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# ON COMPLETELY DECOMPOSABLE AND SEPARABLE MODULES OVER PRÜFER DOMAINS

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ABSTRACT. We generalize known results on summands of completely decomposable and separable torsion-free abelian groups to modules over $h$-local Prüfer domains. Over such domains summands of completely decomposable torsion-free modules are again completely decomposable (Theorem 3.2) and summands of separable torsion-free modules are likewise separable (Theorem 4.2). In addition, a Pontryagin-Hill type theorem is established on countable chains of homogeneous completely decomposable modules over $h$-local Prüfer domains (Theorem 7.1). Several auxiliary results are proved for modules over integral domains that are direct sums of finite or countable rank submodules.

**1. Introduction.** All modules in this note are torsion-free modules over integral domains $R$.

By a *completely decomposable* torsion-free module $M$ is meant a direct sum of rank 1 modules, i.e., of modules that are $R$-isomorphic to submodules of the field $Q$ of quotients of $R$. The cardinal number of the set of summands is called the *rank* of $M$, in notation: $\mathrm{rk}\, M$. This is an invariant of $M$: the cardinality of every maximal independent set in $M$.

By making use of results by Olberding [**15**], recently Goeters [**9**] proved that over an $h$-local Prüfer domain $R$ summands of finite rank completely decomposable torsion-free modules are again completely decomposable. In Theorem 3.2 we extend this theorem to modules of arbitrary ranks. Our approach is different from Goeters' inasmuch as we rely on results by Kolettis [**12**] on homogeneously decomposable

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torsion-free modules. Our theorem generalizes the celebrated Baer-Kulikov-Kaplansky theorem on summands of completely decomposable abelian groups (e.g., Fuchs [**5**, Theorem 86.7]).

We also generalize an old result on abelian groups stating that summands of separable torsion-free groups are again separable (see, e.g., Fuchs [**5**, Theorem 87.5]). Theorem 4.2 asserts that summands of separable torsion-free modules over an $h$-local Prüfer domain $R$ are again separable. The proof is via reduction to the completely decomposable case.

Hill [**10**] established a far-reaching generalization of Pontryagin's criterion on the freeness of abelian groups by proving that the union of a countable ascending chain of pure free subgroups (of any size) is likewise free. This theorem is extended here to countable chains of homogeneous completely decomposable modules over $h$-local Prüfer domains (Theorem 7.1). The hypothesis of purity had to be strengthened: we assume that countable rank RD-submodules in the factors of the chain can be obtained as images of countable rank submodules from the links of the chain (they are called RD$^*$-submodules)–a condition that is automatically satisfied whenever $R$ is a countable domain.

We also show that there is a continuous well-ordered ascending chain with countable rank factors consisting of completely decomposable RD-submodules between a completely decomposable module and a completely decomposable RD$^*$-submodule, a fact that underlines the importance played by countability in the theory of completely decomposable modules. (This phenomenon was first observed by Dugas-Rangaswamy [**4**] for abelian groups.)

Some of our results are proved under more general conditions than needed for our main results: for direct sums of finite or countable rank modules (rather than just for direct sums of rank 1 modules). Besides their independent interest, their proofs also reveal basic ideas on which the results rely.

**2. Preliminaries.** Let $M$ be any module over the domain $R$. Following Hill, we define various families of submodules (see also Fuchs-Salce [**7**]).

By an $H(\aleph_0)$-family in $M$ is meant a collection $\mathcal{H}$ of submodules of $M$ satisfying the following properties:

H1. $0, M \in \mathcal{H}$;

H2. $\mathcal{H}$ is closed under unions, i.e., $M_i \in \mathcal{H}$ $(i \in I)$ implies $\sum_{i\in I} M_i \in H$ for any index set $I$;

H3. if $C \in \mathcal{H}$ and $X$ is any countable subset of $M$, then there is a submodule $B \in \mathcal{H}$ that contains both $C$ and $X$ and is such that $B/C$ is countably generated.

A $G(\aleph_0)$-family $\mathcal{G}$ is defined similarly with H2 replaced by the following weaker condition:

G2. $\mathcal{G}$ is closed under unions of chains.

In this paper we are interested in the rank versions of these families. The $H^*(\aleph_0)$-*family* and the $G^*(\aleph_0)$-family are defined similarly for torsion-free modules $M$ (see Rangaswamy [**16**]): in these cases the submodules in the families are required to be $RD$-submodules and in condition H3 'countable rank' is to be used in place of 'countably generated.' (Recall that a relatively divisible or briefly an $RD$-submodule of $M$ is a submodule $N$ satisfying $rN = N \cap rM$ for all $r \in R$.)

Obviously, every $H^*(\aleph_0)$-family is a $G^*(\aleph_0)$-family, but the converse fails in general. Note that every torsion-free $R$-module $M$ has a $G^*(\aleph_0)$-family of RD-submodules. In fact, select a maximal independent set $X$ in $M$. For a subset $Y$ of $X$, let $M_Y$ denote the smallest RD-submodule of $M$ that contains $Y$. It is readily checked that the set of all $M_Y$ is a $G^*(\aleph_0)$-family. However, this is in general not an $H^*(\aleph_0)$-family, since the sum of two RD-submodules need not be an RD-submodule.

If the $R$-module $M$ is a direct sum of submodules of countable rank, and $M = \oplus_{\alpha\in I} A_\alpha$ with $\operatorname{rk} A_\alpha \leq \aleph_0$ is such a decomposition for an index set $I$, then the standard way of defining an $H^*(\aleph_0)$-family $H$ of summands in $M$ is to consider the set of all partial summands in this decomposition: $H_J = \oplus_{\alpha\in J} A_\alpha$ with $J$ ranging over all subsets of $I$.

It is well known (and easy to check) that the intersection of a finite number of (or of even countably many) $G^*(\aleph_0)$-families is again such a family. The same holds for $H^*(\aleph_0)$-families.

Next we introduce a new concept that will be needed in the sequel, strengthening the RD-property of submodules.

Let $A$ be a submodule of the torsion-free $R$-module $M$, and let $\phi : M \to M/A$ be the canonical map. We say that $A$ is a *strong* $RD^*$*-submodule* of $M$ if

1) it is an RD-submodule, and

2) each finite (and hence countable) rank submodule in $M/A$ has a countable rank preimage in $M$.

For the sake of comparison let us point out that the $RD$-submodule $A$ is balanced in $M$ if every rank one submodule in $M/A$ has a rank one preimage in $M$. Thus the property of being 'RD*' lies between 'RD' and 'balancedness.'

In the following list (a)–(d), $A$, $B$ will denote RD-submodules of the torsion-free $R$-module $M$ such that $A \leq B$. It is straightforward to verify that

(a) direct summands and balanced submodules are RD$^*$-submodules;

(b) if $A$ is an RD$^*$-submodule of $M$, then it is RD$^*$ in $B$ as well;

(c) the property 'RD$^*$' is a transitive relation: if $A$ is an RD$^*$-submodule of $B$ and $B$ is an RD$^*$-submodule of $M$, then $A$ is an RD$^*$-submodule of $M$;

(d) let $A$ be an RD$^*$-submodule of $M$; then $B$ is an RD$^*$-submodule in $M$ if and only if $B/A$ is an RD$^*$-submodule of $M/A$.

**Example 2.1.** Suppose that there exists an uncountably generated rank one torsion-free $R$-module $A$ (e.g., an uncountably generated field of quotients of certain Dedekind domains). If $0 \to H \to F \to A \to 0$ is a free presentation of $A$, then $H$ is RD, but not RD$^*$ in $F$.

**Example 2.2.** It is easy to see that the concept of RD$^*$-submodule is new only if $R$ is uncountable, because if $R$ is a countable domain, then all RD-submodules are automatically RD$^*$-submodules. In fact, if $A$ is RD in $M$ and $\phi : M \to M/A$ is the canonical map, then every countable rank submodule of $M/A$ is countably generated, and the generators are included in $\phi C$ for some countable rank submodule $C$ of $M$.

Next we prove an easy result.

**Lemma 2.3.** *If $A$ is an* RD$^*$*-submodule of the torsion-free $R$-module $M$, then for every $G^*(\aleph_0)$-family $\mathcal{C}$ of* RD*-submodules in $M/A$, $M$*

*admits a $G^*(\aleph_0)$-family $\mathcal{G}$ of* RD*-submodules such that*

$$\mathcal{C} = \{\phi B \mid B \in \mathcal{G}\},$$

*where $\phi$ denotes the canonical projection $M \to M/A$.*

*Proof.* Let $\mathcal{F}$ be the $G^*(\aleph_0)$-family of RD-submodules of $M$ and $\mathcal{C}$ a $G^*(\aleph_0)$-family of RD-submodules in $M/A$. Define $\mathcal{G} = \{B \in \mathcal{F} \mid \phi B \in \mathcal{C}\}$ where $\phi : M \to M/A$ denotes the canonical projection. It is readily seen that $\mathcal{G}$ is as desired. □

We say that two torsion-free $R$-modules, $A$ and $B$, are *quasi-isomorphic* (see Goeters [**9**]) if there exist submodules $A' \leq A$ and $B' \leq B$ such that $A' \cong B$ and $B' \cong A$. Quasi-isomorphism is evidently an equivalence relation on torsion-free $R$-modules.

The equivalence classes of rank 1 torsion-free $R$-modules under quasi-isomorphism are called *types*. The type of a rank 1 torsion-free module $M$ is denoted by the symbol $\tau(M)$. The set of types admits a natural partial order: for types $\sigma$ and $\tau$ we set $\sigma \leq \tau$ if and only if there exist rank 1 $R$-modules $A$ and $B$ with $\tau(A) = \sigma$ and $\tau(B) = \tau$ such that $A$ is a submodule of $B$. The smallest type is the common type of all fractional ideals of $R$, while the largest type is the type of $Q$, the field of quotients of $R$.

Just as for abelian groups, with a given type $\tau$ one can associate two fully invariant submodules, $M(\tau)$ and $M^*(\tau)$, of a torsion-free $R$-module $M$ as follows:

$$M(\tau) = \sum\{X \mid X \leq M; \tau(X) \geq \tau\}$$

and

$$M^*(\tau) = \sum\{X \mid X \leq M; \tau(X) > \tau\}$$

where $X$ stands for rank one submodules. From the definition it is clear that they are submodules of $M$ such that $M(\tau) \geq M^*(\tau)$; furthermore, $M(\sigma) \leq M(\tau)$ and $M^*(\sigma) \leq M^*(\tau)$ whenever $\sigma \geq \tau$.

A torsion-free module $H$ will be called *homogeneous of type $\tau$* if $H(\tau) = H$ and $H^*(\tau) = 0$. Evidently, RD-submodules of homogeneous torsion-free modules are again homogeneous. Projective modules as well as divisible torsion-free modules are homogeneous, and so are direct sums of fractional ideals of $R$.

Kolettis [**12**] calls a torsion-free module $M$ *homogeneously decomposable* if it is a direct sum of homogeneous modules (of equal or different types). He proves that a torsion-free module $M$ of countable rank is homogeneously decomposable if and only if it satisfies the following two conditions: (i) for every type $\tau$, both $M(\tau)$ and $M^*(\tau)$ are summands of $M$; and (ii) every element of $M$ belongs to a direct summand of $M$ that is a finite direct sum of homogeneous modules. Using this characterization, he proves:

**Theorem 2.4** (Kolettis [**12**]). *Summands of a homogeneously decomposable torsion-free $R$-module are themselves homogeneously decomposable.*

**3. Summands of completely decomposable modules.** We repeat the definition: a torsion-free $R$-module $C$ is *completely decomposable* if it is the direct sum of rank 1 submodules. Such a $C$ is homogeneous if it is the direct sum of quasi-isomorphic rank 1 modules. It is clear that completely decomposable modules are homogeneously decomposable.

In the study of completely decomposable modules it is crucial what happens in the finite rank case. It is a classical theorem by Baer [**1**] that a finite rank completely decomposable homogeneous abelian group has the distinguished property that every pure (i.e., RD-)subgroup is a summand, and hence it is likewise completely decomposable. This is not true in general, not even for projective modules. Olberding [**15**] proved that this property is shared by $h$-local Prüfer domains $R$ (recall: a domain $R$ is *$h$-local* if every non-zero element belongs only to finitely many maximal ideals, and every non-zero prime ideal is contained only in a single maximal ideal), moreover:

**Theorem 3.1** (Olberding [**15**]). *The following are equivalent for any integral domain $R$:*

(a) *$R$ is an $h$-local Prüfer domain;*

(b) *every pure submodule of a finite rank completely decomposable homogeneous torsion-free module is a summand;*

(c) *every pure submodule of a finite direct sum of fractional ideals is a summand.*

It is easy to see that in conditions (b) and (c) 'pure submodule' can be replaced by 'RD-submodule' (this strengthens the hypothesis of the difficult implication (c) $\Rightarrow$ (a)). It also follows at once that the summands in (b) and (c) are then completely decomposable.

Using Olberding's theorem, Goeters [**9**] proved that summands of finite rank completely decomposable torsion-free modules over $h$-local Prüfer domains are again completely decomposable. Our present goal is to extend this result to completely decomposable modules of arbitrarily high ranks and to verify the analogue for separable modules (see next section). We call a torsion-free $R$-module $M$ *separable* (in the sense of Baer [**1**]) if 1) every finite set of its elements can be embedded in a finite rank summand of $M$, and 2) finite rank summands of $M$ are completely decomposable. (This is a slightly stronger definition than the one used in Fuchs-Salce [**7**, Chapter 16, Section 5].)

Accordingly, we are now going to prove:

**Theorem 3.2.** *Summands of completely decomposable torsion-free modules over $h$-local Prüfer domains are likewise completely decomposable.*

*Proof.* The proof begins with the reduction to the countable rank case. By the rank version of a well-known theorem by Kaplansky [**11**], summands of modules that are direct sums of countable rank submodules are themselves direct sums of countable rank summands. In view of this, it is straightforward to see that it will suffice to prove that if $M = A \oplus B$ is a countable rank completely decomposable torsion-free $R$-module, then $A$ is also completely decomposable.

Further reduction is possible if we make use of Kolettis's theorem quoted above. Indeed, a completely decomposable module being homogeneously decomposable, from Theorem 2.4 it follows that for the proof of Theorem 3.2 we may assume without loss of generality that $M$ is homogeneous.

The next step in the proof is to show that the summand $A$ of $M$ is separable. So, let $a_1, \dots, a_n$ be elements of $A$. Clearly, there is a finite rank completely decomposable summand $N$ of $M$ that contains all of

$a_1, \dots, a_n$. The RD-submodule $A'$ spanned by the elements $a_1, \dots, a_n$ is by Olberding's theorem a completely decomposable summand of $N$. Thus $A'$ is a completely decomposable summand of $M$, and hence of $A$. This shows that all finite rank RD-submodules are completely decomposable summands, establishing the separability of $A$.

Thus $A$ is the union of a countable chain of finite rank completely decomposable submodules each of which is a summand of the following ones with completely decomposable complements. It follows that $A$ is completely decomposable, completing the proof of the theorem. □

**4. Summands of separable modules.** We start the discussion of separability (defined above) with a general lemma that holds over all integral domains.

**Lemma 4.1.** *A domain $R$ has the property that summands of separable torsion-free $R$-modules are again separable if and only if summands of completely decomposable torsion-free $R$-modules of countable rank are again completely decomposable.*

*Proof.* Before proving the equivalence of the stated conditions, we observe that either implies that summands of finite rank completely decomposable $R$-modules are again completely decomposable. As a consequence, we can argue (as in the final part of the proof of Theorem 3.2) that countable rank separable $R$-modules are completely decomposable.

Necessity follows at once by applying the hypothesis to a completely decomposable module of countable rank noting that countable rank separable modules are completely decomposable.

For sufficiency, assume that summands of completely decomposable torsion-free $R$-modules of countable rank are completely decomposable. Let $M$ be a separable torsion-free $R$-module and $M = A \oplus B$ a direct decomposition of $M$. Given a finite subset $S$ in $A$, we have to show that $S$ is contained in a finite rank summand $H$ of $A$ and finite rank summands of $A$ are completely decomposable.

Let $M_0$ be a finite rank completely decomposable summand of $M$ containing $S$, and let $A_0, B_0$ be finite rank RD-submodules of $A$ and $B$, respectively, such that $M_0 \leq A_0 \oplus B_0$. There is a finite rank completely decomposable summand $M_1$ of $M$ that contains a maximal independent set in $A_0 \oplus B_0$, and hence it contains both $A_0$ and $B_0$. Furthermore, there are finite rank RD-submodules $A_1, B_1$ of $A$ and $B$, respectively, satisfying $M_1 \leq A_1 \oplus B_1$. Continuing this way, we obtain an ascending chain

$$M_0 \leq A_0 \oplus B_0 \leq M_1 \leq A_1 \oplus B_1 \leq \cdots \leq M_n \leq A_n \oplus B_n \leq \cdots \ (n < \omega)$$

where $M_n$ are finite rank summands of $M$, while $A_n, B_n$ are finite rank RD-submodules of $A, B$. The union $M'$ of this chain is a countable rank submodule of $M$ which is completely decomposable as the union of the chain of completely decomposable modules $M_n$ where every module in the chain is a summand in each of the following ones with completely decomposable complement. Moreover, by construction, we have

$$M' = A' \oplus B' \text{ where } A' = \bigcup_n A_n, \ B' = \bigcup_n B_n.$$

By hypothesis, $A', B'$ are completely decomposable as summands of the completely decomposable module $M'$ of countable rank. Therefore, $S$ is contained in a finite rank completely decomposable summand $H$ of $A'$. Then $H$ is a summand of $M'$, and since $H \leq M_k < M'$ for some $k < \omega$, $H$ is a summand of $M_k$, so also of $M$, and hence of $A$.

From our argument it is also clear that finite rank summands of $A$ are summands of a completely decomposable module, so they are themselves completely decomposable. □

Consequently, combining Theorem 3.2 and Lemma 4.1 we can state:

**Theorem 4.2.** *Summands of separable torsion-free modules over an $h$-local Prüfer domain are separable.*

**5. Chains of completely decomposable submodules between completely decomposable submodules.** We would like to call attention to an interesting phenomenon: the existence of chains with countable rank factors between a completely decomposable module and a completely decomposable RD-submodule; see Proposition 5.2. This has been pointed out for abelian groups by Dugas-Rangaswamy [**4**] (cf., also Fuchs-Viljoen [**8**]), and interestingly, it holds over arbitrary integral domains. It provides an additional evidence that complete decomposability is intimately tied to countability even in more general situations.

We phrase the results more generally, for modules that are direct sums of countable rank submodules. The completely decomposable case will then be a simple corollary.

We require a preliminary lemma.

**Lemma 5.1.** *Suppose $B$ is an $R$-module that is a direct sum of modules of countable rank, and $A$ is a submodule of $B$ that is likewise a direct sum of countable rank modules.*

(i) *If $B'$ is a summand of $B$ such that $A' = A \cap B'$ is a summand of $A$, then $A + B'$ is a direct sum of modules of countable rank.*

(ii) *There exist $\mathcal{G}^*(\aleph_0)$-families $\mathcal{A}$ and $\mathcal{B}$ of summands in $A$ and $B$, respectively, such that $\mathcal{A} = \{A \cap X \mid X \in \mathcal{B}\}$.*

*Proof.* (i) By a well-known Kaplansky result [**11**] already mentioned above, summands of a module that is a direct sum of modules of countable rank are again direct sums of modules of countable rank. Consequently, $(A + B')/B' \cong A/A'$ is a module that is a direct sum of modules of countable rank. Furthermore, $B'$ is a summand of $A + B'$, thus $A + B' \cong B' \oplus A/A'$ is likewise a direct sum of submodules of countable rank.

(ii) Fix decompositions of $A$ and $B$ as direct sums of countable rank modules, and let $\mathcal{A}'$ and $\mathcal{B}'$ denote the $\mathcal{H}^*(\aleph_0)$-families of direct sums of subsets of these summands in $A$ and $B$, respectively, The first and the second entries in the pairs $(A', B')$ with $A' = A \cap B'$ $(A' \in \mathcal{A}', B' \in \mathcal{B}')$ yield the desired $\mathcal{G}^*(\aleph_0)$-families $\mathcal{A}$ and $\mathcal{B}$, respectively. □

We can now verify:

**Proposition 5.2.** *Suppose $A$ is an* RD$^*$*-submodule of the torsion-free $R$-module $B$ such that both $A$ and $B$ are direct sums of countable rank submodules.*

(i) *For some ordinal $\tau$, there is a continuous well-ordered ascending chain*

$$A = B_0 \le B_1 \le \cdots \le B_\sigma \le \cdots \le B_\tau = B \tag{1}$$

*of RD-submodules between $A$ and $B$ such that each $B_\sigma$ is a direct sum of submodules of countable rank and $B_{\sigma+1}/B_\sigma$ is torsion-free of rank $\le \aleph_0$, for every $\sigma < \tau$.*

(ii) *If $A$ and $B$ are completely decomposable, then the $B_\sigma$ can be chosen to be completely decomposable as well.*

*Proof.* (i) Select $\mathcal{G}^*(\aleph_0)$-families $\mathcal{A}$ and $\mathcal{B}$ of summands in $A$ and $B$, respectively, as stated in Lemma 5.1 (ii). In view of Lemma 2.3, we can find in $B$ a $\mathcal{G}^*(\aleph_0)$-family $\mathcal{G}$ of RD-submodules $B'$ such that $A + B'$ is always an RD-submodule of $B$. The intersection $\mathcal{B} \cap \mathcal{G}$ is evidently a $\mathcal{G}^*(\aleph_0)$-family, from which we extract a continuous well-ordered ascending chain $0 = B'_0 < B'_1 < \cdots < B'_\sigma < \cdots < B'_\tau = B$ such that all $B'_{\sigma+1}/B'_\sigma$ are of countable rank. Next we form a chain (1) with the RD-submodules $B_\sigma = A + B'_\sigma$ $(\sigma < \tau)$. Lemma 5.1 (i) guarantees that chain (1) will have the desired property, since

$$B_{\sigma+1}/B_\sigma \cong B'_{\sigma+1}/[B'_{\sigma+1} \cap (A + B'_\sigma)] = B'_{\sigma+1}/[(B'_{\sigma+1} \cap A) + B'_\sigma]$$

is a surjective image of $B'_{\sigma+1}/B'_\sigma$.

(ii) In case both $A$ and $B$ are completely decomposable, then the summands $A', B'$ in Lemma 5.1 (i) can be chosen such that all the modules $A/A'$ and $B'$ are completely decomposable. Then the modules $B_\sigma$ of the preceding paragraph will be completely decomposable. □

**6. Chains of finitely decomposable modules.** The classical Pontryagin theorem on torsion-free abelian groups states that the union of an ascending sequence of finite rank free groups is free whenever each group in the sequence is pure in its immediate successor. This important theorem has been generalized by Hill [**10**]: the union of an ascending sequence $0 = A_0 < A_1 < \cdots < A_n < \cdots$ $(n < \omega)$ of free abelian groups (of any size) is free provided that for each $n < \omega$, $A_n$ is pure in $A_{n+1}$. Our next goal is to establish an analogous result for homogeneous completely decomposable modules over an $h$-local Prüfer domain (Theorem 7.1). (A similar result on valuation domains was proved by Rangaswamy [**16**].) In this section, we prove a preparatory result (Theorem 6.3) that might be of independent interest. It is phrased in more general terms than needed in what follows in order to emphasize a main point that makes things work for countable unions.

By a *finitely decomposable* torsion-free $R$-module we mean a module that is the direct sum of finite rank submodules. We call an RD-submodule $A$ of the torsion-free $R$-module $M$ *ultra-balanced* if $A$ is a summand in every RD-submodule $C$ of $M$ that contains $A$ as a finite corank submodule. (Ultra-balanced subgroups of abelian groups have been introduced and discussed by Chao [**2**]. Ultra-balanced submodules are of course balanced.) The meaning of '*ultra-balanced projective*' is evident. It is straightforward to check that the ultra-balanced projective modules are precisely the summands of finitely decomposable modules. They are not necessarily finitely decomposable, not even for abelian groups; this is demonstrated by an example of Corner [**3**]: a countable finitely decomposable torsion-free abelian group that is the direct sum of two indecomposable groups of countable rank.

We now state the crucial lemma (some arguments are similar, e.g., to [**16**, Lemma 5.2]).

**Lemma 6.1.** *Assume that the $R$-module $M$ is the union of an ascending chain*

$$0 = M_0 < M_1 < \cdots < M_n < \cdots (n < \omega) \tag{2}$$

*of torsion-free submodules such that*

(i) *each $M_n$ admits a $G^*(\aleph_0)$-family $\mathcal{D}_n$ of direct summands, and*

(ii) *for each $n < \omega$, $M_n$ is an* RD$^*$*-submodule in $M_{n+1}$.*

*Then there exists a $G^*(\aleph_0)$-family $\mathcal{B}$ of ultra-balanced submodules of $M$ such that for all $n < \omega$ and for all $A \in \mathcal{B}$ we have*

(a) $A \cap M_n \in \mathcal{D}_n$; *and*

(b) $A + M_n$ *is an* RD*-submodule in $M$.*

*Proof.* Assume (2) satisfies hypotheses (i) and (ii). First of all, we claim that the collection

$$\mathcal{B}_n = \{A \in \mathcal{D}_n \mid A + M_k \text{ is RD in } M_n \ (k < n)\}$$

is a $G^*(\aleph_0)$-family of summands in $M_n$. By hypothesis (ii), $M_n$ has a $G^*(\aleph_0)$-family $\mathcal{G}_k$ $(k < n)$ of RD-submodules such that its members project onto RD-submodules of $M_n/M_k$ (see Lemma 2.3). It is readily checked that

$$\mathcal{B}_n = \mathcal{D}_n \cap \mathcal{G}_1 \cap \cdots \cap \mathcal{G}_{n-1}$$

is as desired.

The next step is to show that the collection

$$\mathcal{B} = \{A \leq M \mid A \cap M_n \in \mathcal{B}_n \text{ for each } n < \omega\}$$

is a $G^*(\aleph_0)$-family of RD-submodules in $M$. For details, we refer to the proof of [**6**, Lemma 1.7]. It follows easily that the $G^*(\aleph_0)$-family $\mathcal{B}$ of RD-submodules will have properties (a) and (b). For example, to check condition (b) just observe that the RD-property is transitive and $A = \cup_n (A \cap M_n)$.

It remains to show that the submodules in $\mathcal{B}$ are ultra-balanced in $M$. Suppose $A \in \mathcal{B}$, and let $C$ be an RD-submodule of $M$ such that $A < C$ with $C/A$ of finite rank. Pick a maximal independent set $S = \{c_1, \ldots, c_k\}$ in $C$ mod $A$. There is an index $n$ such that $S \subset M_n$. By (b), $A + M_n$ is an RD-submodule in $M$, and the same is true for $A + (M_n \cap C) = (A + M_n) \cap C$. This RD-submodule contains both $A$ and $S$; consequently, $A + (M_n \cap C) = C$. By (a), $M_n \cap A$ is a summand of $M_n$, say, $M_n = (M_n \cap A) \oplus B$. Therefore, $M_n \cap C = (M_n \cap A) \oplus (B \cap C)$, whence

$$C = A + (M_n \cap A) + (B \cap C) = A + (B \cap C)$$

follows. Since $A \cap B \cap C = A \cap B = A \cap B \cap M_n = 0$, we have $C = A \oplus (B \cap C)$. Here $B \cap C$ is a finite rank RD-submodule of $M$, so $A$ is a summand of every submodule of $M$ in which it is contained with finite corank, i.e., $A$ is ultra-balanced in $M$. □

The countable rank version of Theorem 6.3 is proved separately as our next lemma.

**Lemma 6.2.** *Assume* (2) *is a chain of torsion-free $R$-modules of countable rank such that*

(a) *each $M_n$ is finitely decomposable*;

(b) *$M_n$ is an* RD-*submodule of $M_{n+1}$ for each $n < \omega$.*

*A necessary and sufficient condition that the union $M$ of the chain be finitely decomposable is*

(*) *for every finite set $S$ of elements in $M$ there exist an index $n$ and a finite rank submodule $C$ of $M$ containing $S$ such that $C$ is a summand of $M_m$ for all $m \geq n$.*

*Proof.* Necessity is easy: if $M$ is finitely decomposable, then it must have a finite rank summand $C$ containing a given finite set of elements, and $C$ is necessarily a summand of each $M_n$ in which it is contained.

For the proof of sufficiency, assume the stated condition. Select a maximal independent set $a_0, a_1, \dots , a_n, \dots$ of $M$. We construct a chain $C_0 \leq C_1 \leq \cdots \leq C_n \leq \cdots$ of submodules satisfying the following conditions:

($\alpha$) $a_0, a_1, \dots , a_n \in C_n$ for each $n < \omega$;

($\beta$) $C_n$ is a finite rank summand of all $M_m$ for all $m \geq i_n$ for some $i_n$;

($\gamma$) $i_0 \leq i_1 \leq \cdots \leq i_n \leq \cdots$ .

Hypothesis (*) guarantees that such a chain does exist. Clearly, $C_n$ will be a summand of $C_{n+1}$, because it is a summand of $M_{i_{n+1}}$ containing $C_{n+1}$; say, $C_{n+1} = C_n \oplus B_{n+1}$. Then $M$ will be the direct sum of $C_0$ and the $B_n$'s all of which are of finite rank. Consequently, $M$ is finitely decomposable. □

Observe that the proof of the preceding lemma establishes the necessity of the condition (*) in the following theorem.

**Theorem 6.3.** *Let* (2) *be a chain of torsion-free $R$-modules. Suppose that*

(a) *each $M_n$ is finitely decomposable*;

(b) *$M_n$ is an* RD*-*submodule of $M_{n+1}$ for each $n < \omega$.*

*A necessary and sufficient condition that the union $M$ of the chain be finitely decomposable is condition* (*) *in Lemma* 6.2.

*Proof.* Assuming (*), let $\mathcal{D}_n$ denote an $H^*(\aleph_0)$-family of summands in a fixed direct decomposition of $M_n$ as a direct sum of finite rank submodules. We appeal to Lemma 6.1 to conclude that there is a $G^*(\aleph_0)$-family $\mathcal{B}$ of ultra-balanced submodules of $M$ such that $A \cap M_n \in \mathcal{D}_n$ and $A + M_n$ is an RD-submodule in $M$ for every $A \in \mathcal{B}$ and for every $n < \omega$.

By transfinite induction we construct, for some ordinal $\mu$, a continuous well-ordered ascending chain

$$0 = N_0 < N_1 < \cdots < N_\alpha < \cdots \ (\alpha < \mu) \tag{3}$$

of submodules of $M$ such that, for each $\alpha < \mu$,

(i) $N_\alpha$ is finitely decomposable;

(ii) $N_\alpha \in \mathcal{B}$;

(iii) $N_\alpha$ is a summand in $N_{\alpha+1}$;

(iv) for a finite subset $S$ of $N_\alpha$, $N_\alpha$ has a finite rank summand $C$ of $M$ that contains $S$ and is a summand of $M_m$ for all $m \geq n$, for a suitable $n$;

(v) $N_{\alpha+1}/N_\alpha$ is finitely decomposable of rank $\leq \aleph_0$;

(vi) $M = \cup_{\alpha<\mu} N_\alpha$.

It will suffice to discuss the step from $N_\alpha$ to $N_{\alpha+1}$ for $\alpha < \mu$. So suppose that, for some ordinal $\beta < \mu$, the submodules $N_\alpha$ have been defined for all $\alpha \leq \beta$ satisfying (i)–(v). Pick a countable independent set $a_0, a_1, \ldots, a_n, \ldots$ modulo $N_\beta$ in $M$, and proceed to construct a chain $C_0 \leq C_1 \leq \cdots \leq C_k \leq \cdots$ satisfying conditions $(\alpha)$–$(\gamma)$ for the chosen elements $a_n$. Moreover, in order to satisfy (iv), we require that the $C_k$ be such that

($\delta$) $C_k \cap M_{i_k} \in \mathcal{B}_{i_k}$ for each $k < \omega$.

This can be achieved if we increase the $C_k$ by including an appropriate finite rank summand of $N_\beta$. Then $N_\beta \cap C_k = X_k$ will be a summand of $N_\beta$, say, $N_\beta = X_k \oplus P_k$. Furthermore, by (ii) $N_\beta$ is ultra-balanced in $N_\beta + C_k$, so $N_\beta / P_k \cong X_k$ is ultrabalanced in $(N_\beta + C_k)/P_k$ whence $C_k = X_k \oplus Y_k$ follows for a suitable finite rank submodule $Y_k$ of $M$. Similarly, we obtain $C_{k+1} = X_{k+1} \oplus Y_{k+1}$. Manifestly, these $Y_k$ $(k < \omega)$ form an ascending chain mod$N_\beta$, and we set

$$N_{\beta+1} = \bigcup_{k<\omega} (N_\beta \oplus Y_k).$$

In order to verify (v) for index $\beta$, we show that $Y_k$ is a summand of $Y_{k+1}$ mod $N_\beta$. We argue as follows. Write $C_{k+1} = C_k \oplus D_k$ for $k < \omega$. As $D_k$ is of finite rank, we have $N_\beta + D_k = N_\beta \oplus V_k$ for some finite rank module $V_k$ (again by the ultra-balancedness of $N_\beta$). In addition,

$$N_\beta \oplus Y_{k+1} = N_\beta + C_{k+1} = N_\beta + C_k + D_k = (N_\beta + D_k) + C_k$$
$$= (N_\beta \oplus V_k) + X_k + Y_k = N_\beta + V_k + Y_k.$$

We claim that the last sum is actually a direct sum, and prove this by comparing ranks. If we denote the ranks of $Y_k, Y_{k+1}, V_k$ by $r, s, t$, respectively, then these are also the ranks of $C_k, C_{k+1}, D_k$ modulo $N_\beta$, so from $C_{k+1} = C_k \oplus D_k$ we obtain $s \geq r + t$. This suffices to conclude that $N_\beta + V_k + Y_k = N_\beta \oplus V_k \oplus Y_k$, which implies that $Y_{k+1} \equiv Y_k \oplus V_k \bmod N_\beta$, as desired. The proof can be finished by the same argument as in the proof of Lemma 6.2. □

**7. A main result.** We are now prepared for the proof of a main result (a somewhat weaker form was included in the Ph.D. thesis of the second author [**13**]). It generalizes the Pontryagin-Hill theorem from free abelian groups to homogeneous completely decomposable modules over $h$-local Prüfer domains.

**Theorem 7.1.** *Let $R$ be an $h$-local Prüfer domain and $M$ a torsion-free $R$-module that is the union of a countable ascending chain* (2) *of submodules such that, for every $n < \omega$,*

1°. *$M_n$ is a homogeneous completely decomposable $R$-module of fixed type $\tau$;*

2°. *$M_n$ is an* RD*-*-submodule of $M_{n+1}$.*

*Then $M$ is completely decomposable of type $\tau$.*

*Proof.* Condition (a) of Theorem 6.3 is satisfied by assumption 1°. The stated necessary and sufficient condition (*) in this quoted theorem holds because of Theorem 3.1, so our claim is immediate. □

The following example will show that Theorem 7.1 fails even for abelian groups if the condition of homogeneity is dropped. We use the symbol $\mathbf{Z}/p_1^\infty \cdots p_k^\infty$ to denote the set of all rational numbers in whose denominators only powers of the primes $p_1, \dots, p_k$ occur.

**Example 7.2.** Let $p_1, p_2, \dots, p_n, \dots$ be a list of distinct primes. Define

$$A_0 = \mathbf{Z}, \qquad A_1 = \mathbf{Z}/p_1^\infty \oplus \mathbf{Z}/p_2^\infty,$$
$$A_2 = \mathbf{Z}/p_1^\infty p_3^\infty \oplus \mathbf{Z}/p_1^\infty p_4^\infty \oplus \mathbf{Z}/p_2^\infty p_3^\infty \oplus \mathbf{Z}/p_2^\infty p_4^\infty, \dots$$

where from $A_{n-1}$ we pass to $A_n$ by replacing each summand by two copies of the direct sum of the summand after adjoining to the denominators one of $p^\infty$ for the next two primes $p$ in the list. In this way we get an ascending chain $0 < A_1 < A_2 < \cdots < A_n < \cdots$ of completely decomposable abelian groups if we use the diagonal embeddings (e.g., $A_1 \to A_2$ is induced by identifying $1 \in \mathbf{Z}/p_1^\infty$ with $(1,1) \in \mathbf{Z}/p_1^\infty p_3^\infty \oplus \mathbf{Z}/p_1^\infty p_4^\infty$ and $1 \in \mathbf{Z}/p_2^\infty$ with $(1,1) \in \mathbf{Z}/p_2^\infty p_3^\infty \oplus \mathbf{Z}/p_2^\infty p_4^\infty$). Then each $A_n$ will be a pure subgroup in the following group in the chain. In order to justify our claim that the union $A = \cup_{n<\omega} A_n$ is not completely decomposable, assume by way of contradiction that $A$ is completely decomposable and $J$ is a rank one summand of $A$. Then $J$ is also a summand in the first link $A_m$ of the chain in which it is contained. The rank 1 summands of $A_m$ are fully invariant in $A_m$, so $J$ must be one of the summands in the given decomposition of $A_m$. Manifestly, $J$ has to be a summand in $A_{m+1}$ as well, but the construction shows that this is not the case. Thus, $A$ cannot be completely decomposable.

Finally, we would like to apply our results to projective modules over integral domains $R$.

We consider the case when the projective modules over $R$ are finitely decomposable. It is generally known that projective modules are direct sums of countably generated modules. Over a domain they are finitely decomposable if and only if they are direct sums of finitely generated modules. Rings over which the projective modules are direct sums of finitely generated modules are characterized by McGovern-Puninski-Rothberg [**14**] for all associative rings. The integral domains for which this holds include all Prüfer domains.

**Theorem 7.3.** *Assume that projective modules over the integral domain $R$ are direct sums of finitely generated submodules. Then the union of a countable ascending chain* (2) *of projective $R$-modules $M_n$ subject to condition* (b) *is again projective if and only if condition* (*) *of Theorem* 6.3 *holds.*

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