}, $j, s \geq 1$ such that for each j, $\lambda_{js} = 0$ for $s \gg 1$.

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The missing products in the definition above are determined by the Jacobi identity, more precisely, for $1 \leq k < m$ we have

$$[e_k, e_m] = \sum_{\substack{s \geq 1 \\ 0 \leq j-k \leq m-j-1}} (-1)^{j-k} \binom{m-j-1}{j-k} \lambda_{js} e_{m+k+s}.$$

Moreover, the Jacobi identity imposes conditions on (λ_{js}). This is nicely described in [**4**], where an explicit list of quadratic polynomials is given with common zero-set equal to the set of (λ_{js}) that determines a deformation. The polynomials are obtained by considering a deformation as above as a cochain ψ of degree two in the standard complex $C^*(M_0, M_0)$. The condition on ψ making the new structure defined by ψ to a Lie algebra is in general that ψ satisfies the "deformation equation," $d\psi + [\psi, \psi]/2 = 0$, where $[\cdot, \cdot]$ is the Nijenhuis-Richardson Lie superalgebra structure on $C^*(M_0, M_0)$. It is proved in [**4**] that in this special situation, the deformation equation is equivalent to $d\psi = [\psi, \psi] = 0$ and that all possible infinite filiform Lie algebras may be seen as the set of elements $\psi \in \oplus_{i>0} H^{2,i}(M_0, M_0)$ such that $[\psi, \psi] = 0$.

The solutions are known in two extreme cases. If $\lambda_{js} = 0$ for $j > 1$, then there are no conditions on (λ_{1s}) to be a solution. This gives rise to infinite filiform Lie algebras L satisfying $[L, L]$ abelian, i.e., L is solvable and next to abelian. We will prove in the next section that these Lie algebras consist of all solvable infinite filiform Lie algebras. The isomorphism classes, which are parametrized by infinitely many parameters, have been given by Bratzlavsky [**1**], but the proof there is incomplete. We perform the necessary extra computations in the last section and restate Bratzlavsky's theorem.

The other extreme case where the solutions are known is when $\lambda_{js} = 0$ for $s > 1$. In contrast to the first case, there are only two Lie algebras in this case, denoted M_2 and L_1 in the literature. Together with M_0 these Lie algebras are the only infinite filiform Lie algebras which are $\mathbf{N}$-graded and one-dimensional in each degree ([**2, 5**]). The Lie algebra M_2 belongs to both extreme cases; it has $\lambda_{js} = 0$ for $j, s > 1$. The Lie algebra L_1 is the subalgebra of the Witt algebra generated by $x^{i+1}\partial/\partial x$ for $i \geq 1$.

The subalgebra of the Witt algebra generated by $x^2 d/dx$ and $x^{i+1} d/dx$ for $i \geq k$, where $k > 1$, is also infinite filiform and defined by a defor-

mation of M_0 satisfying $\lambda_{js} = 0$ for $s \neq k-1$. It is conjectured in [**4**] that this gives all solutions in the case when $\lambda_{js} = 0$ for $s \neq s_0$ (together with the solution when only $\lambda_{1s_0} \neq 0$).

We do not know if there are other infinite filiform Lie algebras than those described above. In other words, does there exist a non-solvable infinite filiform Lie algebra which is not a subalgebra of the Witt algebra?

3. Criterion for solvability.

Lemma 2.1. *Let L be an infinite filiform Lie algebra. Then*

$$L \text{ is solvable} \iff L^k \text{ is abelian for some } k.$$

Proof. Let $L^{(1)} = L$ and $L^{(2k)} = [L^{(k)}, L^{(k)}]$. Then by induction and Jacobi identity $L^{(2k)} \subset L^{2k}$ and hence the implication to the left follows.

To prove the implication in the other direction, suppose L is solvable and $L^2 = [L, L] \neq 0$. Then there is a non-zero abelian ideal I contained in L^2. Let $e_k + r \in I$, where $r \in L^{k+1}$.

Claim: for all $j \geq k$ for all N, $\quad e_j \in I + L^N$.

Proof of Claim: By applying ad_a^{j-k} to $e_k + r \in I$ one gets $e_j + r_1 \in I$, where $r_1 \in L^{j+1}$. Suppose $r_1 = \lambda e_{j+1} + r'$, where $r' \in L^{j+2}$. Since $\lambda e_{j+1} + \lambda[a, r_1] \in I$, we get $e_j + r_2 \in I$ for some $r_2 \in L^{j+2}$. Continuing in this manner, the claim follows. Now, let $i, j \geq k$. Then for all $N \geq 1$

$$[e_i, e_j] \in [I, I] + L^N = L^N.$$

But $\cap_{N \geq 1} L^N = 0$. Hence L^k is abelian. □

Theorem 2.2. *Let L be a solvable infinite filiform Lie algebra over a field of characteristic zero. Then $[L, L]$ is abelian.*

Proof. According to the lemma, we may suppose that L^{n+1} is abelian and $n \geq 2$. We will prove that L^n is abelian. This gives the theorem by induction.

We know that L has a basis $\{a, e_1, e_2, \dots\}$ such that $[a, e_i] = e_{i+1}$ and $[e_i, e_j] \in \operatorname{span}\{e_k;\ k \geq i+j+1\}$. We have $L^i = \operatorname{span}\{e_j;\ j \geq i\}$. Hence, by assumption

$$[e_i, e_j] = 0 \quad \text{for } i, j \geq n+1.$$

Let

$$[e_{n-1}, e_n] = \lambda_0 e_{2n} + \lambda_1 e_{2n+1} + \cdots$$
$$[e_n, e_{n+1}] = \mu_0 e_{2n+2} + \mu_1 e_{2n+3} + \cdots$$

For $j > n+1$, we get

$$[e_n, e_j] = [e_n, [a, e_{j-1}]] = -[e_{n+1}, e_{j-1}] + [a, [e_n, e_{j-1}]] = [a, [e_n, e_{j-1}]].$$

Hence, by induction

$$[e_n, e_j] = \mu_0 e_{n+j+1} + \mu_1 e_{n+j+2} + \cdots,$$

for all $j \geq n+1$. Moreover,

$$\begin{aligned}[e_{n-1}, e_{n+1}] &= [e_{n-1}, [a, e_n]] = -[e_n, e_n] + [a, [e_{n-1}, e_n]] \\ &= \lambda_0 e_{2n+1} + \lambda_1 e_{2n+2} + \cdots.\end{aligned}$$

Suppose

$$[e_{n-1}, e_{n+j}] = (\lambda_0 - (j-1)\mu_0) e_{2n+j} + (\lambda_1 - (j-1)\mu_1) e_{2n+j+1} + \cdots.$$

Then,

$$\begin{aligned}[e_{n-1}, e_{n+j+1}] &= -[e_n, e_{n+j}] + [a, [e_{n-1}, e_{n+j}]] \\ &= (\lambda_0 - j\mu_0) e_{2n+j+1} + (\lambda_1 - j\mu_1) e_{2n+j+2} + \cdots,\end{aligned}$$

and hence, by induction, the last formula holds for all $j \geq 0$. Since L^{n+1} is abelian, we have

$$\begin{aligned}0 &= [[e_{n-1}, e_n], e_{n+1}] \\ &= \sum_{j\geq 0} \mu_j [e_{n-1}, e_{2n+2+j}] - \sum_{k\geq 0} \lambda_k [e_n, e_{2n+1+k}] \\ &= \sum_{j\geq 0} \mu_j \sum_{k\geq 0} (\lambda_k - (n+j+1)\mu_k) e_{3n+j+2+k} \\ &\quad - \sum_{j,k\geq 0} \lambda_k \mu_j e_{3n+k+2+j} \\ &= -\sum_{j,k\geq 0} (n+j+1)\mu_j \mu_k e_{3n+2+j+k}.\end{aligned}$$

Suppose $i \geq 0$ and $\mu_j = 0$ for $j \leq i-1$. Then from the above $-(n+i+1)\mu_i^2 = 0$. Hence, $\mu_i = 0$ and by induction it follows that $\mu_i = 0$ for all $i \geq 0$. Hence L^n is abelian. □

Corollary 2.3. *Suppose a set (λ_{js}) of parameters determines a deformation L of M_0 as in the introduction and suppose $\lambda_{js} = 0$ for all but finitely many j, s. Then $\lambda_{js} = 0$ for $j > 1$.*

Proof. The assumption gives that L^k is abelian for some k. Hence, by Lemma 2.1 and Theorem 2.2, it follows that $[L, L]$ is abelian which implies that $\lambda_{js} = 0$ for $j > 1$, since $[L, L]$ has a basis $\{e_2, e_3, \dots\}$ and $[e_j, e_{j+1}] = \sum_{s \geq 1} \lambda_{js} e_{2j+1+s}$. □

3. The solvable case. Suppose $(\lambda) = (\lambda_s)_{s \geq 1}$ is an arbitrary sequence of constants. We define a Lie algebra $L_{(\lambda)}$ as follows. $L_{(\lambda)}$ has basis $\{a, e_1, e_2, \dots\}$ and multiplication

$$\begin{aligned} [a, e_i] &= e_{i+1} \quad \text{for } i \geq 1 \\ [e_1, e_i] &= \sum_{j \geq 1} \lambda_j e_{1+i+j} \quad \text{for } i \geq 2 \\ [e_i, e_j] &= 0 \quad \text{for } i, j \geq 2. \end{aligned}$$

It follows that $[a, [e_1, e_i]] = [e_1, e_{i+1}]$ for $i \geq 2$. From this it is easy to see that Jacobi identity holds, so $L_{(\lambda)}$ is indeed an infinite filiform Lie algebra such that $L_{(\lambda)}^2$ is abelian and any infinite filiform Lie algebra L such that $[L, L]$ is abelian is obtained in this way. Moreover, $L_{(\lambda)}$ is obtained from M_0 by performing the deformation defined by (λ_{js}), where $\lambda_{js} = 0$ for $j > 1$ and $\lambda_{1s} = \lambda_s$.

In order to investigate when $L_{(\lambda)}$ and $L_{(\lambda')}$ are isomorphic, we will study automorphisms ϕ of $L_{(\lambda)}$ and determine the structure vector (λ') in the new basis $\{\phi(a), \phi(e_1), \phi(e_2), \dots\}$. Such a map is determined by

$$\begin{aligned} \phi(a) &= c_0 a + c_1 e_1 + c_2 e_2 + \cdots \\ \phi(e_1) &= d_0 a + d_1 e_1 + d_2 e_2 + \cdots, \end{aligned}$$

since then $\phi(e_{i+1}) = \phi([a, e_i]) = [\phi(a), \phi(e_i)]$ is determined inductively. It is easily seen that $c_0 \neq 0$ and $[\phi(e_1), \phi(e_2)] = c_0^{-1} d_0 \phi(e_3) + \cdots$. Hence, $\{\phi(a), \phi(e_1), \phi(e_2), \dots\}$ is a basis of the same kind as

$\{a, e_1, e_2, \dots\}$ (i.e., $[e_1, e_2] \in L^4$) only if $d_0 = 0$. Then ϕ is an automorphism if and only if $c_0 d_1 \neq 0$. Following [**3**] we may decompose any automorphism as a composition of three types of automorphisms:

$$\begin{aligned} \nu(c_0, c_1, d_1): \quad & \phi(a) = c_0 a + c_1 e_1 \\ & \phi(e_1) = d_1 e_1 \\ \sigma(d, k),\ k \geq 2: \quad & \phi(a) = a \\ & \phi(e_1) = e_1 + d e_k \\ \tau(c, k),\ k \geq 2: \quad & \phi(a) = a + c e_k \\ & \phi(e_1) = e_1. \end{aligned}$$

The proof of the fact that the automorphisms of type σ and τ do not change (λ) is missing in [**1**]. The following lemma completes the proof in [**1**].

Lemma 3.1. *Two Lie algebras $L_{(\lambda)}$ and $L_{(\lambda')}$ are isomorphic if and only if (λ') is obtained from (λ) by performing an automorphism of type ν.*

Proof. Consider first the case $\sigma(d, k)$, $k \geq 2$. Then

$$\begin{aligned} \phi(e_2) &= [a, e_1 + d e_k] = e_2 + d e_{k+1} \\ \phi(e_3) &= [a, e_2 + d e_{k+1}] = e_3 + d e_{k+2}, \end{aligned}$$

and in general, by induction

$$\phi(e_i) = [a, e_{i-1} + d e_{k+i-2}] = e_i + d e_{k+i-1}.$$

Hence,

$$\begin{aligned} [\phi(e_1), \phi(e_2)] &= [e_1 + d e_k, e_2 + d e_{k+1}] \\ &= \sum_{i \geq 1} \lambda_i e_{3+i} + d \sum_{i \geq 1} \lambda_i e_{k+2+i} \\ &= \sum_{i \geq 1} \lambda_i (e_{3+i} + d e_{k+2+i}) = \sum_{i \geq 1} \lambda_i \phi(e_{3+i}). \end{aligned}$$

It follows that the automorphism of type $\sigma(d, k)$, $k \geq 2$, does not change the vector (λ_{js}).

Now, consider the case $\tau(c,k)$, $k \geq 2$. Then

$$\begin{aligned}\phi(e_2) &= [a + ce_k, e_1] = e_2 - c\sum_{i\geq 1}\lambda_i e_{k+1+i}\\ \phi(e_3) &= [a + ce_k, e_2 - c\sum_{i\geq 1}\lambda_i e_{k+1+i}]\\ &= e_3 - c\sum_{i\geq 1}\lambda_i e_{k+2+i},\end{aligned}$$

and in general, by induction

$$\phi(e_j) = [a + ce_k, e_{j-1} - c\sum_{r\geq 1}\lambda_r e_{k+j-2+r}] = e_j - c\sum_{r\geq 1}\lambda_r e_{k+j-1+r}.$$

Hence,

$$\begin{aligned}[\phi(e_1),\phi(e_2)] &= \left[e_1, e_2 - c\sum_{i\geq 1}\lambda_i e_{k+1+i}\right]\\ &= \sum_{i\geq 1}\lambda_i e_{3+i} - c\sum_{i\geq 1}\lambda_i\sum_{r\geq 1}\lambda_r e_{k+2+i+r}\\ &= \sum_{i\geq 1}\lambda_i\left(e_{3+i} - c\sum_{r\geq 1}\lambda_r e_{k+2+i+r}\right) = \sum_{i\geq 1}\lambda_i\phi(e_{3+i}).\end{aligned}$$ □

It is easy to see that an automorphism of type $\nu(c_0, 0, d_1)$ transforms the sequence $(\lambda_1, \lambda_2, \dots)$ into the sequence $(c_0^{-2}d_1\lambda_1, c_0^{-3}d_1\lambda_2, \dots)$. This gives the possibility to choose the first two non-zero $\lambda_i : s$ to be 1.

Suppose that $\lambda_i = 0$ for $i < t$ and $\lambda_t = 1$, and consider an automorphism ϕ of type $\nu(0, c, 0)$. Then it is easy to see that

$$\begin{aligned}[\phi(e_1),\phi(e_2)] = \phi(e_{3+t}) + \lambda_{t+1}\phi(e_{3+t+1}) + \lambda_{t+2}\phi(e_{3+t+2}) + \cdots\\ + (\lambda_{2t} - (1+t)c)\phi(e_{3+2t}) + \text{higher terms.}\end{aligned}$$

Hence, if $c = \lambda_{2t}/(1+t)$ then (λ) is transformed to (λ') where $\lambda'_i = 0$ for $i < t$, $\lambda'_t = 1$ and $\lambda'_{2t} = 0$. In all, any (λ) may be transformed to a sequence (μ) with the following properties

$$\begin{aligned}&\mu_i = \mu_j = 1 \quad \text{for some } 1 \leq i < j,\ j \neq 2i\\ &\mu_r = 0 \qquad\quad \text{for } r < j,\ r \neq i\\ &\mu_{2i} = 0.\end{aligned}$$

Also, if (μ) and (μ') are two sequences of this kind, such that an automorphism of type $\nu(c_0, c_1, d_1)$ transforms (μ) to (μ'), then this forces $c_0 = 1, c_1 = 0, d_1 = 1$ and hence $\mu = \mu'$.

Hence, we get the following version of Bratzlavsky's theorem (in combination with Theorem 2.2) in the case when the ground field is the complex numbers.

Theorem 3.2. *Suppose L is a solvable infinite filiform Lie algebra over the complex numbers. Then there are unique integers $1 \leq r < s$, $s \neq 2r$ and complex numbers λ_t, $t > s$, $t \neq 2r$ such that L is isomorphic to the infinite filiform Lie algebra given by the equations*

$$\begin{aligned} [a, e_i] &= e_{i+1} && \text{for } i \geq 1 \\ [e_1, e_i] &= e_{1+i+r} + e_{1+i+s} + \sum_{\substack{t>s,\\ t\neq 2r}} \lambda_t e_{1+i+t} && \text{for } i \geq 2 \\ [e_i, e_j] &= 0 && \text{for } i, j \geq 2. \end{aligned}$$

REFERENCES

1. F. Bratzlavsky, *Classification des algèbres de Lie nilpotentes de dimension n, de classe $n-1$, dont l'idéal dérivé est commutatif*, Acad. Roy. Belg. Bull. Cl. Sci. **560** (1974), 858–865.

2. A. Fialowski, *On the classification of graded Lie algebras with two generators*, Moscow Univ. Math. Bull. **38** (1983), 76–79.

3. J.R. Gómez, A. Jimenéz-Merchán and Y. Khakimdjanov, *Low-dimensional filiform Lie algebras*, J. Pure Appl. Algebra **130** (1998), 133–158.

4. D.V. Millionshchikov, *The variety of Lie algebras of maximal class*, arXiv:0904.3181v1 [math.RA] 21 Apr 2009.

5. A. Shalev and E.I. Zelmanov, *Narrow Lie algebras: A coclass theory and a characterization of the Witt algebra*, J. Algebra **189** (1997), 294–331.

6. M. Vergne, *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes*, Bull. S.M.F. **98** (1970), 81–116.

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