JOURNAL OF COMMUTATIVE ALGEBRA
Volume 2, Number 3, Fall 2010

DECOMPOSITIONS OF IDEALS INTO IRREDUCIBLE IDEALS IN NUMERICAL SEMIGROUPS

VALENTINA BARUCCI

ABSTRACT. It is proved that each ideal I of a numerical semigroup S is in a unique way a finite irredundant intersection of irreducible ideals. The same result holds if "irreducible ideals" are replaced by "**Z**-irreducible ideals." The two decompositions are essentially different and, if $n(I)$ and $N(I)$ respectively are the number of irreducible or **Z**-irreducible components, it is $n(I) \leq N(I) \leq e$, where e is the multiplicity of S. However, if I is a principal ideal, then $n(I) = N(I) = t$, where t is the type of S.

1. Introduction. In one of her famous papers, [**8**], Emmy Noether shows that each proper ideal of a Noetherian ring admits a representation as an irredundant intersection of finitely many irreducible ideals. Such a representation is not unique, but the number of components is uniquely determined by the ideal. The present paper deals with numerical semigroups, which are mathematical objects much simpler than Noetherian rings. So it is not surprising that the results of decomposition of an ideal as intersection of irreducible ideals are stronger. Such a decomposition in fact, if irredundant, is unique, as can be easily proved (cf. Theorem 3.3). On the other hand, the irreducibility of ideals in rings can also be considered in terms of fractional ideals. In a ring R, with total ring of quotients Q, a fractional ideal J is said to be Q-irreducible if it is not the intersection of two fractional ideals properly containing it (cf. [**5**]). The concepts of ideal and fractional ideal in rings have natural correspondences in numerical semigroups. In fact, similarly to Q-irreducible fractional ideals, **Z**-irreducible relative ideals in a numerical semigroup can be defined. It turns out that a relative ideal of a numerical semigroup S is **Z**-irreducible if and only if it is of the form $z + \Omega$, for some $z \in \mathbf{Z}$, where Ω is the canonical ideal of S. Theorem 4.4 shows that a relative ideal of a numerical semigroup S is in a unique way an irredundant intersection of **Z**-irreducible ideals. However, given an ideal I of S, $I \subset S$, the two decompositions as irre-

Received by the editors on September 14, 2009, and in revised form on March 5, 2010.

DOI:10.1216/JCA-2010-2-3-281

dundant intersection of irreducible and of **Z**-irreducible ideals respectively are essentially different. The number of components, $n(I)$ and $N(I)$ respectively, can be valuated and in general $n(I) \leq N(I) \leq e$, where e is the multiplicity of the semigroup. However, it turns out that, in case of a principal ideal I, $n(I) = N(I)$ equals the type of the semigroup. Some similar results for rings are recalled in the last short section.

Numerical semigroups have been the matter of my first cooperation with Ralf Fröberg and I want to thank him for introducing me in this subject, mostly discussing and deepening the implications of the nice report [**4**] he wrote several years ago with some colleagues of Stockholm University. Working or–as I would say–"playing" with numerical semigroups is not only fun, but it is often useful for making, denying or proving conjectures on numerical semigroup rings or, more generally, on one-dimensional local Cohen Macaulay rings.

2. Generalities for numerical semigroups. We fix for all the paper the following notation. S is a *numerical semigroup*, i.e., a subsemigroup of $\mathbf{N}$, with zero and with a finite complement in $\mathbf{N}$. The numerical semigroup generated by $d_1, \dots, d_\nu \in \mathbf{N}$ is $S = \langle d_1, \dots, d_\nu \rangle = \{\sum_{i=1}^{\nu} n_i d_i; n_i \in \mathbf{N}\}$. $M = S \setminus \{0\}$ is the *maximal ideal* of S, e is the *multiplicity* of S, that is, the smallest positive integer of S, f is the *Frobenius number* of S, that is, the greatest integer which does not belong to S.

A *relative ideal* of S is a nonempty subset I of $\mathbf{Z}$ (which is the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$. A relative ideal which is contained in S is an *integral ideal* of S.

If I, J are relative ideals of S, then the following are relative ideals too:

$$I \cap J$$
$$I \cup J$$
$$I + J = \{i + j;\ i \in I, j \in J\}$$
$$I -_{\mathbf{Z}} J = \{z \in \mathbf{Z} \mid z + J \subseteq I\}$$
$$I -_S J = (I -_{\mathbf{Z}} J) \cap S = \{s \in S \mid s + J \subseteq I\}.$$

If $z \in \mathbf{Z}$, $z + S = \{z + s;\ s \in S\}$ is the principal relative ideal generated by z and it is easy to check that $I -_{\mathbf{Z}} (z + S) = I - z = \{i - z;\ i \in I\}$.

Moreover, the ideal generated by $z_1, \dots, z_h \in \mathbf{Z}$ is

$$(z_1 + S) \cup \cdots \cup (z_h + S)$$

If I is a relative ideal of S, and $s \in S$, $s \neq 0$, then $\mathrm{Ap}_s(I) = I \setminus (s+I)$ is the set of the s smallest elements in I in the s congruence classes mod s and is called the *Apery set* of I (with respect to s). In particular, $\mathrm{Ap}_e(S)$ is the Apery set of S with respect to the multiplicity e. Since f is the greatest gap of S, $f+s$ is the largest element in $\mathrm{Ap}_s(S)$.

The following lemma corresponds to Nakayama's lemma for local rings. For numerical semigroups the proof is very easy.

Lemma 2.1. *If I is a relative ideal of S, then the unique minimal set of generators of I is $I \setminus (M+I)$.*

Since $e + I \subseteq M + I$, then $I \setminus (M+I) \subseteq I \setminus (e+I) = \mathrm{Ap}_e(I)$ and by Lemma 2.1 each relative ideal I of S needs at most e generators.

Recall also that $t = \#\{(S -_{\mathbf{Z}} M) \setminus S\}$ is the *type* of the semigroup S.

3. Decomposition into irreducible ideals. Let I be a proper integral ideal of a numerical semigroup S.

I is *irreducible* if it is not the intersection of two integral ideals which properly contain I.

Consider the partial order on S given by

$$(\star) \qquad s_1 \preceq s_2 \iff s_1 + s_3 = s_2, \text{for some } s_3 \in S$$

and for $x \in S$, set

$$B(x) = \{s \in S \mid s \preceq x\}.$$

Lemma 3.1. it If I is a proper integral ideal of S, then the following conditions are equivalent:

(1) *I is irreducible.*

(2) *I is completely irreducible, i.e., is not the intersection of any set of integral ideals which properly contain I.*

(3) *I is maximal as an integral ideal with respect to the property of not containing an element x, for some $x \in S$.*

(4) *$I = S \setminus B(x)$, for some $x \in S$.*

Proof. Conditions (1) and (2) are equivalent because I has finite complement in $\mathbf{N}$.

(2) $\Rightarrow$ (3). Let H be the intersection of all the integral ideals properly containing I. Then there is an $x \in H \setminus I$, so I is maximal with respect to the property of not containing x.

(3) $\Rightarrow$ (2). Each integral ideal J properly containing I contains x, so I is not the intersection of all such ideals J and it is completely irreducible.

(3) $\Leftrightarrow$ (4) is trivial. $\square$

The fact that, in any commutative monoid S, an ideal of the form $S \setminus B(x)$, for some $x \in S$ is irreducible was observed in [**9**].

Example. Let $S = \langle 5, 6, 8 \rangle = \{0, 5, 6, 8, 10, \rightarrow\}$. Here the arrow means that each integer $z \geq 10$ is in the set. The same notation will be used several times in the sequel. If $x = 12$, then $B(x) = \{0, 6, 12\}$ and $I = S \setminus B(12) = \{5, 8, 10, 11, 13, \rightarrow\}$ is an irreducible ideal of S.

Lemma 3.2. *If I is a proper integral ideal of S, then*:

(1) *The irreducible integral ideals containing I are exactly the ideals of the form $S \setminus B(x)$, with $x \in S \setminus I$.*

(2) *The irreducible integral ideals containing I and minimal over I are exactly those of the form $S \setminus B(x)$, with $x \in S \setminus I$ and x maximal (with respect to $\preceq$) in $S \setminus I$.*

Proof. (1) Let $x \in S \setminus I$. We show that $I \subseteq S \setminus B(x)$. In fact, if $i \in I$, then $i \in S$ and $i \notin B(x)$ because otherwise $i + s = x$, for some $s \in S$; hence $x \in I$, a contradiction.

Conversely, if $x \in I$, then $I \nsubseteq S \setminus B(x)$; in fact, $x \in B(x)$.

(2) It is enough to observe that, for $x, y \in S$, we have $x \preceq y$ if and only if $B(x) \subseteq B(y)$, which is equivalent to $S \setminus B(x) \supseteq S \setminus B(y)$. $\square$

Thus, we get:

Theorem 3.3. *If I is a proper integral ideal of a numerical semigroup S and if $(I -_S M) \setminus I = \{x_1, \dots, x_n\}$, then*

$$I = (S \setminus B(x_1)) \cap \cdots \cap (S \setminus B(x_n))$$

is the unique irredundant decomposition of I into integral irreducible ideals.

Proof. Using Lemma 3.2 (2), it is enough to show that, if I is an integral proper ideal of S and $x \in S$, then x is maximal in $S \setminus I$ (with respect to $\preceq$) if and only if $x \in (I -_S M) \setminus I$. In fact, x is maximal in $S \setminus I$ if and only if $x \notin I$ and $x + m \in I$, for each $m \in M$, that is, if and only if $x \in (I -_S M) \setminus I$. $\square$

We denote by $n(I)$ the number of components of the unique irredundant decomposition of an integral ideal I of S into integral irreducible ideals, which by Theorem 3.3 equals $\#\{(I -_S M) \setminus I\}$.

Example. Let $S = \langle 5, 6, 8 \rangle$. If $I = \langle 6, 15 \rangle = \{6, 11, 12, 14, \rightarrow\}$, then $(I -_S M) \setminus I = \{10, 13\}$. So $I = (S \setminus B(10)) \cap (S \setminus B(13))$ is the unique irredundant decomposition of I into integral irreducible ideals.

The following corollary was proved in a different way in [**1**, Theorem 4.2].

Corollary 3.4. *If $I = i + S$ is a proper principal integral ideal of S, then*

$$I = (S \setminus B(x_1)) \cap \cdots \cap (S \setminus B(x_t))$$

is the unique decomposition of I into integral irreducible ideals, where, for $h = 1, \dots, t$, x_h is maximal in $\mathrm{Ap}_i(S)$. Moreover, the number t of components equals the type of the semigroup S.

Proof. By Lemma 3.2 (2), we have to consider the components of the form $S \setminus B(x_h)$ where x_h is maximal in $S \setminus (i+S) = \mathrm{Ap}_i(S)$. Observing that

$$((i + S) -_S M) = ((i + S) -_{\mathbf{Z}} M) = i + (S -_{\mathbf{Z}} M)$$

we have that x_h is maximal in $S \setminus (i+S) = \mathrm{Ap}_i(S)$ if and only if

$$x_h \in ((i+S) -_S M) \setminus (i+S) = i + (S -_{\mathbf{Z}} M) \setminus (i+S),$$

that is, if and only if $x_h - i \in (S -_{\mathbf{Z}} M) \setminus S$. Moreover, $\#\{(S -_{\mathbf{Z}} M) \setminus S\} = t$ is the *type* of the semigroup S. $\square$

Example. Let $S = \langle 5, 6, 8 \rangle$. The type of S is 2, in fact, $(S -_{\mathbf{Z}} M) \setminus S = \{7, 9\}$. If $I = 5 + S$, then $(I -_S M) \setminus I = \{12, 14\}$. So $I = (S \setminus B(12)) \cap (S \setminus B(14))$ is the unique decomposition of I into integral irreducible ideals.

4. Decomposition into Z-irreducible ideals. A relative ideal I of a numerical semigroup S is $\mathbf{Z}$-*irreducible* if it is not the intersection of two relative ideals which properly contain I. Of course, if I is a proper integral ideal of S which is $\mathbf{Z}$-irreducible, it is also irreducible.

Lemma 4.1. *If I is a relative ideal of S, then the following conditions are equivalent*:

(1) *I is $\mathbf{Z}$-irreducible.*

(2) *I is completely irreducible, i.e., is not the intersection of any set of relative ideals which properly contain I.*

(3) *I is maximal as relative ideal with respect to the property of not containing an element z, for some $z \in \mathbf{Z}$.*

Proof. Let m be the smallest element of I with respect to the natural order of $\mathbf{Z}$. Then all the relative ideals of S containing I, except a finite number, also contain $S - (f + 1 - m)$. So the relative ideals minimal over I are finitely many and (1) is equivalent to (2). For the equivalence between (2) and (3), the same argument for the equivalence between (2) and (3) in Lemma 3.1 can be applied. $\square$

The relative ideal Ω of S maximal with respect to the property of not containing f, the Frobenius number of S, is called the *canonical ideal* of S. Thus, setting $B(f) = \{z \in \mathbf{Z} \mid z \preceq f\} = \{f - s; s \in S\}$, we have

$$\Omega = \mathbf{Z} \setminus B(f) = \{f - x;\ x \in \mathbf{Z} \setminus S\}.$$

Calling an integer z *symmetric* to x if $z = f - x$, Ω consists of the integers which are symmetric to the gaps of the semigroup, and we have $S \subseteq \Omega \subseteq \mathbf{N}$.

By Lemma 4.1 the only $\mathbf{Z}$-irreducible relative ideals of S are the relative ideals maximal with respect to the property of not containing $z + f$, for some $z \in \mathbf{Z}$, i.e., just the translations $z + \Omega$ of Ω. Thus, we have the following fact (cf. [**1**, Proposition 3.5] for a different proof):

Proposition 4.2. *Let J be a relative ideal of S. Then J is $\mathbf{Z}$-irreducible if and only if $J = z + \Omega$, for some $z \in \mathbf{Z}$.*

We want to emphasize that, differently from the case of irreducible integral ideals, we have essentially (i.e., modulo translations) a unique $\mathbf{Z}$-irreducible relative ideal in a numerical semigroup, the canonical ideal Ω.

The following are well-known properties of the canonical ideal (cf, e.g., [**1**]). In the next proposition and for all the rest of the section, if I, J are relative ideals, the notation $I - J$ always means $I -_{\mathbf{Z}} J$.

Proposition 4.3. (1) *For each relative ideal I of S, $\Omega - (\Omega - I) = I$. In particular $\Omega - (\Omega - S) = \Omega - \Omega = S$.*

(2) *If $I \subseteq J$ are relative ideals of S, then $\#\{J \setminus I\} = \#\{(\Omega - I) \setminus (\Omega - J)\}$.*

(3) *For each set $\{I_h\}_{h \in H}$ of relative ideals of S,*

$$\Omega - \bigcap_{h \in H} I_h = \bigcup_{h \in H} (\Omega - I_h).$$

(4) *The cardinality of a minimal set of generators of Ω is the type t of S.*

For the decomposition of a relative ideal into $\mathbf{Z}$-irreducible ideals we get:

Theorem 4.4. (1) *Each relative ideal J of S is in a unique way an irredundant intersection of $\mathbf{Z}$-irreducible relative ideals.*

(2) *The number of components of such decomposition equals the cardinality of a minimal set of generators of* $\Omega - J$, *which is also equal to* $\#\{(J-M)\setminus J\}$.

Proof. (1) Suppose I is a relative ideal of S minimally generated by $i_1, \dots, i_h$, $I = \langle i_1, \dots, i_h\rangle = (i_1+S)\cup\cdots\cup(i_h+S)$ then

$$\Omega - I = \Omega - \bigcup_{j=1}^{h}(i_j+S) = \bigcap_{j=1}^{h}(\Omega-(i_j+S)) = \bigcap_{j=1}^{h}(\Omega - i_j),$$

which, by Proposition 4.2, is a decomposition into **Z**-irreducible relative ideals.

Moreover, the intersection is irredundant: if $\cap_{j\neq k}(\Omega - i_j)\subseteq(\Omega - i_k)$, then, applying Proposition 4.3 (3),

$$i_k \in \Omega - (\Omega - i_k) \subseteq \Omega - \bigcap_{j\neq k}(\Omega - i_j) = \bigcup_{j\neq k}(i_j+S),$$

which is a contradiction with the minimality of the set of generators for I. Finally, observe that each relative ideal J is of the form $\Omega - I$, in fact $\Omega - (\Omega - J) = J$.

(2) We have seen above that the number of components for an irredundant decomposition of $J = \Omega - I$ equals the number h of minimal generators of $I = \Omega - (\Omega - I) = \Omega - J$, which applying Lemma 2.1 and Proposition 4.3 (2), is

$$\begin{aligned}\#\{(\Omega - J)\setminus((\Omega-J)+M)\} &= \#\{(\Omega - ((\Omega-J)+M))\setminus J\}\\ &= \#\{((\Omega-(\Omega-J))-M)\setminus J\} = \#\{(J-M)\setminus J\}. \qquad \square\end{aligned}$$

Example. Let $S = \langle 5,6,8\rangle$ and consider the ideal $J = \langle 6, 15\rangle = \{6, 11, 12, 14, \rightarrow\}$. We have $(J-M)\setminus J = \{9, 10, 13\}$, $I = \Omega - J = \langle -4, -1, 0\rangle$ and so

$$J = (\Omega + 4)\cap(\Omega+1)\cap\Omega$$

is the unique irredundant decomposition of J as intersection of **Z**-irreducible relative ideals of S.

By Theorem 4.4, we also get another characterization of the canonical ideal Ω and its translations:

Corollary 4.5. *Let J be a relative ideal of S. Then J is* **Z***-irreducible if and only if* $\#\{(J-M)\setminus J\}=1$. *In that case* $J=\Omega+z$, *for some* $z\in\mathbf{Z}$, *and* $(J-M)\setminus J=\{f+z\}$.

Proof. The first part follows from Theorem 4.4 (2). For the second part we know, by Proposition 4.2 that $J=\Omega+z$, for some $z\in\mathbf{Z}$. Moreover, $f+z\notin\Omega+z$, because the Frobenius number f is symmetric to $0\in S$ and so $f\notin\Omega$. Finally, $f+z\in(\Omega+z)-M$ because $f+M\subseteq\Omega$. □

In particular, if $J=i+S$ is a principal relative ideal, since

$$\#\{((i+S)-M)\setminus(i+S)\}=\#\{(S-M)\setminus S\}$$

and $\#\{(S-M)\setminus S\}$ is the type of S, we get:

Corollary 4.6 [**1**, Proposition 4.4, (ii)]. *If J is a relative principal ideal of S, then the unique decomposition of J as intersection of* **Z***-irreducible relative ideals has t components, where t is the type of S.*

Example. Let $S=\langle 5,6,8\rangle$. S is of type $t=2$. The principal ideal $8+S$ has the following unique irredundant decomposition as intersection of **Z**-irreducible relative ideals of S

$$8+S=(8+\Omega)\bigcap(6+\Omega)$$

Given a proper integral ideal I of a numerical semigroup, we want to estimate and compare the two natural numbers $n(I)$ and $N(I)$, i.e., the number of components of the irredundant decomposition of I as intersection of irreducible integral ideals and **Z**-irreducible relative ideals respectively.

Combining Corollaries 3.4 and 4.6 we get:

Corollary 4.7. *If I is a proper principal integral ideal of S, then*

$$n(I)=N(I)=t,$$

where t is the type of S.

Remark. The converse of Corollary 4.7 is not true as the following example shows. Let $= \langle 4,5,7\rangle$, which is a semigroup of type 2, and consider the integral ideal $I = \langle 7,8,9\rangle = \{7,8,9,11,\rightarrow\}$. Since $(I -_S M) \setminus I = (I - M) \setminus I = \{4, 10\}$, we have $n(I) = N(I) = 2$, but I is not principal.

Proposition 4.8. *If I is a proper integral ideal of S, then*

$$1 \leq n(I) \leq N(I) \leq e$$

where e is the multiplicity of S.

Proof. The second inequality is because $(I -_S M) \setminus I \subseteq (I - M) \setminus I$ (cf. Theorems 3.3 and 4.4 (2)) and the third is because $N(I)$ equals the cardinality of a minimal set of generators of $\Omega - I$ which is a relative ideal of S and hence needs at most e generators. □

Remark. Observe that $n(I) = N(I) = e$ can be realized in each numerical semigroup S. In fact for any integral ideal I of S of the form $\{a, a+1, \rightarrow\} = \langle a, a+1, \dots, a+e-1\rangle$, with $a \geq f + 1 + e$, we have $\#\{(I -_S M) \setminus I\} = \#\{(I - M) \setminus I\} = \{a, a-1, \dots, a-e+1\}$; thus, $n(I) = N(I) = e$.

Recall that, if $S = \langle d_1 = e, d_2, \dots, d_\nu\rangle$, the blowup of the maximal ideal M is the semigroup $\mathcal{B}(M) = \langle e, d_2 - e, \dots, d_\nu - e\rangle$, which is the smallest semigroup containing $M - e$.

Corollary 4.9. *Let J be a relative ideal of S. Then the following conditions are equivalent*:

(1) $N(J) = e$.

(2) $\Omega - J$ *needs e generators.*

(3) $\Omega - J$ *is an ideal of $\mathcal{B}(M)$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 4.4 (2) and the equivalence (2) $\Leftrightarrow$ (3) is proved in [**3**, Proposition 8]. □

Consider now some particular cases. If $I = M$ is the maximal ideal of S, then M is trivially irreducible as integral ideal. That agrees with

Theorem 3.3, in fact $M = S \setminus B(0)$, since $(M -_S M) \setminus M = S \setminus M = \{0\}$. On the other hand:

Proposition 4.10. *The maximal ideal M of S is $\mathbf{Z}$-irreducible if and only if $S = \mathbf{N}$. If $S \neq \mathbf{N}$ and Ω is minimally generated by $z_1, \dots, z_t$, then*

$$M = (\Omega - z_1) \cap \cdots \cap (\Omega - z_t) \cap (\Omega - f)$$

is the unique irredundant decomposition of M into $\mathbf{Z}$-irreducible ideals. In particular, $N(M) = t + 1$.

Proof. If $S = \mathbf{N}$, it is easy to see that $M = \Omega + 1$ is $\mathbf{Z}$-irreducible. Let $S \neq \mathbf{N}$. Arguing as in the proof of Theorem 4.4, we have to look for a minimal set of generators of $\Omega - M$. By Lemma 4.5, $(\Omega - M) \setminus \Omega = \{f\}$, thus $(\Omega - M)$ is minimally generated by $z_1, \dots, z_t, f$ and the conclusion follows as in the proof of Theorem 4.4. In particular, $N(M) \geq 2$ and M is not $\mathbf{Z}$-irreducible. □

Example. $S = \langle 5, 6, 8 \rangle$ has canonical ideal $\Omega = \langle 0, 2 \rangle$ and Frobenius number $f = 9$. Thus,

$$M = \Omega \cap (\Omega - 2) \cap (\Omega - 9)$$

is the unique irredundant decomposition of M into $\mathbf{Z}$-irreducible ideals.

If $I = C = \{f+1, f+2, \rightarrow\}$ is the conductor of S, then $(C - M) \setminus C = \{f, f-1, \dots, f+1-e\}$. This is a set with e elements, which agrees by Theorem 4.4 with the fact that $\Omega - C = \mathbf{N} = \langle 0, 1, \dots, e-1 \rangle$ is a relative ideal minimally generated by e elements. Thus $N(C) = e$. On the other hand, by Theorem 3.3, $n(C)$ equals the number of elements of $\{f, f-1, \dots, f+1-e\}$ which are in S. Hence:

Proposition 4.11. *If C is the conductor of S, then*

$$1 \leq n(C) \leq e - 1 \leq N(C) = e.$$

If $n(C) = 1$, i.e., if C is irreducible (this happens for example in a semigroup of the form $S = \langle e, e+1, \dots, 2e-1 \rangle$), then the maximal distance $N(C) - n(C) = e - 1$ is realized.

5. Rings. In this section R is assumed to be a ring with total ring of quotients Q, $R \neq Q$, such that each regular ideal (i.e., an ideal containing a nonzerodivisor) is generated by its set of regular elements. As usual, $I : J$ means $\{x \in Q \mid xJ \subseteq I\}$, for I, J fractional ideals of R and $I :_R J = (I : J) \cap R$.

If (R, M) is a Noetherian local ring and I is an M- primary ideal, it is well known that the number $n(I)$ of components of an irredundant decomposition of I are

$$n(I) = l_R(I :_R M/I) = \dim_{R/M} \mathrm{Socle}\,(R/I).$$

In particular this holds for each regular ideal, if R is one-dimensional, and Theorem 3.3 can be seen as an analogy of that for numerical semigroups.

For a ring theoretic result similar to Theorem 4.4, we have to consider local rings (R, M), of total ring of quotients Q, where a completely Q-irreducible fractional ideal exists. Following the terminology of [**5**], a completely Q-irreducible fractional ideal is a fractional ideal that is not an intersection of any set of fractional ideals properly containing it. This concept is close to that of an m-canonical ideal. An m-canonical ideal of a ring R is a fractional ideal ω such that $\omega : (\omega : I) = I$, for each regular ideal I of R. It turns out that if (R, M) is a (not necessarily Noetherian) local ring possessing an m-canonical ideal ω, then ω is completely Q-irreducible and each completely Q-irreducible ideal is of the form $x\omega$ for some regular element $x \in Q$ (cf., e.g., [**2**, Proposition 2.1]).

It is well known that if (R, M) is a Noetherian local ring, which has an m-canonical ideal, then R is one-dimensional. In particular, each analytically unramified one-dimensional local ring, e.g., the ring of an algebraic curve singularity, has an m-canonical ideal.

The following result appears in [**2**]. For convenience of the reader, we include here the proof which is the multiplicative version of the proof of Theorem 4.4 for numerical semigroups.

Proposition 5.1. *Suppose that the local ring (R, M) of total ring of quotients Q has an m-canonical ideal ω. Then*:

(1) *Each regular fractional ideal J of R is an irredundant intersection of completely Q-irreducible fractional ideals.*

(2) *The number* $N(I)$ *of components of an irredundant decomposition of* J *is finite if and only if the ideal* $(\omega : J)$ *is finitely generated and in this case*

$$N(I) = l_R(J : M/J).$$

Proof. Let $\{i_h\}$ be a (non necessarily finite) set of regular elements of Q. Then:

$$\sum i_h R = \sum i_h(\omega : \omega) = \sum(\omega : i_h^{-1}\omega) = \omega : \bigcap i_h^{-1}\omega$$

where, for the last equality, a property of the canonical ideal is applied (cf., [**7**, Lemma 2.2 (e)].

Now, each regular fractional ideal of J of R is of the form $\omega : I$, in fact $\omega : (\omega : J) = J$, and, by the above equalities, $I = \sum i_h R$ if and only if $\omega : I = \cap i_h^{-1}\omega$. It follows that $\{i_h\}$ is a minimal system of generators for $I = \omega : J$ if and only if the decomposition $J = \cap i_h^{-1}\omega$ into Q-irreducible fractional ideals is irredundant and that $I = \omega : J$ is finitely generated if and only if such an irredundant decomposition has finitely many components $N(J)$.

Moreover, if $N(J)$ is finite, it equals the cardinality of a minimal set of generators of $\omega : J$ which is

$$l_R(\omega : J/(\omega : J)M) = l_R(\omega : ((\omega : J)M/J) = l_R(J : M/J). \qquad \square$$

Remark. Observe that if the ring R of Proposition 5.1 is Noetherian (and thus necessarily one-dimensional) then $\omega : J$ is finitely generated and a minimal set of generators has at most e elements, where e is the multiplicity of the ring (cf., [**6**]). Thus, if I is a proper integral regular ideal of R, then, with the notation above, similarly to numerical semigroups we have

$$1 \le n(I) = l_R(I :_R M/I) \le l_R(I : M/I) = N(I) \le e.$$

REFERENCES

1. V. Barucci, *On propinquity of numerical semigroups and one-dimensional local Cohen-Macaulay rings*, in *Commutative algebra and its applications*, M. Fontana, S. Kabbaj, O. Olberding and I. Swanson, eds., de Gruyter, Berlin, 2009.

2. V. Barucci, *Local rings of minimal length*, J. Pure Appl. Algebra **213** (2009), 991–996.

3. ———, *Numerical semigroup algebras*, in *Multiplicative ideal theory in commutative algebra*, J.W. Brewer, S. Glaz, H. Heinzer and B. Olberding, eds., Springer, Berlin, 2006.

4. R. Fröberg, C. Gottlieb and R. Häggkvist, *Semigroups, semigroup rings and analytically irreducible rings*, Reports Dept. Math. Univ. Stockholm **1**, 1986.

5. L. Fuchs, W. Heinzer and B. Olberding, *Commutative ideal theory without finiteness conditions: irreducibility in the quotient field*, in *Abelian groups, rings, modules, and homological algebra*, Lect. Notes Pure Appl. Math. **249**, Chapman and Hall/CRC, Boca Raton, FL, 2006.

6. C. Gottlieb, *On generarors of ideals in one-dimensional local rings*, Comm. Algebra **21** (1993), 421–425.

7. W. Heinzer, J. Huckaba and I. Papick, *m-canonical ideals in integral domains*, Comm. Algebra **26** (1998), 3021–3043.

8. E. Noether, *Idealtheorie in Ringbereichen*, Math. Ann. **83** (1921), 24–66.

9. J.C. Rosales, P.A. García-Sanchez and J.I. García-García, *Irreducible ideals of finitely generated commutative monoids*, J. Algebra **238** (2001), 328–344.

Dipartimento di Matematica, Sapienza, Università di Roma, Piazzale Aldo Moro 2, 00185 Roma, Italy
Email address: barucci@mat.uniroma1.it