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COMMUTING NILPOTENT MATRICES AND ARTINIAN ALGEBRAS

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ABSTRACT. Fix an $n \times n$ nilpotent matrix B whose Jordan blocks are given by the partition P of n. Consider the ring $\mathcal{C}_B \subset \mathrm{Mat}_n(\mathsf{k})$ of $n \times n$ matrices with entries in an algebraically closed field k that commute with B, and its subset, the variety $\mathcal{N}_B \subset \mathcal{C}_B$ of those that are nilpotent. Then $\mathcal{N}_B$ is an irreducible algebraic variety: so there is a Jordan block partition $Q(P)$ of the generic matrix $A \in \mathcal{N}_B$, that is greater than any other Jordan partition occurring for elements of $\mathcal{N}_B$. What is $Q(P)$? We here introduce an algebra $\mathcal{E}_B$ whose radical is $\mathcal{U}_B$, a maximal nilpotent subalgebra of $\mathcal{N}_B$. We study the poset $\mathcal{D}_P$, related to the digraph used by Oblak and Košir [**13**]. Using our results, we give new, simpler proofs for much of what is known about $Q(P)$, often clarifying or reducing the assumptions needed.

1. Introduction. We denote by k an algebraically closed field, and by $R = \mathsf{k}\{x, y\}$ the completed regular local ring–the power series ring–in two variables. We fix an n-dimensional k-vector space V, and consider the ring $\mathrm{End}_{\mathsf{k}}(V)$ of k-endomorphisms of V. We denote by $P \to n$ a partition $P = (p_1, \dots, p_t), p_1 \geq \cdots \geq p_t$ of $n = |P| = \sum p_i$. We will use the alternate notation $P = \{n_i\} = (n_{p_1}, \dots, n_1)$ where $n_i = \#$ parts equal to i, $t = \sum n_i$ and $n = \sum i n_i$. The notation $P = (4^2, 2^3)$ denotes $n_4 = 2, n_2 = 3$, so $P = (4, 4, 2, 2, 2)$. We denote by $S_P = \{i \mid n_i > 0\}$. We denote by J_P a nilpotent endomorphism of V whose Jordan decomposition has blocks given by the partition P: we will fix later a particular basis $\mathcal{V}$ of V, in which J_P has Jordan block matrix B. We denote by $\mathcal{C}_B$ the centralizer of B in the matrix ring $\mathrm{Mat}_n(\mathsf{k})$, and by $\mathfrak{J}_B$ the Jacobson radical of $\mathcal{C}_B$. We denote by $\mathcal{N}_B$ the

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algebraic subset of $\mathcal{C}_B$ comprised of the nilpotent matrices commuting with B, and by $\mathcal{U}_B$ a maximal nilpotent subalgebra of $\mathcal{N}_B$, as follows. There is a natural projection

$$\pi : \mathcal{C}_B \longrightarrow \mathcal{C}_B/\mathfrak{J}_B = \mathcal{M}_B \cong \Pi_{i \in S_P} \mathrm{Mat}_{n_i}(\mathsf{k}) \tag{1.1}$$

with kernel $\mathfrak{J}_B$ onto the semisimple quotient $\mathcal{M}_B$; since the field k is algebraically closed, $\mathcal{M}_B$ is the product of matrix rings (Lemma 2.2). We will choose a maximal nilpotent subalgebra $\mathcal{T} \subset \mathcal{M}_B$ comprised of products whose components are strictly upper triangular matrices (see Remark 2.1 ff. below), and we set $\mathcal{U}_B = \pi^{-1}(\mathcal{T}) \subset \mathcal{N}_B$. Then $\mathcal{U}_B$ is a maximal nilpotent subalgebra of $\mathcal{C}_B$. Up to isomorphism $\mathcal{U}_B$ depends only on the partition P. We will, however, describe $\mathcal{U}_B$ explicitly in terms of the basis $\mathcal{V}$ (Theorem 2.3).

For a nilpotent matrix A, we denote by P_A the partition given by the sizes of the blocks of its Jordan form. It is well known that $\mathcal{N}_B$ is an irreducible algebraic subvariety of affine n^2-space ([**3**, Lemma 2.3], [**4**, Lemma 1.5]); also any element of $\mathcal{N}_B$ is conjugate to one in $\mathcal{U}_B$ by a unit of $\mathcal{C}_B$ (Lemma 2.2). It follows that there is a partition $Q(P)$ giving the Jordan block decomposition P_A of the generic element A of $\mathcal{N}_B$, and for this purpose we may assume A is a generic element of $\mathcal{U}_B$. Here for any $A \in \mathcal{N}_B$, $Q(P) \geq P_A$ in the orbit closure partial order (3.13).

Problem. Given P, determine $Q(P)$.

The map: $P \to Q(P)$ has been studied in [**4, 6, 13, 15, 16**]. Previously, the study of pairs of commuting nilpotent $n \times n$ matrices had been connected to that of the punctual Hilbert scheme parametrizing length-n Artinian algebra quotients of $\mathsf{k}[x, y]$ by Baranovsky, Basili and Premet [**2, 3, 18**].

We introduce an algebra $\mathcal{E}_B$ that we consider fundamental in understanding this problem. We then derive many of the previously shown results about the problem using this algebra and its properties, including its relation to a poset $\mathcal{D}_P$ whose vertices correspond to a basis $\mathcal{V}$ of V. The algebra $\mathcal{E}_B$ is the path algebra of $\mathcal{D}_P$ mod relations. We give new combinatorial results about $\mathcal{D}_P$ and its relation to a certain weighted poset $\mathcal{B}_P$. We use these to give new, simpler proofs of some known results, often reducing the assumptions needed. We introduce an involution τ on $\mathcal{D}_B$, $\mathcal{E}_B$ and $\mathcal{B}_P$, and use it to reprove Theorem 3.1 and Theorem 3.3 of Oblak: that the largest part–the index–of $Q(P)$, is the maximum length of certain U shaped chains of $\mathcal{D}_P$. For the proof we work within $\mathcal{B}_P$. We end with a brief discussion of $Q(P)$ and sequences of chains in $\mathcal{D}_P$.

2. The algebra $\mathcal{E}_B \subset \mathcal{C}_B$ and the poset $\mathcal{D}_P$. We define an algebra $\mathcal{E}_B$ with radical $\mathcal{U}_B$. Given the nilpotent Jordan block matrix B of partition P, we have the B-module decomposition

$$V = \oplus V_{i,k}, \quad i \in S_p,\ 1 \leq k \leq n_i \tag{2.1}$$

into B-invariant subspaces

$$V_{i,k} \cong \mathsf{k}[B]/(B^i). \tag{2.2}$$

Specifying the identification of the space $V_{i,k}$ with $\mathsf{k}[B]/(B^i)$, we choose a B-generator $(1, i, k)$ of $V_{i,k}$, and let

$$(u, i, k) = B^{u-1}(1, i, k), \quad u = 1, \ldots, i. \tag{2.3}$$

We define the basis $\mathcal{V}$ of V:

$$\mathcal{V} = \{(u, i, k) \mid i \in S_P,\ 1 \leq u \leq i,\ 1 \leq k \leq n_i\}. \tag{2.4}$$

Remark 2.1 (Projection $\pi : \mathcal{C}_B \to \mathcal{M}_B$). Let $C \in \mathcal{C}_B$, the centralizer of B. Let $\{W\} = \cup_{i \in S_P}\{W_i\}$, where

$$\{W_i\} = \{(1, i, k) \mid 1 \leq k \leq n_i\}. \tag{2.5}$$

Evidently, $\{W\}$ is a minimal set of $\mathsf{k}[B]$ generators of V. Denote by W_i, W, respectively, the spans of $\{W_i\}, \{W\}$. Denote by π_i the projection $\mathcal{C}_B \to \mathrm{End}_{\mathsf{k}} W_i$, the endomorphism ring, obtained by restricting C to W_i then projecting to W_i. The following lemma is a result of Basili and others [**3, 19**]. Compare also [**11**, Theorem 6].

Lemma 2.2. *The canonical homomorphism $\pi : \mathcal{C}_B \to \mathcal{M}_B \cong \Pi \mathrm{Mat}_{n_i}(\mathsf{k})$ of* (1.1) *to the semisimple quotient satisfies*:

$$\pi = \prod \pi_i. \tag{2.6}$$

For each $A \in \mathcal{N}_B$, there is a unit C of $\mathcal{C}_B$ such that $CAC^{-1} \in \mathcal{U}_B$.

Proof. The second statement follows directly from (2.6) and that a nilpotent matrix in the image of π_i is conjugate to another that is strictly upper triangular. □

When $n_i > 1$ we order the elements of $\{W_i\}$, by $(1,i,1) < (1,i,2) < \cdots < (1,i,n_i)$.[1] We have from (2.6) the following result.

Theorem 2.3 [**3**, Lemma 2.3]. A. *The nilpotent commutator $\mathcal{N}_B$ is the set of $C \in \mathcal{C}_B$ for which each $\pi_i(C) \mid i \in S_P$ is nilpotent.*

B. *The maximal nilpotent subalgebra $\mathcal{U}_B \subset \mathcal{C}_B$ is comprised of those elements $C \in \mathcal{N}_B$ for which $\pi_i(C) \mid n_i > 1$, is strictly upper triangular.*

C. *Let $C \in \mathcal{N}_B$. Then $C \in \mathcal{U}_B$ if and only if C satisfies*

$$C \circ (1,i,k) \,|\, (1,i,k') = 0, \quad \text{for } k' \leq k \leq n_i, \tag{2.7}$$

that is, C acting on $(1,i,k)$ has zero component on each such $(1,i,k')$.

Proof. (A), (B) is Basili's result. (C) translates (B) to the restriction of C to W_i. □

We define the idempotent $\varepsilon_{i,k} \in \mathcal{C}_B$ projecting V to $V_{i,k}$, and let

$$\{E\} = \{\varepsilon_{i,k} \mid i \in S_P, 1 \leq k \leq n_i\}, \quad E = \langle \{E\} \rangle. \tag{2.8}$$

Definition 2.4 (**Algebra** $\mathcal{E}_B$). Given the basis $\mathcal{V}$ of V, we let

$$\mathcal{E}_B = E \oplus \mathcal{U}_B \subset \mathcal{C}_B. \tag{2.9}$$

We denote by $\mathfrak{D}_P$ the quiver associated to the algebra $\mathcal{E}_B$.

Note that $\mathcal{U}_B = \operatorname{rad} \mathcal{E}_B$, and that $\{E\} = \{\varepsilon_{i,k}\}$ are a complete set of primitive orthogonal idempotents of $\mathcal{E}_B$. We similarly label $\{(i,k), i \in S_P, 1 \leq k \leq n_i\}$ the t vertices of the quiver $\mathfrak{D}_P$.

Basis for $\mathcal{U}_B/{\mathcal{U}_B}^2$. For each pair $(V_{i,k}, V_{i',k'})$, $i > i'$ denote by $\beta_{i,i'}$ the canonical $B-$module surjection

$$\beta_{i,i'} : V_{i,k} \twoheadrightarrow V_{i',k'} \text{ satisfying } \beta_{i,i'}(1,i,k) = (1,i',k'). \tag{2.10}$$

and for $i < i'$ denote by $\alpha_{i,i'}$ the canonical B-module inclusion

$$\alpha_{i,i'} : V_{i,k} \hookrightarrow V_{i',k'} \text{ satisfying } \alpha_{i,i'}(1,i,k) = (1+i'-i, i', k'). \tag{2.11}$$

Evidently

$$\varepsilon_{i,k}\mathcal{E}_B\varepsilon_{i',k'} \subset \operatorname{Hom}(\mathsf{k}[B]/(B^i)), \mathsf{k}[B]/(B^{i'})) \cong \mathsf{k}[B]/(B^{\min(i,i')}), \tag{2.12}$$

and we have

$$\mathcal{U}_B/{\mathcal{U}_B}^2 = \oplus\varepsilon_{i,k}\, \mathcal{U}_B/\mathcal{U}_{\mathcal{B}}^2\, \varepsilon_{i',k'}. \tag{2.13}$$

By definition the quiver $\mathfrak{D}_P$ has

$$\dim_k\langle\varepsilon_{i,k}(\mathcal{U}_B/(\mathcal{U}_B)^2)\varepsilon_{i',k'}\rangle$$

arrows from $(i,k) \to (i',k')$ (see [**1**, Definition 3.1]).

For $i \in S_P$ we denote by i^+ the next larger element of S_P, and by i^- the next smaller element, if they exist. We say that i is *isolated* in S_P if both $i^- \neq i-1$ and $i^+ \neq i+1$. Thus 4 is isolated in $S_P = \{4,2,1\}$ for $P = (4,4,2,1)$.

In the following theorem, a homomorphism specified from $V_{i,k}$ to $V_{i',k'}$ is zero on the other direct summand components of V in (2.1). The homomorphism J_i: $V_{i,n_i} \to V_{i,1}$ corresponds to multiplication by the Jordan $i \times i$ single block matrix J_i; we may regard this after identifications, as multiplication by B on the vector space $\mathsf{k}[B]/(B^i)$:

$$J_i = m_B : V_{i,n_i} \cong \mathsf{k}[B]/(B^i) \longrightarrow V_{i,1} \cong \mathsf{k}[B]/(B^i). \tag{2.14}$$

Theorem 2.5 [**5**, Lemma 1.29, Section 3.1 and Lemma 3.4, Theorem 3.13] (**Basis for** $\mathcal{U}_B/\mathcal{U}_{\mathcal{B}}^2$). *The vector space $\langle\varepsilon_{i,k}\, \mathcal{U}_B/\mathcal{U}_{\mathcal{B}}^2\, \varepsilon_{i',k'}\rangle$ is zero or has dimension one. When non-zero it has as a basis the class in $\mathcal{U}_B/\mathcal{U}_{\mathcal{B}}^2$ of the following homomorphism in $\mathcal{U}_B$:*

i. *When $i' = i^-$, the homomorphism $\beta_{i,i'}$ from $V_{i,n_i} \to V_{i',1}$.*

ii. *When $i' = i^+$ the homomorphism $\alpha_{i,i'}$ from $V_{i,n_i} \to V_{i',1}$.*

iii. *When $i' = i$, and $n_i > 1$, the identity homomorphism $e_{i,k}$ from $V_{i,k} \to V_{i,k+1}, k = 1, \ldots n_i - 1$.*

iv. *When $i' = i$, and i is isolated, the homomorphism J_i (Jordan nilpotent block) from $V_{i,n_i} \to V_{i,1}$.*

We define the representation $\mathcal{M}\mathfrak{D}_P$ of $\mathfrak{D}_P$, by mapping the t vertices of $\mathfrak{D}_P$ to the corresponding vector spaces $V_{i,k}$, and the arrows given in (i)–(iv) above to the corresponding homomorphisms from $V_{i,k}$ to $V_{i',k'}$.

For $i \in S_P$ we let $j_i = \max(n_i + n_{i+1}, n_i + n_{i-1})$. Thus $j_i = n_i$ if i is isolated.

Corollary 2.6. *The dimension of $\mathcal{U}_B/{\mathcal{U}_B}^2$ satisfies*

$$\dim_{\mathsf{k}} \mathcal{U}_B/\mathcal{U}_B^2 = t + 2(\#S_P - 1) - \#\{i \in S_P \mid j_i > n_i\}. \tag{2.15}$$

Proof. The dimension count from iii. and iv. of Theorem 2.5 is

$$\sum_i (n_i - 1) + \#\{\text{ isolated } i \text{ in } S_P\} = t - \#\{i \in S_P \mid j_i > n_i\}; \tag{2.16}$$

the dimension count from i. and ii. is $2(\#S_P - 1)$. □

Example 2.7. When $P = (3,3)$ the quiver $\mathfrak{D}_P$ has two vertices, labeled (3,1) and (3,2) and two arrows, corresponding to the identity I from $V_{3,1}$ to $V_{3,2}$ and J_3 from $V_{3,2}$ to $V_{3,1}$. We have $J_3 \circ I = I \circ J_3 = J_3$, hence[2]

$$(J_3 \circ I)^3 = (I \circ J_3)^3 = 0. \tag{2.17}$$

Example 2.8. For $P = (3,1,1)$, the quiver $\mathfrak{D}_P$ has a loop on the vertex $(3,1)$. See Figure 1.

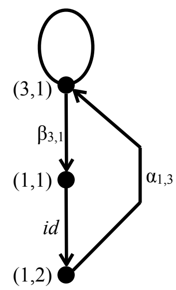

FIGURE 1. Quiver $\mathfrak{D}_P$ for $P = (3,1,1)$.

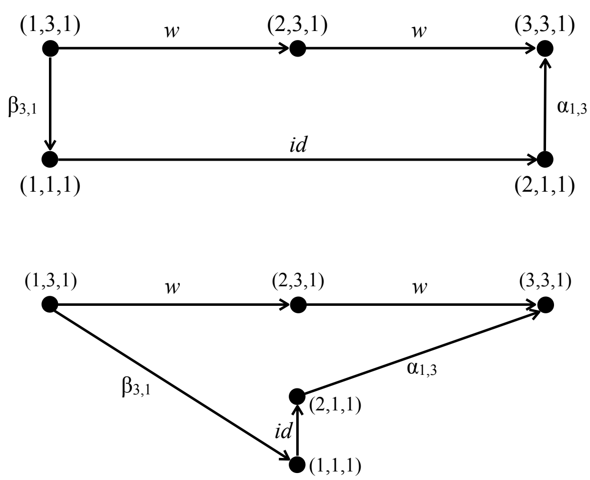

FIGURE 2. Poset $\mathcal{D}_P$ and maps for $P = (3,1,1)$.

2.1. The $\mathsf{k}[B]$ module V and the poset $\mathcal{D}_P$. Recall from (2.1) that $V = \oplus V_{i,k} \mid i \in S_P,\ 1 \leq k \leq n_i$ and $V_{i,k}$ (the kth copy of V_i) has cyclic vector $(1,i,k)$ and basis $\{(u,i,k) = B^{u-1}(1,i,k), \mid 1 \leq u \leq i\}$, the whole comprising the basis $\mathcal{V}$ for V. For $v' \in \mathcal{V}$ denote by $v\ |_{v'}$ the component of v on v', in this basis. We now define a poset $\mathcal{D}_P$ whose vertices are these basis elements, and such that

$$v \leq v' \iff \exists A \in \mathcal{U}_B \text{ such that } Av|_{v'} \neq 0. \tag{2.18}$$

We restrict $\beta_{i,i'}$ to the context of Theorem 2.5. Thus, from (2.10) the map $\beta_{i,i'}$ for $i > i'$ maps (u, i, n_i) to $(u, i', 1), 1 \leq u \leq i'$, and is zero on (u, j, k) for $j \neq i$ or $k \neq n_i$. Similarly, from (2.11) $\alpha_{i,i'}$, $i < i'$ maps (u, i, n_1) to $(u + i' - i, i', 1) \mid u \leq i$ and is zero on other summands of (2.1), and $e_{i,u,k} : (u, i, k) \to (u, i, k + 1)$, $1 \leq k < n_i$, and is zero on all other summands.

Definition 2.9 (Maps and poset $\mathcal{D}_P$ associated to P). a. Vertices ${\mathcal{D}_P}^o$ of $\mathcal{D}_P$: For each pair $(u, i) \mid i \in S_P$, $1 \leq u \leq i$, there is a column of n_i vertices (u, i, k), $1 \leq k \leq n_i$. The vertices in the column are simply ordered so that $(u, i, 1)$ is least (at the bottom), and (u, i, n_i) is maximum (at the top).

b. The adjacent vertices $v < v'$ and edges ${\mathcal{D}_P}^1$ of D_P are determined by the following maps.[3]

i. $\beta_{i,i-}$ from the top vertex (u, i, n_i) of any column in the i row to the bottom vertex $(u, i^-, 1)$ in the next reachable position in the i^- row;

ii. $\alpha_{i,i+}$ from the top vertex (u, i, n_i) of any column in the i row to the bottom vertex $(u + i^+ - i, i^+, 1)$ in the next reachable position in the i^+ row;

iii. identity mapping $e_{u,i,k}$ upward from (u, i, k) to $(u, i, k + 1), 1 \leq k < n_i$.

iv. When i is isolated (when both $i - 1 \notin S_P$ and $i + 1 \notin S_P$), and $1 \leq u < i$, the map w_i corresponding to the Jordan block J_i, from the top vertex (u, i, n_i) of the u column in the i row to the bottom vertex $(u + 1, i, 1)$ of the next column in that row.

In Figure 2 we give two visualizations of the poset $\mathcal{D}_P$ and maps for $P = (3, 1, 1)$. All compositions of maps corresponding to chains ending to the right of the $n = 5$ depicted vertices are zero. For example, $w^3 = \beta_{3,1} \cdot w = 0$.

Remark 2.10. Now (2.18) follows from Theorem 2.5 and Definition 2.9. The small quiver $\mathfrak{D}_P$ has cycles (Example 2.7). The poset

$\mathcal{D}_P$ is acyclic; the paths in $\mathcal{D}_P$ correspond to the arrows in the digraph studied by Oblak and by Košir and Oblak [**13, 15**]. A similar arrangement of n vertices is discussed in [**5**][4], and is also found in a note by Oblak et al., from June 2008 [**7**], and appears to be envisioned earlier by Oblak and Košir. The two sided ideal I of the path algebra $K\mathcal{D}_P$ is defined by all commutativity relations on $\mathfrak{U}_P$; that is, any two paths from a vertex (u,i,k) to another (u',i',k') have difference in I.[5] This is an admissible ideal, defining a bound quiver algebra $K\mathcal{D}_P/I$ that is *basic*: the canonical semisimple quotient E of $K\mathcal{D}_P/I$ is a product of fields [**1**, Lemma 2.10].

Lemma 2.11. *We have $\mathcal{E}_B \cong K\mathcal{D}_P/I$. The algebra $\mathcal{E}_B$ has the filtration*

$$\mathcal{E}_B \supset \mathcal{U}_B \supset {\mathcal{U}_B}^2 \supset \cdots \supset {\mathcal{U}_B}^{i(Q(P))} = 0,$$

where $i(Q(P))$ is the index of $Q(P)$.

Although $K\mathcal{D}_P$ is naturally graded by path length, the algebra $\mathcal{E}_B$ is not graded: the basis–generating set–for the ideal I of relations includes binomials corresponding to the differences of two paths from a vertex i to a vertex j in the poset $\mathcal{D}_P$: these paths in general have different lengths. Theorem 2.5 gives the basis of $\mathcal{U}_B/(\mathcal{U}_B)^2$, and implies

Corollary 2.12. *The algebra $\mathcal{U}_B$ is generated as $\mathsf{k}[B]$ algebra by the maps*

$$\alpha_{i,i'}, e_{u,i,k}, \beta_{i,i'}, w_i, \tag{2.19}$$

restricted as in Theorem 2.5 *and Definition* 2.9.

2.2. The action of $A \in \mathcal{U}_B$ on $\mathcal{V}$ and on the poset $\mathcal{D}_P$. For each $i \in S_P$, the poset $\mathcal{D}_P$ has a (vertical) sheet comprised of n_i rows, each of length i. The kth row (counting from the bottom) of this sheet is

$$((1,i,k), \dots, (u,i,k), \dots (i,i,k)).$$

We visualize this row as situated symmetrically about a center $u = (i+1)/2$. We define the statistic $\nu(u,i,k) = \nu(u,i) = 2u - i - 1$.[6]

Lemma 2.13. *Let* $(u', i', k') \geq (u, i, k)$ *in* $\mathcal{D}_P$. *Then*

$$\begin{aligned} u' &\geq u, \\ u' - (i'+1) &\geq u - (i+1), \text{ and} \\ \nu(u', i') &\geq \nu(u, i) + |i' - i|. \end{aligned} \tag{2.20}$$

Proof. We use (2.18). Since $(u, i, k) = B^{u-1} \cdot (1, i, k)$, we have

$$A \cdot (u, i, k) = B^{u-1} A \cdot (1, i, k) \subset B^{u-1} V \Longrightarrow u' > u.$$

Since $B^{i+1-u}(u, i, k) = 0$ we have

$$\begin{aligned} B^{i+1-u} A \cdot (u, i, k) &= A B^{i+1-u}(u, i, k) = 0 \\ &\Longrightarrow u' - (i'+1) \geq u - (i+1). \end{aligned}$$

These imply (2.20). □

A *saturated chain* in a poset is a chain that is not the proper subchain of a longer chain with the same endpoints. For $p = (u, i, k) \in \mathcal{D}_P$ we denote by $\iota(p) = i \in S_P$ the projection to S_P and likewise we define the sequence $\iota(C)$ for a chain C of $\mathcal{D}_P$. A *corner vertex* $c \in C$ is a vertex such that $\iota(c)$ is a local min or local max of $\iota(C)$.

Proposition 2.14. *Let* $C' = (c_0, c_1, \dots, c_s), c_0 \leq c_1 \leq \cdots \leq c_s$ *denote the corner points of a saturated chain* $C \subset \mathcal{D}_P$. *Then*

$$\nu(c_s) = \nu(c_0) + \sum_{1 \leq i \leq s} \left|\iota(c_i) - \iota(c_{i-1})\right| + \varepsilon, \tag{2.21}$$

where $\varepsilon = 2\#\{c_i \mid \iota(c_i) = \iota(c_{i+1})\}$ *for* $\iota(c_i)$ *isolated in* S_P.

Proof. This is immediate from (2.20). □

Definition 2.15. We define an involution τ on the vertices of the poset $\mathcal{D}_P$ as follows:

$$\tau(u, i, k) = (i + 1 - u, i, n_i + 1 - k). \tag{2.22}$$

For an arrow $p \to p'$ of $\mathcal{D}_P$ we set

$$\tau(p \to p') = \tau(p)' \to \tau(p). \tag{2.23}$$

We extend τ to the algebra $\mathcal{E}_B$ as follows: We set

$$\tau(\alpha_{i,i'}) = \beta_{i',i}, \quad \tau(w_i) = w_i, \quad \tau(e_{u,i,k}) = e_{i+1-u,i,n_i-k}. \tag{2.24}$$

We note that τ is a generalized transpose:

Lemma 2.16. *For $A, A' \in \mathcal{E}_B$ we have*

$$\tau(A \circ A') = \tau(A') \circ \tau(A). \tag{2.25}$$

2.3. Almost rectangular subpartitions of P. An *almost rectangular* partition of n is one whose largest and smallest part differ by at most one [**4, 13**]. They are the Jordan block partitions of powers $(J_n)^k$ of nilpotent *regular* matrices J_n, having a single $n \times n$ block. For example $(J_5)^2$ has Jordan blocks $(3, 2)$, and $(J_5)^3$ has Jordan blocks $(2, 2, 1)$. We will show

Theorem 2.17 [**3**, Proposition 2.4]. *The number of parts in $Q(P)$ is the minimum number $r(P)$ of partitions $P_1, \dots, P_r$ in a decomposition of P as union of almost rectangular subpartitions.*

Example. Let $P = (5, 4, 3, 3, 2, 1)$: then $P = P_1 \cup P_2 \cup P_3$, where $P_1 = (5, 4), P_2 = (3, 3, 2)$ and $P_3 = (1)$ is such a minimal decomposition, and $r(P) = 3$.

Recall that the number of parts in the Jordan block partition P_A for a nilpotent matrix A is $\dim_{\mathsf{k}} \ker(A)$.

Lemma 2.18. *Let $A \in \mathcal{U}_B$. Then $\dim_{\mathsf{k}} \ker(A) \geq r(P)$.*

Proof. Let $P_1 \cup \dots \cup P_r, r = r(P)$ be a minimal length decomposition of P into disjoint almost rectangular subpartitions, arranged so that the parts of P_i are smaller than those of P_{i+1}. For $a = 1, \dots, r$, we denote by

$$\mathcal{D}_P(a) = \{(i+1-a, i, k),\ i \in S_P,\ 1 \leq k \leq n_i.\} \tag{2.26}$$

the ath saturated chain from the right of the poset $\mathcal{D}_P$, and by

$$\mathcal{V}(a) = \langle \mathcal{D}_P(a) \rangle \cong ((0 : B^a)/(0 : B^{a-1})) \cap \mathcal{V} \tag{2.27}$$

its span. We let i_a be the smaller part of P_a, and we denote by $\kappa_a = (i_a + 1 - a, i_a, 1)$ the minimum element in $\mathcal{D}_P(a) \cap \iota^{-1}(i_a)$, $a = 1, \dots, r$. Let $\overline{\kappa} = \{v \in \mathcal{D}_P \mid v \geq \kappa\}$, and set

$$\begin{aligned} \mathcal{V}'(a) &= \mathcal{V}(a) \cap \overline{\kappa_a} \\ \mathcal{V}' &= \mathcal{V}'(1) \cup \dots \cup \mathcal{V}'(r), \\ \mathcal{V}'' &= \mathcal{V}' - \{\kappa_1, \dots, \kappa_r\} \end{aligned} \tag{2.28}$$

We have for $a = 1, \dots, r-1$,

$$\begin{gathered} \nu(\kappa_a) = i_a + 1 - 2a, \quad \nu(\kappa_{a+1}) = i_{a+1} + 1 - 2(a+1), \text{ so} \\ \nu(\kappa_{a+1}) - \nu(\kappa_a) = i_{a+1} - i_a - 2. \end{gathered} \tag{2.29}$$

The assumption that $\{P_1, \dots, P_r\}$ is a disjoint AR decomposition, implies that for each $a = 1, \dots, r-1$ we have $i_{a+1} - i_a \geq 2$. Hence, from (2.29)

$$\nu(\kappa_{a+1}) \geq \nu(\kappa_a). \tag{2.30}$$

Now (2.20), (2.30) and $i_{a+1} - i_a > 0$ imply that no vector v with non-zero component on κ_a can be in the image of $\langle \overline{\kappa_{a+1}} \rangle$ under A. Hence,

$$A \circ \langle \mathcal{V}' \rangle \subset \langle \mathcal{V}'' \rangle, \tag{2.31}$$

showing that A has at least $r = r(P)$-dimensional kernel. □

Proof of Theorem 2.17. Let $P_1 \cup \dots \cup P_r$ be a decomposition of P into $r = r(P)$ almost rectangular subpartitions. The Jordan block matrix J_{P_i} commutes with a conjugate of $J_{|P_i|}$, so $B = J_P$ commutes with a conjugate A of the nilpotent Jordan matrix with blocks of sizes

$$\begin{array}{ccccc}
* & * & \bullet & 2 & 1 \\
 & * & * & 2 & 1 \\
 & & * & \bullet & 1 \\
 & & & * & 1 \\
 & & & * & 1 \\
 & & & & \bullet
\end{array}$$

FIGURE 3. $P = (5,4,3,2,2,1)$, $\dim_{\mathsf{k}} \ker(A) \geq 3$.

$|P_1|, \ldots, |P_r|$, and rank $\ker(A) = r$. Since kernel rank being less than $r+1$ is a Zariski open condition on $\{A \in \mathcal{N}_B\}$, we have that the number of parts of a generic $A \in \mathcal{N}_B$ is no more than $r(P)$. This and Lemma 2.18 complete the proof of Theorem 2.17. □

Example 2.19. Let $P = (5,4,3,2,2,1)$, $P_1 = (2,2,1)$, $P_2 = (4,3)$, $P_3 = (5)$. Then the key vertices are $\kappa_1 = (1,1,1)$, $\kappa_2 = (2,3,1)$, $\kappa_3 = (3,5,1)$ ($\bullet$'s in Figure 3). Here $\mathcal{V}'(1) = \mathcal{V}(1)$, the span of $\mathcal{D}_P(1)$, the rightmost chain of $\mathcal{D}_P$; $\mathcal{V}'(2) = \langle (2,3,1), (3,4,1), (4,5,1) \rangle$, $\mathcal{V}'(3) = \langle (3,5,1) \rangle$, while $\mathcal{V}''$ is the span of the vertices labeled "1" or "2." Here $\dim_{\mathsf{k}} \mathcal{V}' = 10$, $\dim_{\mathsf{k}} \mathcal{V}'' = 7$, and $A\mathcal{V}' \subset \mathcal{V}''$. Of course, κ_a is not necessarily in $\ker A$.

2.4. The Gorenstein property of $F[A,B]$. We now prove a result implying Theorem 2.23 of Košir and Oblak that $\mathsf{k}[A,B]$ is Gorenstein for generic A [**13**]. Our proof of Theorem 2.20 is closely related to that in [**13**]. But we show more precisely that if a certain minimal set of entries of A over a field F are non-vanishing, then $F[A,B]$ is Gorenstein and has a cyclic vector.

The homomorphism $A \in \mathcal{U}_B$. We define an extension T of k which we will use here, and later to state Conjecture 3.16. A *maximal* consecutive subsequence (MCS) of S_P is one not properly contained in a larger consecutive subsequence. We define the subset $S_P{}'' \subset S_P$:

(2.32)

$\ell \in S_P{}'' \iff (\ell, \ell+1, \ldots, \ell+\lambda_\ell - 1)$ is an MCS of S_P of *odd* length λ_ℓ.

For example $\ell \in S_P''$ if $\ell \in S_P$ is *isolated*: both $\ell - 1 \notin S_P$ and $\ell + 1 \notin S_P$–then ℓ is the unique element of an MCS. The field $T = \mathsf{k}(\{s_i, t_i, t_{j,k}, z_\ell\})$ is an extension of k obtained by adjoining variables indexed as below, and over $F = T$ we let

$$A = \sum_{i \in S_P - p_t} (s_i\beta_{i,i-} + t_i\alpha_{i-,i}) + \sum{}' t_{j,k}e_{j,u,k} + \sum{}'' z_l w_l \tag{2.33}$$

where $\sum'$ is the sum over triples $(u, j, k) \mid k < n_i$, and $\sum''$ denotes the sum over indices $\ell \in S_P''$.

We fix a field $F = \theta(T)$ and work in $\mathcal{U}_B$ over F.

Theorem 2.20. *Let $A \in \mathcal{U}_B$, and suppose that A satisfies, when written in the format* (2.33)*, that for all i, for all (i, k),*

$$\theta(s_i), \theta(t_i), \theta(t_{i,k}) \neq 0. \tag{2.34}$$

The $F[A, B]$-module V has cyclic vector the source $(1, p_1, 1)$ of $\mathcal{D}_P$, and has as a cocyclic vector the sink (p_1, p_1, n_{p_1}). The algebra $\mathcal{A} = F[A, B]$ is a length-n Gorenstein Artinian algebra over the field F.

Proof. The vector space W of (2.5) is the span of vertices $\{W\}$ on the left side of the poset $\mathcal{D}_P$. It satisfies

$$V = W \oplus BV = W \oplus \langle B\mathsf{k}[B] \cdot W\rangle. \tag{2.35}$$

Denote by $C = (c_0, \ldots, c_{t-1}), t = \sum n_i$ the ordered chain in $\mathcal{D}_P$ whose vertices are W, and by $\pi_C : \mathsf{k}[A] \to \mathrm{Mat}_t(W)$ the projection to $\mathrm{Mat}_t(W)$ using the direct sum decomposition (2.35). As with any saturated chain of $\mathcal{D}_P$ between two given vertices, here the source $(1, p_1, 1)$ and $(1, p_t, n_{p_t})$, the assumed nonvanishing of the s_i's and the $t_{i,k}$'s of (2.33)–coefficients of the β's and the identity maps $e_{u,i,k}$ along the chain C–implies that we have for each i, $0 \leq i \leq \ell - 1$.

$$\pi_C(A^i) = \eta_i c_i + \varepsilon_i, \text{ with } \eta_i \in F, \text{ and } \varepsilon_i \in \langle c_{i+1}, \ldots c_\ell\rangle. \tag{2.36}$$

Thus, $\pi_C(A)$ has triangular form on the ordered basis for W given by C. Since $B\,\mathsf{k}[B]W$ is an A–module, this suffices to show that the image $\pi_C \cdot k[A] \supset W$. We conclude from $\mathsf{k}[B] \cdot W = V$ that $F[A,B] \cdot (1, p_1, 1) = V$, so $(1, p_1, 1)$ is indeed a cyclic vector for $F[A, B]$.

Next, note that $(0 : B) \subset V$ is the span of right hand endpoints of $(0 : B) = \mathcal{V}(1) = \langle \mathcal{D}_P(1) \rangle$ where

$$\mathcal{D}_P(1) = \{(i, i, k) \mid i \in S_P, 1 \leq k \leq n_i\}. \tag{2.37}$$

That $A \in \mathcal{U}_B$ and satisfies (2.34) implies that A satisfies $\mathcal{A}\mathcal{D}_P(1) \subset \mathcal{D}_P(1)$ and only moves basis elements upward along the chain $\mathcal{D}_P(1)$. Thus, in the $\mathcal{A}$ module V we have

$$(0 : A) \cap (0 : B) = \langle (p_1, p_1, n_{p_1}) \rangle,$$

implying V has the cocyclic vector $(0 : (A, B)) = \langle (p_1, p_1, n_{p_1}) \rangle$, that is unique up to scalar, as claimed. Since V is a cyclic $\mathcal{A}$-module and $\dim_F \mathcal{A} = \dim_F V$ the socle of $\mathcal{A}$ has dimension one as F vector space, implying that $\mathcal{A}$ is Gorenstein. □

Remark 2.21. The two parts of the statement and proof are dual under the action of the involution τ on $\mathcal{D}_P$ and its associated objects. The proof gives a quite weak condition for $F[A, B]$ to be Gorenstein. There are examples of A satisfying this condition, but such that the Jordan block partition p_A does not have parts differing pairwise by at least 2 (Example 3.17). It follows that in such cases $P_A \neq P(H)$, the partition associated below to the Hilbert function $H(\mathcal{A}), \mathcal{A} = F[A, B]$. However, by Theorem 2.22 a generic element of the pencil $A + \lambda B$ will have the partition $P(H)$.

2.5. The ring $\mathsf{k}[A, B]$ and stability of $Q(P)$. We resume briefly some results of [**4, 13**]. We denote by $I_{A,B}$ the kernel of

$$\phi : \mathsf{k}\{x, y\} \longrightarrow \mathsf{k}[A, B], x \longrightarrow A, y \longrightarrow B, \tag{2.38}$$

and denote by $\mathcal{A}_A = \mathsf{k}\{x, y\}/I_{A,B}$ the Artinian quotient algebra (when A is understood, we will shorten this to $\mathcal{A}$). We denote by $\mathfrak{m}$ the maximal ideal $\mathfrak{m} = (x, y)$ of R, or of $\mathcal{A}$. The associated graded algebra

$$\mathcal{A}^* = Gr_{\mathfrak{m}}(\mathcal{A}) = \oplus_i \mathcal{A}_i,\ \mathcal{A}_i = \mathfrak{m}^i\mathcal{A}/\mathfrak{m}^{i+1}$$

satisfies

$$\mathcal{A}_i \cong R_i/I_i,\ I_i = (\mathfrak{m}^i \cap I + \mathfrak{m}^{i+1})/(\mathfrak{m}^{i+1}).$$

The *Hilbert function* of $\mathcal{A}$ is the sequence (here $h_0 = 1$)

$$H = (h_0, , h_1, \dots), h_i = \dim_{\mathsf{k}} \mathcal{A}_i. \tag{2.39}$$

We will usually list just the finite number of nonzero entries of H. We define the partition $P(H)$ to be the dual partition to the sequence of values $\{h_0, h_1, \dots\}$ (see [**4**, Definition 1.7]). For $H = (1, 2, 3, 2, 1, 1), P(H) = (6, 3, 1)$. We denote by m_A the operator multiplication by A on the ring $\mathcal{A} = \mathsf{k}[A, B]$. The first statement below follows from Lemma 2.2 and Theorem 2.20.

Theorem 2.22 [**4**, Theorem 2.21]. *The partition $Q(P)$ is the partition giving the Jordan blocks of m_A on $\mathcal{A}$ for A generic in $\mathcal{U}_B$. Assume* k *has characteristic zero, or* $\operatorname{char} \mathsf{k} > n$. *Then $Q(P) = P(H)$, the partition dual to the Hilbert function $H(\mathcal{A})$.*

Theorem 2.23 [**13**]. *The ring $\mathcal{A} = \mathsf{k}[A, B]$ for A generic in $\mathcal{N}_B$ is Gorenstein.*

We will say P is *stable*, or (after D.I. Panyushev) *self-large* if $Q(P) = P$. It was known that P is stable in this sense if and only if the parts of P differ pairwise by at least two (see [**4**, Theorem 1.12] or [**16**, Example 2.5(a)]). We next connect Theorem 2.23 to the stability of $Q(P)$. Recall that the Hilbert function $H = H(\mathcal{A})$ of codimension two Artinian algebras–$I_{A,B} \in \mathfrak{m}^2$–satisfy, with j the *socle degree* of $\mathcal{A}$

$$H = (1, 2, \dots, d, h_d, h_{d+1}, \dots, h_j, 0),\ d \geq h_d \geq \cdots \geq h_j > h_{j+1} = 0. \tag{2.40}$$

(Recall, the *codimension* of $\mathcal{A}$ is $\dim_{\mathsf{k}} \mathcal{A}_1$.) When $A \in \mathsf{k}[B]$, $\mathcal{A}$ has codimension one, and $H(\mathcal{A}) = (1, 1, \dots, 1)$. Macaulay's lemma below characterizes the Hilbert functions of codimension two Gorenstein Artinian algebras as those sequences (2.40) whose successive values drop by at most one (see [**4**, Lemma 2.25ff] for references).

Lemma 2.24 [**14**]. *A sequence H satisfying* (2.40) *is the Hilbert function of some codimension two Gorenstein Artinian algebra if and only if*

$$\text{for all } i \geq d-1,\ h_i - h_{i+1} \leq 1. \tag{2.41}$$

He also showed that $\mathcal{A}$ is codimension two Gorenstein if and only if $\mathcal{A}$ is a *complete intersection*: that is, if $I_{A,B}$ has two generators. From Theorems 2.22, 2.23 and Lemma 2.24 we conclude

Theorem 2.25 [**13**]. *When* $\operatorname{char} \mathsf{k} = 0$ *or* $\operatorname{char} \mathsf{k} > n$, *$Q(P)$ is stable: $Q(P)$ has parts differing pairwise by at least* 2.

Example 2.26. When $P = (7, 5, 3, 1, 1)$, we have $H = H(\mathsf{k}[A, B])$ satisfies $H = (1, 2, 3, 4, 3, 2, 1, 1)$ and $Q(P) = P(H) = (8, 5, 3, 1)$.

3. The index of $Q(P)$. In [**15**], Oblak associated to P a digraph, essentially the poset $\mathcal{D}_P$ whose vertices can be viewed as the Ferrers graph of P (Definition 2.9). She used results of Gansner [**10**] in her proof of Theorem 3.1 below. See also [**9, 17**] for a general discussion of nilpotent matrices defined from a poset. The poset $\mathcal{D}_P$ and the nilpotent matrices A are quite special: in particular the poset has an involution (Definition 2.15), and the nilpotent matrices A, due to their commuting with B are Toeplitz matrices [**3**].

Given an almost rectangular (AR) subpartition P' of P, let

$$s(P, P') = \#\{\text{parts of } P \text{ greater than any part of } P'\}.$$

Oblak introduced a statistic we will term the *Oblak invariant*,

$$\operatorname{Ob}(P, P') = |P'| + 2s(P, P'). \tag{3.1}$$

We let

$$\operatorname{Ob}(P) = \max\{\operatorname{Ob}(P, P') \mid P' \text{ almost rectangular in } P\}. \tag{3.2}$$

The *index* $i(Q)$ of a partition Q is its largest part. Oblak showed

Theorem 3.1 [**15**]. *Suppose* k *has characteristic zero. The index* $i(Q(P))$ *satisfies* $i(Q(P)) = \mathrm{Ob}\,(P)$.

This was subsequently shown, for k algebraically closed of arbitrary characteristic in [**5**]. We need a key concept introduced by Oblak, an AR chain with two tails, which we term an "Oblak U-chain" or simply "U chain" (for its shape). Let S be a subset of S_P. We denote by $\iota^{-1}(S)$ the subposet of $\mathcal{D}_P$ comprised of all vertices of parts $i \in S$, and the edges among them.

Definition 3.2. Let $a \in S(P)$. An *Oblak U-chain* $C(a) = AR(C) \cup T(C)$ between a vertex $v = (1, j, 1)$ and its conjugate vertex $v' = \tau(v) = (j, j, n_j)$, is a saturated, symmetric chain comprised as follows.

a. The *almost rectangular* portion $AR(C) = \iota^{-1}\{a, a-1\}$ for some $a \in S_P$.

b. The *tail* portion $T(C) = T_1 \cup \tau(T_1)$ is comprised of the saturated chain T_1 at the left of $\mathcal{D}_P$ with vertices $\{(1, i', k) \mid j \geq i' > a\}$, and the saturated chain $\tau(T_1)$ at the right of $\mathcal{D}_P$ with vertices $\{(i', i', k) \mid j \geq i' > a\}$.

c. We include in $C(a)$ the two edges connecting $AR(C)$ to T_1 and to T_2.

Denote the length of a chain C by $|C|$. Let $P(a)$ be the subpartition of P comprised of all parts equal to a or $a-1$. Then we have from (3.1) and Definition 3.2

$$\mathrm{Ob}\,(P, P(a)) = |C(a)|.$$

Oblak shows Theorem 3.1 as a consequence of the following graph-theoretic result, which we state in terms of $\mathcal{D}_P$.

Theorem 3.3 [**15**]. *The maximum length chains between a vertex* $v = (1, j, 1)$ *and* $\tau(v) = (j, j, n_j)$ *of* $\mathcal{D}(P)$ *include a U-chain.*

Example. Let $P = (5, 4, 3, 3, 2, 1)$, $P' = (3, 3, 2)$, $s(P, P') = 2$, $|P'| = 8$, so $\mathrm{Ob}\,(P, P') = 12$, which is maximal, and $Q(P) = (12, 5, 1)$. Here $P'' = (4, 3, 3)$ also satisfies $\mathrm{Ob}\,(P, P'') = 12$: the almost rectangular subpartition P' yielding a maximum $\mathrm{Ob}\,(P, P')$ need not be unique.

Lemma 3.4. *Let $1 \leq u \leq j$, and suppose $n_j \geq 1$. The section of the poset $\mathcal{D}_P$ between $(1, j, 1)$ and (u, j, n_j) is isomorphic to $\mathcal{D}_{P'}$ where P' is obtained from P by subtracting $j - u$ from each part of P.*

3.1. Weighted poset $\mathcal{B}_P$. We denote by $\mathbf{Z}$ the integers, by $\mathbf{N}$ the natural numbers, and by $[a, b]$ the closed interval of integers i, $a \leq i \leq b$. Given positive integers $p < q$, we denote by $P_{p,q}$ the partition $P_{p,q} = (p, p+1, \ldots, q)$ whose parts have multiplicity one.

Definition 3.5 (Weighted poset $\mathcal{B}_P$). Fix integers $p < q$, and for each i, $p \leq i \leq q$ an integer $n_i \in \mathbf{Z}$, the *weight* of i. This is the *data* P. We denote by $\mathcal{B}_P \subset \mathbf{Z} \times [p, q] \subset \mathbf{Z} \times \mathbf{N}$ a weighted poset isomorphic as poset to $\mathcal{D}_{P_{p,q}}$. The i row of $\mathcal{B}_P$ is $(\nu(1, i), i) \ldots (\nu(i, i), i)$, obtained from the i-row of $\mathcal{D}_{P_{p,q}}$, so is

$$(-i+1, i), (-i+3, i), \ldots, (i-1, i), \tag{3.4}$$

centered at $x = 0$. The edges of $\mathcal{B}_P$ are from (i, u) to $(i+1, u+1)$ and from (i, u) to $(i-1, u+1)$. The involution τ, defined via $\mathcal{D}_{P_{a,b}}$ satisfies $\tau(c, i) = (-c, i)$, and

$$\tau((c, i) \to (c+1, i+1)) = (-c-1, i+1) \to (-c, i).$$

Each vertex of the i row has weight $w(u, i) = n_i$. The weight $w(C)$ of a chain C in $\mathcal{B}_P$ is $w(C) = \sum_{v \in C^o} w(v)$. We denote by $v_0 = (-q+1, q)$, the minimum element of $\mathcal{B}_P$ and $\tau(v_0) = (q-1, q)$, the maximum element.

Note that we allow zero or negative n_i. We may think of $\mathcal{B}_P$ as a kind of quotient of $\mathcal{D}_P$, but it is closer to an avatar: when P is a partition, so each $n_i \geq 0$, the weighted poset $\mathcal{B}_P$ determines $\mathcal{D}_P$ and vice-versa. Given a poset $\mathcal{D}$ we denote by $\mathcal{C}(\mathcal{D})$ the poset of saturated chains of $\mathcal{D}$ under inclusion. For the weighted poset $\mathcal{B}_P$ we denote by $\mathcal{C}'(\mathcal{B}_P) \subset \mathcal{C}(\mathcal{B}_P)$ the subposet of chains that begin and end in a vertex having positive weight.

Map $\mathfrak{v} : \mathcal{C}(\mathcal{D}_P) \to \mathcal{C}'(\mathcal{B}_P)$. Fix P a partition with maximum part q and minimum part p. We denote by $\mathfrak{v}$ the "collapsing" map $\mathfrak{v} : \mathcal{D}^0_{\mathcal{P}} \to \mathcal{B}^0_{\mathcal{P}}$ from the vertices $(u, i, k) \in \mathcal{D}_P$ for which $n_i > 0$ to $(\nu(u, i), i) \in \mathcal{B}_P$, that has n_i sheets over the i-row of $\mathcal{B}_P$. We may extend $\mathfrak{v}$ to a map $\mathcal{C}(\mathcal{D}_P)$ to $\mathcal{C}'(\mathcal{B}_P)$ as follows. For those edges of $\mathcal{D}_P$ corresponding to maps $\beta_{u,i} : (u, i, n_i) \to (u, i^-, 1)$ the image is the unique minimal saturated chain in $\mathcal{C}'(\mathcal{B}_P)$ from $(\nu(u, i), i)$ to $(\nu(u, i), i^-)$ that includes an intermediate chain through vertices of zero weight; and analogously for edges corresponding to maps $\alpha_{u,i}$. The image of a map $e_{u,i,k}$ is the identity, which we suppress, since we collapse the column $\{(u, i, k), 1 \leq k \leq n_i\}$ to a single vertex of $\mathcal{B}_P$. When i is isolated in $\mathcal{S}_P$, the image $\mathfrak{v}(w)$ where $w : (u, i, n_i) \to (u + 1, i, 1)$, is the v-shaped chain

$$\mathfrak{v}(w) = (\nu(u, i), i) \longrightarrow (\nu(u, i) + 1, i - 1) \longrightarrow (\nu(u, i) + 2, i),$$

unless $i = p$, when we must take the intermediate vertex to be $(\nu(u, i) + 1, i + 1)$ (see Example 3.7). We say a chain $C \subset \mathcal{D}_P$ is *complete* if

$$(u, i, k) \in C \Longrightarrow (u, i, k') \in C \text{ for all } k' \mid 1 \leq k' \leq n_i. \tag{3.5}$$

The following properties of $\mathfrak{v}$ are readily verified.

Lemma 3.6. *For any chain $C \in \mathcal{C}'(\mathcal{B}_P)$ there is a unique complete chain preimage $\mathfrak{v}^{-1}(C) \subset \mathcal{C}(\mathcal{D}_P)$ satisfying $|\mathfrak{v}^{-1}(C)| = w(C)$, and $\mathfrak{v}(\mathfrak{v}^{-1}(C)) = C$. Each saturated complete chain C of $\mathcal{D}_P$ satisfies, there is a saturated chain $C' = \mathfrak{v}(C) \subset \mathcal{B}_P$ such that $\mathfrak{v}^{-1}(C') = C$ and $|C| = w(C')$. Every chain of $\mathcal{D}_P$ having maximum length is saturated and complete.*

Example 3.7. For the partition data $P = (4, 2)$ $(n_4 = 1, n_3 = 0, n_2 = 1)$ and the edge e corresponding to $\beta_{4,2} = (1, 4, 1) \to (1, 2, 1) \in \mathcal{D}_{\mathcal{P}}$, we have

$$\mathfrak{v}(e) = \text{ chain } (-3, 4) \longrightarrow (-2, 3) \longrightarrow (-1, 2) \subset \mathcal{C}'(\mathcal{B}_P),$$

adding the zero-weight vertex $(-2, 3)$ at height three.

For $P = (6, 4, 2)$; the edge e arising from $w = J_4 : (1, 4, 1) \to (2, 4, 1)$ of $\mathcal{D}_P$ has image $\mathfrak{v}(e)$ the chain $(-3, 4) \to (-2, 3) \to (-1, 4)$ of $\mathcal{B}_P$, by definition. However $C'' = (-3, 4) \to (-2, 5) \to (-1, 4)$ also satisfies $\mathfrak{v}^{-1}(C'') = C$.

The following generalizes the problem of finding the longest chain in $\mathcal{D}_P$:

Problem 3.8. Given data P, find a chain from v_0 to $\tau(v_0)$ in $\mathcal{B}_P$ of maximum weight.

The statistic $\nu(u, i, k) = 2u - (i + 1)$ is by Lemma 2.13 nondecreasing on any chain C of the poset $\mathcal{D}_P$. It separates the vertices v of $\mathcal{D}_P$ into the *left* where $\nu(v) < 0$, the *middle* where $\nu(v) = 0$, and the *right* where $\nu(v) > 0$. Likewise, we define the left, middle, and right portions $L(C), M(C), R(C)$, respectively, of a chain C in $\mathcal{D}_P$, including the edges within each portion. We denote by C^0, $\mathcal{D}^0$ the vertices of a chain or poset.

Lemma 3.9. *Let $C \in \mathcal{C}(\mathcal{D}_P)$ be a saturated chain between $v = (1, j, 1)$ and $\tau(v) = (j, j, n_j)$. Then there is a symmetric saturated chain C' between v and $\tau(v)$ whose length is at least that of C.*

Proof. The chain $\mathfrak{v}(C) \in \mathcal{C}'(\mathcal{B}_P)$ is between $\mathfrak{v}(v) = (-j + 1, j)$ and $\tau(\mathfrak{v}(v)) = (j - 1, j)$. Since $\mathfrak{v}(C)$ is saturated, it has by equation (2.20) a unique middle vertex $(0, k)$. Replacing the right side of $\mathfrak{v}(C)$ by $\tau(L(C))$ or vice-versa, one obtains a symmetric chain $B \in \mathcal{B}_P$ with weight at least that of $\mathfrak{v}(C)$. By Lemma 3.6 $\mathfrak{v}^{-1}(B)$ is a symmetric, saturated, complete chain of length at least that of C. □

We will say $c \in S_P$ is *small* in P if

$$\exists a \in S_P,\ a > c \text{ such that } n_a + n_{a-1} > n_c + n_{c-1}. \tag{3.6}$$

The *spread* $S(D) \supset \iota(D)$ of a subposet of $\mathcal{D}_P$ is the interval of integers

$$S(D) = (\min\{\iota(D)\}, \max\{\iota(D)\}).$$

The invariant $s(D) = \#S(D)$. Given $D \subset \mathcal{B}_P$ we define the spread $S(D)$ and $s(D)$ similarly; also given $S \subset S_P$ we may define $\iota^{-1}(S) \subset \mathcal{B}_P$ as before, and we likewise define U-chains of $\mathcal{B}_P$. The following lemma is key.

Lemma 3.10. *Let $C(c)$ be a U-chain of $\mathcal{D}_P$ (respectively, of $\mathcal{B}_P$) connecting v to $\tau(v)$, and suppose that c is small in P. Then there is a longer chain C' of $\mathcal{D}_P$ (respectively, higher weight chain of $\mathcal{B}_P$) connecting v and $\tau(v)$, such that C' lies on or to the right of $C(c)$, and it satisfies $S(C') \subset S(C)$.*

Assume $v = v_0 = (1, p_1, 1)$ and that (3.6) is satisfied for some $a > c$. Assume further that Theorem 3.3 is valid for partitions of largest part a. Then there is a U-chain $C(b)$ between v and $\tau(v)$ with $a \geq b > c$, with $\iota(C(b)) \subset \iota(C(c))$ and such that $|C(b)| > |C(c)|$.

Example 3.11. Take $P = (6, 5^{11}, 4^5, 3^6)$. Then the pair $c = 4, a = 6$ satisfy (3.6) so 4 is small in P. However in $\mathcal{D}_P$ we have $|C(6)| = 61 < |C(4)| = 62$. Since $|C(5)| = 77$, we take $b = 5$ in Lemma 3.10.

Proof of Lemma 3.10. We work in $\mathcal{B}_P$. Then $C(c)$ has a portion

$$\begin{aligned} C(c) \supset (-a+1, a), (-a+2, a-1), (-a+3, a-2), \dots ;\\ (-c+1, c), (-c+2, c-1), (-c+3, c), \dots ,\\ (c-2, c-1), (c-1, c), (c, c+1), \dots , (a-1, a). \end{aligned}$$

Consider C' formed from $C(c)$ by adding a jog to the right to this portion of $C(c)$ (see Figure 4):

$$\begin{aligned} C' \supset (-a+1, a), (-a+2, a-1), (-a+3, a), (-a+4, a-1),\\ (-a+5, a-2), \dots ; (-c+3, c), (-c+4, c-1), \dots ,\\ (c-2, c-1), (c-1, c), (c, c+1), \dots (a-1, a). \end{aligned}$$

We have $\Delta w = w(C') - w(C(c))$ satisfies

$$\begin{aligned} \Delta w &= w(-a+3, a) + w(-a+4, a-1)\\ &\quad - w(-c+1, c) - w(-c+2, c-1)\\ &= n_a + n_{a-1} - (n_c + n_{c-1}) > 0 \end{aligned} \tag{3.7}$$

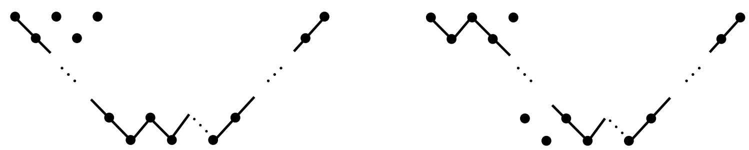

FIGURE 4. $C(c)$ vs C'.

since c is small. Then the complete saturated chain $\mathfrak{v}^{-1}(C')$ evidently satisfies the conditions of the first assertion of the lemma. One can symmetrize C' to a chain at least as long; using Lemma 3.4 we find a U-chain $C(b)$ satisfying $|C(b)| \geq |C'|$, as required in the second assertion. □

Proof of Theorem 3.3. The proof is trivial for partitions of largest part 1, 2 or 3. We assume it is true for partitions of largest part less than q, and let P be a partition of largest part q. By Lemma 3.6 and Lemma 3.9 we may assume that $C' \subset \mathcal{D}_P$ is a τ-symmetric path in $\mathcal{D}_P$ and that $C' = \mathfrak{v}^{-1}(C)$ where $C \subset \mathcal{B}_P$ is a path from $v_0 = (-q+1, q)$ to $\tau(v_0)$ with $w(C) = |C'|$. We need to find an Oblak U-chain $E \subset \mathcal{B}_P$ between v_0 and $\tau(v_0)$ with $w(E) \geq w(C)$.

Omitting the vertices v_0 and $\tau(v_0)$ from C we have a τ-symmetric path $D \subset \mathcal{B}_P$ between $v_1 = (-q+2, q-1)$ and $\tau(v_1) = (q-2, q-1)$, which may, however, return to a part q. Such a return is the main complication, and we will use Lemma 3.10.

Case 1. Assume that D remains in parts less or equal $q-1$. Then by induction, $w(D)$ is majorized by $w(D_1)$, D_1 a U-chain between v_1 and $\tau(v_1)$. Including the vertices v_0, $\tau(v_0)$ and corresponding edges give the required U-chain E in $\mathcal{B}_P$.

Case 2. Assume that D begins $v_1 \to v_2 = (-q+3, q)$. Then consider the section $\mathcal{B}'$ of $\mathcal{B}_P$ between v_2 and $\tau(v_2)$. Then $\mathcal{B}' \cong B_{P'}$ where P' is the partition with parts no greater than $q-2$, satisfying $n'_i(P') = n_{i+2}(P)$. By induction, the portion D' of D from v_2 to $\tau(v_2)$ is a chain whose weight is majorized by the weight of a U-chain $D'(c')$ in $\mathcal{B}'_P$ from v_2 to $\tau(v_2)$. Let $c = c' + 2$ (so we work in $\mathcal{B}' \subset \mathcal{B}$).

By Lemma 3.10 c' is not small in $\mathcal{B}_{P'}$; hence, c is not small in $\mathcal{B}$: in particular $n_c + n_{c-1} \geq n_a + n_{a-1}$ for $q \geq a \geq c$. The U-chain $E = C(c)$ in $\mathcal{B}_P$ from v_0 to $\tau(v_0)$ omits the jog in C up to level q and satisfies, analogously to (3.7),

$$w(C(c)) - w(C) = n_c + n_{c-1} - (n_q + n_{q-1}) \geq 0,$$

so $w(E) \geq w(C)$.

Case 3. Since C is assumed symmetric, the remaining case is that the chain $D \subset \mathcal{B}_P$ first returns to level q in the left (or centerline) of $\mathcal{B}_P$, but to the right of v_2, so at a vertex $v_3 = (-q+4+b, q)$, where $0 \leq b \leq q-4$. We consider the section $\mathcal{B}'$ of $\mathcal{B}_P$ between v_3 and $\tau(v_3)$, which is congruent to $\mathcal{B}_{P'}$, where now P' has maximum part $q-3-b$. By the induction assumption, the portion D' of D from v_3 to $\tau(v_3)$ is a chain whose weight is majorized by the weight of a U-chain $D'(c')$ from v_3 to $\tau(v_3)$ in $\mathcal{B}_{P'}$, and we replace the former by the latter. Letting $c = c' + b + 3$, we have by Lemma 3.10 that c is not small, compared to each $a, c < a \leq q$. Here $c \leq q$.

A similar argument can be applied to the portion of C between v_0 and v_3, which is isomorphic to a suitable $\mathcal{B}_{P'}$. We may replace this portion by a chain of equal or greater weight that is a U-chain for the smaller poset, so C now descends from v_0 to a minimum part $d-1$ where d is not small with respect to each $a, d < a \leq q$, so

$$n_d + n_{d-1} \geq n_a + n_{a-1} \text{ for all } a, \quad d < a \leq q. \tag{3.8}$$

We symmetrize, so similarly replace the portion from $\tau(v_3)$ to $\tau(v_0)$. Here $d < q$.

Thus, in Case 3 the symmetric chain C has been replaced by one of equal or higher weight, having a modified"UUU" shape with *flats*–AR chains involving only two adjacent $i \in S_P$–at the minima and interior maxima. The three minima are at heights $d-1$, $c-1$ and $d-1$, respectively, and the four maxima are at height q. Denote now by G the portion of this chain C between the leftmost lowest point $v_4 = (-d, d-1)$ and $\tau(v_4)$. G comprises most of C, including all minima.

Subcase 3a. If $c \geq d$, then G corresponds to a chain $G' \subset \mathcal{B}_M$ where M has largest part $d-1$; we may assume by induction that

$w(G) = w(G') \leq w(G'(a'))$ where $G'(a')$ is a U-chain for $\mathcal{B}_M$ and satisfies a' is not small for $\mathcal{B}_M$. Translating–inverting this–we have replaced G by a new τ-symmetric G of equal or higher weight, coming from $G'(a')$. The new G rises from v_4 at height $d-1$ to a flat at maximum height $a \geq d$ that satisfies

$$n_a + n_{a-1} \geq n_d + n_{d-1}. \tag{3.9}$$

If $a < q$ the new symmetric C containing G is in Case 1. If $a = q$ we have from (3.8) and (3.9),

$$n_q + n_{q-1} = n_d + n_{d-1} \geq n_b + n_{b-1} \text{ for all } b, d \leq b \leq q. \tag{3.10}$$

This and $d \leq c$ implies that we may replace C by the U-chain $E = C(q)$, which will have weight at least that of C.

Subcase 3b. If $c < d$ denote by H the portion of C between v_4 and the vertex $v_5 = (b+5-d, d-1)$. Here v_5 is the first vertex of C at level $d-1$ on the subchain from v_3 to the central minimum of C at height c. H comprises a single lobe of C, an inverted U. An argument similar to that of Case 3a shows that we may replace H by an inverted U-chain, that rises from v_3 to height $a \geq d$ and descends to v_5, and that (3.9) is satisfied: we may symmetrize this to form a new τ-symmetric chain of equal or higher weight than the original C. Again, if $a < q$ we are in Case 1. If $a = q$ then again (3.10) is satisfied, and we may replace the portion of C between v_0 and v_3 by a flat–AR chain–at maximum height q, and symmetrize, yielding a new chain C of equal or higher weight that is in Case 2 (it still has the dip to height $c-1$ in the middle). This completes the induction step, and the proof. □

Corollary 3.12. *Let* k *be algebraically closed. Then the index* $i(Q(P)) = \mathrm{Ob}\,(P)$.

Proof. Let ℓ be the length of the longest chain in $\mathcal{D}_P$: these chains are from v_0 to $\tau(v_0)$. We have

$$A^{\ell-1} \cdot v_0 = \sum_{\{C \mid |C| = \ell\}} \mu_C, \tag{3.11}$$

where μ_C is a monomial, the product of the entries of A corresponding to the edges of the chain C. By Theorem 3.3, $\ell = \mathrm{Ob}\,(P)$; that is, there exists $c \mid |C(c)| = \ell$. Denote by a the maximum such c. For A generic as in (2.33), $\mu_{C(a)} = \mu_{C'} \Rightarrow C' = C(a)$. This implies that $A^{\ell-1} \neq 0$, but evidently $A^{\ell} = 0$, implying that the index $i(Q(P)) = \ell$. □

We now answer Problem 3.8 for non-negative weights.

Corollary 3.13. *Let $\mathcal{B}_P$ of data P (weights n_i for $p \leq i \leq q$) with non-negative weights be a weighted poset as in Definition* 3.5*, and let r be an integer, $p \leq r \leq q$. Then the set of maximum weight chains between $v_0 = (-r+1, r)$ and $\tau(v_0) = (r-1, r)$ contains a U-chain or an inverted U-chain.*

Proof. We prove the Corollary by induction on the width $2r-1$ of the relevant portion of the poset. When $r = 1$ or 2 the statement is true. We may assume that a maximum weight chain $C = (c_1, \dots, c_s)$ between v_0 and $\tau(v_0)$ is symmetric (analogue for $\mathcal{B}_P$ of Lemma 3.9). Assume the Corollary for widths smaller than $2r-1$. The saturated subchain between c_2 and $\tau(c_2) = c_{s-1}$ has width $2r-3$, so by induction may be replaced by an equal or higher-weight chain C' that is U-shaped or inverted U-shaped. If the deformed chain C'' from v to $\tau(v)$ that includes C' is not U or inverted U shape, then after possible reflection about a horizontal axis, the current path C'' has a jog up to level $r+1$ followed by a U-chain $C(c)$. By Lemma 3.10 c is not small for $P_{\leq r+1}$, so $n_{r+1} + n_r \leq n_c + n_{c-1}$. Thus, we may omit the jog up, making a new path D of equal or higher weight that is a U-path from v_1 to v_2–with an extra sawtooth at level $c, c-1$ on the left and on the right of the almost rectangular part, compared to C''. This completes the induction step, and the proof. □

Remark 3.14. The hypothesis for Corollary 3.12 could be weakened to k an infinite field. The hypothesis for Corollary 3.13 can be changed to arbitrary weights (including negative values) provided we restrict to maximum weight or minimum weight chains that are assumed to be saturated. The need to restrict to saturated chains is seen in the case $P = (2,1), n_2 > 0, n_1 < 0$.

When the multiplicities n_i for $p \leq i \leq q$ alternate between two values, then all saturated chains through these values have the same weight.

For example, for $P = (5^2, 4^3, 3^2)$ when $n_5 = n_3 = 2, n_4 = 3$, the τ-symmetric chain beginning

$$(-4,5) \rightarrow (-3,4) \rightarrow (-2,3) \rightarrow (-1,4) \rightarrow (0,5) \rightarrow \cdots$$

has weight 22, as do the two U-chains $C(5)$ and $C(3)$.

The process of the proof could be used to make a recursive algorithm to create from any chain of $\mathcal{D}_P$ a U-chain that is at least as long.

We can apply a similar argument to obtain the analogous result to Theorem 3.3 for weighted posets $\mathcal{B} \subset [-q,q] \times [p-q, q-p]$ where $0 < p < q$: the row of $\mathcal{B}$ at height 0 has q vertices, at height a has $q - |a|$ vertices, so form a diamond shape.

3.2. The Oblak conjecture. Oblak's beautiful recursive conjecture concerning $Q(P)$ is stated in [**7**], and is based on her index Theorem 3.1. We state a variation. Let the field $F = \theta(T)$ with T from (2.33).

Definition 3.15. A is *adequate* if it has general enough (or even generic) entries on the following pairs of vertices:

i. Each pair of vertices corresponding to an edge of $\mathcal{D}_P$;

Each pair of vertices corresponding to the set S''_P of (2.32).

In general, in order for $P_A = Q(P)$ we will need at least one appropriate non-zero diagonal entry in the block matrix for A, for each odd consecutive sequence in S_P. This can be "seen" from the conjecture by noting that when S_P has an odd length consecutive sequence, eventually, after removing enough successive U-chains, the remainder partition will have an isolated part: and one needs to include in A a corresponding w term. This is the reason for our introducing condition (ii) above for adequate A.

Conjecture 3.16.[7] *Let $A \in \mathcal{U}_B$ be general enough among adequate elements of $\mathcal{U}_B$. Then $P_A = Q(P)$. Furthermore, let C be any maximum length U-chain in $\mathcal{D}_P$, and let $P-C$ be the partition obtained from the vertices of $\mathcal{D}_P$ by removing the vertices in C. Then*

$$Q(P) = Q(P-C) \cup |C|. \tag{3.12}$$

In [**6**] we give a proof using standard bases in some quite special cases. We rely heavily on the form of the conjecture to define the standard basis–generators for the ideal $I_{A,B} \subset \mathsf{k}\{x,y\}$ defining $\mathsf{k}[A,B]$. We believe this approach will extend to a proof of Conjecture 3.16. For references on standard bases, see [**4, 8, 12**].

Example 3.17.

a. Let $P = (2)$ and $A = 0$. Then $\mathsf{k}[A,B] \cong \mathsf{k}[x,y]/(x,y^2)$, which is Gorenstein. The partition $P_A = (1,1)$, but $P_{A+tB} = (2)$ for $t \neq 0$ and $Q(P) = (2)$. Here A is not adequate.

b. Let $P = (3,2)$, and let $A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ in block matrix form, where $\alpha = \alpha_{2,3}$ is $3 \times 2, \beta = \beta_{3,2}$ is 2×3, and $\alpha \circ \beta = J_3$, $\beta \circ \alpha = J_2$, then A has rank 4, corank one. $A^4 = \begin{pmatrix} (J_3)^2 & 0 \\ 0 & 0 \end{pmatrix}$, $A^5 = 0$ so $P_A = (5) = Q(P)$ and $\mathsf{k}[A,B] \cong \mathsf{k}[x,x^5]$, which is Gorenstein. Here A is adequate.

c. Let $P = (4,3,2)$, and let $A = \begin{pmatrix} 0 & \alpha_{3,4} & 0 \\ \beta_{4,3} & 0 & \alpha_{2,3} \\ 0 & \beta_{3,2} & 0 \end{pmatrix}$ in block matrix form. Then $\mathcal{A} = \mathsf{k}[A,B]$ is Gorenstein by Theorem 2.20 and $H(\mathcal{A}) = (1,2,2,1,1,1,1)$; however, A is not adequate, and $p_A = (7,1,1)$. By Theorem 2.22 $A + \lambda B$ for λ general (here λ nonzero suffices), has partition $P(H(\mathcal{A})) = (7,2)$, which is $Q(P)$. Replacing $A_{3,3} = 0$ by J_2 gives an adequate matrix A' with $p_{A'} = Q(P) = (7,2)$.

We will consider a succession $\mathfrak{C} = (C_1, \ldots, C_{r(P)})$ of maximum length U-chains in successive P_u chosen as in Conjecture 3.16. Here $C_1 \subset \mathcal{D}_P$, $P_1 = P$ and

$$C_u \subset \mathcal{D}_{P_u}, P_u = P - (C_1^0 \cup \ldots \cup {C_{u-1}}^0) \text{ for } u = 2, \ldots, r_P.$$

We will denote by $Q(\mathfrak{C})$ the partition $(|C_1|, \ldots, |C_{r_P}|)$ of n. Denote by a_u the larger part arising from the AR section of C_u, in the pullback C'_u of the chain C_u to $\mathcal{D}_P^0$. Note that C'_u need not be a chain of $\mathcal{D}_P$. However, at each stage one may identify chains $O_1(C_1, \ldots, C_u), \ldots, O_u(C_1, \ldots, C_u)$ by choosing for O_1 the outside perimeter of $C'_1 \cup \cdots \cup C'_u$, and for O_k the k-th from outside perimeter. We denote by $\#(C_1 \cup \cdots \cup C_u)$ the number of distinct vertices of $\mathcal{D}_P$ covered by the union, equivalently the total weight of the image vertices in $\mathcal{B}_P$ under $\mathfrak{v}$.

Proposition 3.18. *Let $\mathfrak{C}$ be a choice of maximum length chains in successive $\mathcal{D}_{P_i}$ as above. The set $\cup_{k=1}^{u} C'_k = \cup_{k=1}^{u} O_k(C'_1, \dots, C'_u)$ is a union of disjoint chains in $\mathcal{D}_P$, and $\# \cup_1^u (O_k) = \sum_{k=1}^{u} |C_k|$.*

Proof. The point of this is that the chains O_k in the original poset $\mathcal{D}_P$ form a shelling of the union $\cup C'_k$ of original vertices contributing to $\cup_{k=1}^{u} C_u$. □

Example 3.19. Let $P = (6, 5^4, 4^3, 3, 1)$. Then in $\mathcal{B}_P$,

$$C_1 : (1,6) \longrightarrow \iota^{-1}(5,4) \longrightarrow (6,6).$$

In $\mathcal{B}_{P_1}, P_1 = (4,3,1)$, $C_2 = \iota^{-1}(4,3)$ and $C_3 = (1,1) \subset \mathcal{B}_{P_2}$, $P_2 = (1)$. Then we have $w(C_1) = 34$, $w(C_2) = 7$, $w(C_3) = 1$ and $(a_1, a_2, a_3) = (5, 6, 1)$, for (C_1, C_2) the outside shells $O_1 = O_1(C_1, C_2)$ and $O_2 = O_2(C_1, C_2)$ satisfy, in $\mathcal{B}_P$

$$O_1 = (1,6) \to (1,5) \to \iota^{-1}(4,3) \to (5,5) \to (6,6), \text{ and}$$
$$O_2 = (2,6) \to (2,5) \to (3,6) \to (3,5) \to (4,6) \to (4,5) \to (5.6).$$

Here O_1, O_2 are chains of the original $\mathcal{B}_P$ having weights $w(O_1) = 25$, $w(O_2) = 16$. The union $O_1 \cup O_2 = C'_1 \cup C'_2$ and $\#(O_1 \cup O_2) = \#(C'_1 \cup C'_2) = 41$.

Recall that for two partitions $P \to n$, $P' \to n$ the orbit closure partial order is

$$P \geq P' \iff \forall k \sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} p'_i. \tag{3.13}$$

The generic matrix A of the following Corollary of Proposition 3.18 does not necessarily commute with B, i.e., does not satisfy the Toeplitz condition, so $A \notin \mathcal{U}_B$. Thus the Corollary is rather weaker than that predicted by Conjecture 3.16 above.

Corollary 3.20. *Let A be a generic $n \times n$ nilpotent matrix (not necessarily commuting with B), whose nonzero entries correspond to pairs of vertices $v < v'$ in $\mathcal{D}_P$. Let $\mathfrak{C}$ be a successive choice of maximum length chains in successive P_i as above. Then the Jordan block partition $P_A \geq Q(\mathfrak{C})$, in the orbit closure order on partitions.*

Proof. This is a consequence of Proposition 3.18 and the Gansner theory [**9, 10**], applied to the subchains $O_1, \dots, O_u$ of $\mathcal{D}_P$. □

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ENDNOTES

1. This is the partial order arising from $\mathcal{D}_P$, see Definition 2.9.

2. The maps are in a representation $\mathcal{M}\mathfrak{D}_P$ of $\mathfrak{D}_P$, and the relation is determined by commutativity relations on a larger quiver $\mathfrak{A}_P$ with points corresponding to vertices of the poset $\mathcal{D}_P$. We do not explore this viewpoint further.

3. This may be also viewed as an avatar of the large quiver $\mathfrak{A}_P$, since we are giving a poset and maps. We will speak of the *path algebra* $K\mathcal{D}_P$ or, equivalently of $\mathfrak{A}_P$.

4. See [**5**, pages 25ff, subsection "A lattice version of the almost rectangular moves of [**7**]]."

5. This includes the stipulation that composition of maps α's, β's, w_i's landing apparently outside of $\mathcal{D}_P$ are also in I; for example, $w_i^i \in I$.

6. Using the statistic we may visualize the rows as centered at $x = 0$; we use ν later in defining the weighted poset $\mathcal{B}_P$ (Definition 3.5).

7. P. Oblak's conjecture in [**7**] chooses at each step the highest U-chain $C(a)$ giving a maximum length chain.

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