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A REFINEMENT OF SHARPLY F-PURE AND STRONGLY F-REGULAR PAIRS

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ABSTRACT. We point out that the usual argument used to prove that R is strongly F-regular if and only if R_Q is strongly F-regular for every prime ideal $Q \in \operatorname{Spec} R$, does not generalize to the case of pairs $(R, \mathfrak{a}^t)$. The author's definition of sharp F-purity for pairs $(R, \mathfrak{a}^t)$ suffers from the same defect. We therefore propose different definitions of sharply F-pure and strongly F-regular pairs. Our new definitions agree with the old definitions in several common contexts, including the case that R is a local ring.

1. Introduction. The notion of a strongly F-regular ring was introduced by Hochster and Huneke in [**10**] because it was easily seen to be well behaved with respect to localization (this is in contrast to weak F-regularity). It later was discovered that strongly F-regular rings (in characteristic $p > 0$) were closely related to rings with Kawamata log terminal singularities (in characteristic 0), see [**4, 7, 17**]. However, the notion of Kawamata log terminal singularities extends to pairs $(R, \mathfrak{a}^t)$ where $\mathfrak{a} \subseteq R$ is an ideal and $t > 0$ is a real number. Therefore, it was natural to ask whether there is an analogous notion of strong F-regularity for pairs $(R, \mathfrak{a}^t)$.

In [**16**], Takagi gave such a definition and proved that it satisfied many properties similar to Kawamata log terminal singularities (also see [**7, 19**]). In fact, by using this characteristic $p > 0$ definition, Takagi was able to prove remarkable results in characteristic zero for which there are still no known characteristic zero proofs, see for example [**16**, Theorem 4.1]. We now state this definition:

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Definition 1.1. Suppose that R is an F-finite reduced ring in characteristic $p > 0$, $\mathfrak{a} \subseteq R$ is an ideal and $t > 0$ is a real number. Then we say that the pair $(R, \mathfrak{a}^t)$ is *strongly F-regular* if, for every $d \in R^{\circ}$, there exists an integer $e > 0$ and an element $a \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$ such that the inclusion $R \hookrightarrow R^{1/p^e}$, defined by $1 \mapsto (da)^{1/p^e}$ splits as a map of R-modules.

The author of this note also defined a notion, for pairs, which he called sharp F-purity $(R, \mathfrak{a}^t)$, see Definition 2.2. The reader should also compare with the notion of F-purity for pairs as defined in [**7, 16, 19**]. Roughly speaking, $(R, \mathfrak{a}^t)$ is sharply F-pure if it satisfies the condition used to define strongly F-regular pairs in the case that $d = 1$; see [**14**] for details.

Takagi's definition of strongly F-regular pairs and the author's definition of sharply F-pure pairs both work extremely well in the case that R is a local ring. Furthermore, strongly F-regular pairs have been studied largely in that context. However, there are certain ways in which both definitions are unsatisfactory in the case that R is a non-local ring. For example, the author expects that $(R, \mathfrak{a}^t)$ being strongly F-regular (respectively, sharply F-pure) is a different condition than the localized pair $(R_{\mathfrak{m}}, \mathfrak{a}^t_{\mathfrak{m}})$ being strongly F-regular (respectively sharply F-pure) for every maximal ideal $\mathfrak{m}$ of $\operatorname{Spec} R$. On the other hand, in the classical non-pair setting, R is strongly F-regular if and only if $R_{\mathfrak{m}}$ is strongly F-regular for every maximal ideal of $\operatorname{Spec} R$. Note that Hara and Watanabe's definition of strong F-regularity for a pair (R, Δ), see [**7**], does not suffer from this issue. However, I do not know if their definition of F-purity localizes well (although the issue is somewhat different than the one described above).

Therefore, the main purpose of this paper is to state a refined definition of strong F-regularity and sharp F-purity for pairs, which satisfies the above localization criterion. Our new definition for strong F-regularity is stated below.

Definition 1.2. Suppose that R is an F-finite reduced ring in characteristic $p > 0$, $\mathfrak{a} \subseteq R$ is an ideal and $t > 0$ is a real number. Then we say that the pair $(R, \mathfrak{a}^t)$ is *locally strongly F-regular* if, for every $d \in R^{\circ}$, there exists an $e > 0$, and a map

$$\phi \in \operatorname{Hom}_R(R^{1/p^e}, R) \cdot \left(d\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right)^{1/p^e}$$

such that $\phi(1) = 1$ (or equivalently, that ϕ is surjective).

The point is that the ϕ in Definition 1.2 might be equal to a sum

$$\phi(\_\_) = \sum \phi_i((da_i)^{1/p^e} \_\_)$$

for $\phi_i \in \operatorname{Hom}_R(R^{1/p^e}, R)$ and $a_i \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$, whereas in Definition 1.1, one would only consider sums with a single term. In particular, a "strongly F-regular" pair is "locally strongly F-regular." We also state a better version of sharp F-purity for pairs, see Definition 3.2. In fact, we state these definitions in greater generality: we state them for triples $(R, \Delta, \mathfrak{a}^t)$ where Δ is an effective $\mathbf{Q}$-divisor on $X = \operatorname{Spec} R$.

For a pair $(R, \mathfrak{a}^t)$, Definition 1.1 and Definition 1.2 are equivalent under any of the following conditions (likewise for sharply F-pure pairs):

(i) R is a local ring, or

(ii) R is an $\mathbf{N}$-graded ring, $\mathfrak{a}$ is a graded ideal and $\Delta = 0$ or

(iii) $\mathfrak{a}$ is a principal ideal, or

(iv) $\operatorname{Hom}_R(R^{1/p^e}, R)$ is a free R^{1/p^e}-module for some e greater than zero. (This occurs, for example, if R is sufficiently local and $\mathbf{Q}$-Gorenstein with index not divisible by p).

It follows from (i) that Definition 1.2 is equivalent to requiring that Definition 1.1 holds after localizing at every maximal ideal, see Corollary 5.4.

This note also corrects a minor misstatement in the author's paper, [**13**, Corollary 4.6], where the author assumed that strong F-regularity for pairs was characterized locally. See Remark 6.8 for details.

Throughout this paper, all rings will be assumed to be commutative with unity, Noetherian, and contain a field of characteristic $p > 0$. Furthermore, all rings will be assumed to be reduced and F-finite.

2. Why Definition 1.1 does not seem to localize well. We first begin by reminding the reader of why Hochster and Huneke's original definition localizes well. It is easy to see that if R is strongly F-regular, then R_Q is strongly F-regular for every $Q \in \operatorname{Spec} R$. This direction also holds for pairs $(R, \mathfrak{a}^t)$. Therefore, we will sketch the converse in the classical non-pair setting.

For any $d \in R^\circ$ and for each $e \geq 0$, consider the map

$$\Phi_{d,e} : \operatorname{Hom}_R(R^{1/p^e}, R) \longrightarrow R$$

which is the evaluation map at d^{1/p^e} (that is, $\phi \mapsto \phi(d^{1/p^e})$). It is easy to see that R is strongly F-regular if and only if for every d, $\Phi_{d,e}$ is surjective for some $e > 0$. Since R is F-finite, this is equivalent to requiring that $(\Phi_{d,e})_\mathfrak{m}$ is surjective after localization at each maximal ideal $\mathfrak{m} \in \operatorname{Spec} R$.

Therefore, the only question is whether we can find a common e so that the statement holds after localization at each maximal ideal $\mathfrak{m} \subset R$. To this end, observe that if $(\Phi_{d,e_0})_\mathfrak{m}$ is surjective, it is also surjective for all $e > e_0$ (since a strongly F-regular local ring is also F-pure).

Now, as e increases, the support of the modules $R/\operatorname{Image}(\Phi_{d,e})$ (which is also well behaved with respect to localization), is a decreasing set of closed subsets of $\operatorname{Spec} R$. On the other hand, each point of $\operatorname{Spec} R$ is not contained in $\operatorname{Supp}(R/\operatorname{Image}(\Phi_{d,e}))$ for e sufficiently large. Thus, we must have that $\operatorname{Supp}(R/\operatorname{Image}(\Phi_{d,e})) = \varnothing$ for $e \gg 0$ since R is Noetherian. This implies that $R = \operatorname{Image}(\Phi_{d,e})$ for $e \gg 0$.

Consider now a pair $(R, \mathfrak{a}^t)$. Let us try to argue in the same way we did for the original definition of a strongly F-regular ring, see Definition 1.1. In that case, we are restricting the map $\Phi_{d,e}$ to the set S of maps $\phi : R^{1/p^e} \to R$ that can be written in the form $\phi(\underline{\quad}) = \psi(a^{1/p^e}\underline{\quad})$ for some $\psi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ and some $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$. The problem is, the set S is not necessarily an R-submodule (or even a subgroup) of $\operatorname{Hom}_R(R^{1/p^e}, R)$. Thus, we cannot say that $\Phi_{d,e}|_S$ is surjective if and only if $\Phi_{d,e}|_S$ is surjective after localizing at every maximal ideal.

Remark 2.1. While the set of the maps ϕ of the form $\phi(\underline{\quad}) = \psi(a^{1/p^e}\underline{\quad})$ are not necessarily a R^{1/p^e}-submodule of $\operatorname{Ho}_R(R^{1/p^e}, R)$, they do generate the submodule $\mathrm{Hom}_R(R^{1/p^e}, R)\cdot(\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$. This will be useful later.

We also recall the author's original definition of sharply F-pure pairs.

Definition 2.2 [**14**]. Suppose that R is an F-finite reduced ring in characteristic $p > 0$, $\mathfrak{a} \subseteq R$ is an ideal and $t > 0$ is a real number. Then we say that the pair $(R, \mathfrak{a}^t)$ is *sharply F-pure* if there exists an $e > 0$ and an element $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$, such that the inclusion $R \hookrightarrow R^{1/p^e}$ which sends $1 \mapsto (da)^{1/p^e}$ splits as a map of R-modules.

It is easy to see that this definition suffers from the same defect that Definition 1.1 suffers from.

3. A "better" definition. Before we give our refined definition, we first fix some notation.

Definition 3.1. A triple $(R, \Delta, \mathfrak{a}^t)$ is the combined information of

(1) an F-finite reduced ring R,

(2) an ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^\circ \neq \varnothing$,

(3) a real number $t > 0$.

Furthermore, if R is a normal domain, we also consider

(4) Δ an effective $\mathbf{Q}$-divisor on $X = \operatorname{Spec} R$.

If R is not a normal domain, we assume $\Delta = 0$.

If $\mathfrak{a} = R$ (respectively, if $\Delta = 0$) then we call the triple $(R, \Delta, \mathfrak{a}^t)$ a pair and denote it by (R, Δ) (respectively, by $(R, \mathfrak{a}^t)$). Note that if R is strongly F-regular, then R is normal, so condition (4) is not so restrictive. On the other hand, little is lost in this paper if one always assumes that $\Delta = 0$.

Given an effective integral divisor D on $X = \operatorname{Spec} R$, we use the notation $R(D)$ to denote the global sections of the $\mathcal{O}_X$-module $\mathcal{O}_X(D)$. Also note that for any effective divisor D, there is a natural map

$R \to R(D)$. Therefore, we have natural maps

$$\pi_{\Delta,e} : \mathrm{Hom}_R((R(\lceil (p^e-1)\Delta\rceil))^{1/p^e}, R) \longrightarrow \mathrm{Hom}_R(R^{1/p^e}, R).$$

These maps are always injective. Of course, if $\Delta = 0$, then $\pi_{\Delta,e}$ is the identity.

The notation

$$\begin{aligned}&\mathrm{Image}\,(\pi_{\Delta,e})\cdot\left(J^{1/p^e}\right)\\ &\qquad = \mathrm{Image}\,\Big(\mathrm{Hom}_R((R(\lceil (p^e-1)\Delta\rceil))^{1/p^e}, R)\\ &\qquad\qquad\qquad \longrightarrow \mathrm{Hom}_R(R^{1/p^e}, R)\Big)\cdot\left(J^{1/p^e}\right)\end{aligned}$$

will be used to denote the R^{1/p^e}-submodule of $\mathrm{Hom}_R(R^{1/p^e}, R)$ obtained by multiplying the R^{1/p^e}-submodule

$$\mathrm{Image}\,(\pi_{\Delta,e}) \subseteq \mathrm{Hom}_R(R^{1/p^e}, R)$$

by the R^{1/p^e}-ideal J^{1/p^e}. It is important to note that the elements of this new submodule are still elements of $\mathrm{Hom}_R(R^{1/p^e}, R)$.

Definition 3.2. Suppose that $(R, \Delta, \mathfrak{a}^t)$ is a triple.

- We say that $(R, \Delta, \mathfrak{a}^t)$ is *locally strongly F-regular* if, for every $d \in R^\circ$, there exists an $e > 0$, and a map $\phi \in \mathrm{Image}\,(\pi_{\Delta,e})\cdot(d\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ where $\phi : R^{1/p^e} \to R$ is surjective.
- We say that $(R, \Delta, \mathfrak{a}^t)$ is *locally sharply F-pure* if there exists an $e > 0$, and a map $\phi \in \mathrm{Image}\,(\pi_{\Delta,e})\cdot(\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ where $\phi : R^{1/p^e} \to R$ is surjective.

We now state several equivalent definitions.

Lemma 3.3. *Suppose that $(R, \Delta, \mathfrak{a}^t)$ is a triple. Then the following are equivalent*:

(a) *The triple $(R, \Delta, \mathfrak{a}^t)$ is locally strongly F-regular.*

(b) *For every $d \in R^\circ$, there exists some $e > 0$ and some $\phi \in$ $\mathrm{Image}\,(\pi_{\Delta,e})\cdot(d\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ that splits the natural map $R \to R^{1/p^e}$.*

(c) *For every* $d \in R^{\circ}$, *there exists some* $e > 0$ *and some* $\phi \in \operatorname{Image}(\pi_{\Delta,e}) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ *such that* $\phi(d^{1/p^e}) = 1$.

(d) *The map,*

$$\operatorname{Image}(\pi_{\Delta,e}) \cdot \left(d\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right)^{1/p^e} \longrightarrow R,$$

which evaluates an element ϕ *at* 1*, is surjective for some* $e > 0$.

Furthermore, the following are also equivalent:

(a′) *The triple* $(R, \Delta, \mathfrak{a}^t)$ *is locally sharply* F*-pure.*

(b′) *For some* $e > 0$*, there exists a* $\phi \in \operatorname{Image}(\pi_{\Delta,e}) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ *that splits the natural map* $R \to R^{1/p^e}$.

(c′) *For some* $e > 0$*, there exists a* $\phi \in \operatorname{Image}(\pi_{\Delta,e}) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ *such that* $\phi(1) = 1$.

(d′) *The map,*

$$\operatorname{Image}(\pi_{\Delta,e}) \cdot \left(\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right)^{1/p^e} \longrightarrow R,$$

which evaluates an element ϕ *at* 1*, is surjective for some* $e > 0$.

Proof. Note first that condition (b) certainly implies condition (a). Conversely, if $\phi \in \operatorname{Image}(\pi_{\Delta,e}) \cdot (d\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ is surjective, then there exists an $x \in R$ such that $\phi(x^{1/p^e}) = 1$. But then the map $(\phi \cdot x^{1/p^e})$ sends 1 to 1 and so condition (b) is satisfied. We will leave the equivalence of (b), (c) and (d) to the reader as they are similarly straightforward. The equivalence of (a′) through (d′) is essentially the same. $\square$

In Section 5, we will prove that if R is local, then $(R, \Delta, \mathfrak{a}^t)$ is locally strongly F-regular (respectively, locally sharply F-pure) if and only if the localized triple $(R_Q, \Delta|_{\operatorname{Spec} R_Q}, \mathfrak{a}_Q^t)$ is strongly F-regular (respectively, sharply F-pure) for every $Q \in \operatorname{Spec} R$. This justifies the *"locally" strongly* F*-regular* terminology. However, the author feels that it would be better if the word "locally" was removed from future work (but that Definition 3.2 was still used). Regardless, in this note, because there are two definitions, we will use the word "locally" to distinguish the new version.

Remark 3.4. Definition 3.2 can easily be generalized by replacing $\mathfrak{a}^t$ with a graded system of ideals $\mathfrak{a}_\bullet$, see [**5, 13**]. We won't do this here however.

Question 3.5. *Is there an example of a pair* $(R, \mathfrak{a}^t)$ *that is locally strongly* F*-regular* (*respectively, locally sharply* F*-pure*) *but not strongly* F*-regular* (*respectively, sharply* F*-pure*)?

It seems that such an example may be difficult to construct (as there would be infinitely many conditions to check).

4. The "better" definition behaves well with respect to localization. In this section, we show that locally strongly F-regular (respectively, locally sharply F-pure) triples can be characterized locally. First however, we need a lemma.

Lemma 4.1. *Suppose that we have maps*:

$$\phi \in \text{Image } (\pi_{\Delta,e}) \cdot \left(\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right)^{1/p^e}, \ \psi \in \text{Image}, (\pi_{\Delta,d}) \cdot \left(\mathfrak{a}^{\lceil t(p^d-1)\rceil}\right)^{1/p^d}.$$

Then $\phi \circ (\psi^{1/p^e})$ *is contained in*

$$\text{Image } (\pi_{\Delta,e+d}) \cdot \left(\mathfrak{a}^{\lceil t(p^{e+d}-1)\rceil}\right)^{1/p^{d+e}}.$$

Proof. It is sufficient to check this for some ϕ of the form $\phi(\underline{\quad}) = \phi'(x^{1/p^e}\underline{\quad})$ where $\phi' \in \text{Image}\,(\pi_{\Delta,e})$ and $x \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$. This is because of two facts:

(1) Every element of $\text{Image}\,(\pi_{\Delta,e}) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ is a sum of elements of the form $\phi_i(\underline{\quad}) = \phi_i'(x_j^{1/p^e}\underline{\quad})$ for $\phi_i' \in \text{Image},(\pi_{\Delta,e})$ and $x_j \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$.

(2) A composition of a sum of maps with a map (i.e., $(\phi_1+\phi_2)\circ\psi^{1/p^e}$) is a sum of compositions of maps (i.e., $\phi_1 \circ \psi^{1/p^e} + \phi_2 \circ \psi^{1/p^e}$).

Likewise, we may assume that ψ is of the form $\psi(\underline{\quad}) = \psi'(y^{1/p^d}\underline{\quad})$ for some $\psi' \in \text{Image}\,(\pi_{\Delta,d})$ and some $y \in \mathfrak{a}^{\lceil t(p^d-1)\rceil}$.

Now, $\phi'(x^{1/p^e}(\psi'(y^{1/p^d}\_\_))^{1/p^e}) = \phi'(\psi'^{1/p^e}(x^{1/p^e}y^{1/p^{d+e}}\_\_))$. But we have that

$$\begin{aligned} x^{1/p^e}y^{1/p^{d+e}} &= (x^{p^d}y)^{1/p^{d+e}} \\ &\in \left(\mathfrak{a}^{p^d\lceil t(p^e-1)\rceil}\mathfrak{a}^{\lceil t(p^d-1)\rceil}\right)^{1/p^{d+e}} \\ &\subseteq \left(\mathfrak{a}^{\lceil t(p^{d+e}-1)\rceil}\right)^{1/p^{d+e}}. \end{aligned}$$

Therefore, it is sufficient to show that $\phi' \circ (\psi')^{1/p^e} \in \mathrm{Image}\,(\pi_{\Delta,d+e})$.

If $\Delta = 0$, we are done, so we may assume $\Delta \neq 0$ and that R is a normal domain. Therefore, it is sufficient to check this at height one primes of R since the modules $\mathrm{Image}\,(\pi_{\Delta,e+d})$ and $\mathrm{Hom}_R(R^{1/p^{e+d}}, R)$ are rank 1 reflexive $R^{1/p^{e+d}}$-modules. However, at a height one prime $Q \in \mathrm{Spec}\, R$, the pair $(R_Q, \Delta|_{\mathrm{Spec}\, R_Q})$ can be identified with a pair $(R_Q, (f)^{1/n})$ where $n\Delta$ is integral and f is a local defining equation for $n\Delta$ at Q. Then the argument follows as in the case above, also see [**17**, Proof of Lemma 2.5]. □

Remark 4.2. Suppose that $(R, \Delta, \mathfrak{a}^t)$ is locally sharply F-pure (respectively, locally strongly F-regular), due to the existence of some $\phi \in \mathrm{Image}\,(\pi_{\Delta,e}) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ with $1 \in \phi(R^{1/p^e})$ (respectively, with $1 \in \phi(d^{1/p^e}R^{1/p^e})$)). Lemma 4.1 implies that for every $n \geq 1$, we can find

$$\phi_n \in \mathrm{Image}\ (\pi_{\Delta,ne}) \cdot \left(\mathfrak{a}^{\lceil t(p^{ne}-1)\rceil}\right)^{1/p^{ne}}$$

with $1 \in \phi_n(R^{1/p^{ne}})$ (respectively, with $1 \in \phi_n(d^{1/p^{ne}}R^{1/p^{ne}})$).

Theorem 4.3. *A triple $(R, \Delta, \mathfrak{a}^t)$ is locally strongly F-regular (respectively locally sharply F-pure) if and only if $(R_Q, \Delta|_{\mathrm{Spec} R_Q}, \mathfrak{a}^t_Q)$ is locally strongly F-regular (respectively, locally sharply F-pure) for every ideal $Q \in \mathrm{Spec}\, R$.*

Proof. This may be obvious to experts, but because the original definition seems to lack this property, we prove it carefully here. First we note that the direction $(\Rightarrow)$ is straightforward and not substantially different from the classical non-pair setting. Thus we only prove the

($\Leftarrow$) direction. Suppose that for each $Q \in \operatorname{Spec} R$, $(R_Q, \Delta|_{\operatorname{Spec} R_Q}, \mathfrak{a}_Q^t)$ is locally strongly F-regular (respectively, locally sharply F-pure). Fix a $d \in R^\circ$ (or set $d = 1$, if one is checking the sharply F-pure case). By Lemma 3.3 (d), we see that for each $Q \in \operatorname{Spec} R$, there exists an $e_Q > 0$ so that the map which evaluates at 1,

$$E_{e_Q,Q} : \operatorname{Image}\left(\pi_{\Delta,e_Q}\right)_Q \cdot \left(d\mathfrak{a}^{\lceil t(p^{e_Q}-1)\rceil}\right)_Q^{1/p^{e_Q}} \longrightarrow R_Q,$$

is surjective. But then for each Q, this holds in an affine neighborhood U_Q of Q. We can cover $X = \operatorname{Spec} R$ by a finite collection of such neighborhoods $U_1, \ldots, U_n$ with corresponding surjective evaluation maps E_{e_i,U_i}. Of course, the particular e_i's associated to each neighborhood may vary. However, Lemma 4.1 implies that if E_{e_i,U_i} is surjective, then so is E_{ne_i,U_i} for every $n > 0$. Thus, increasing the e_i if necessary, we can find a common e that works on all U_i. But then we are done, since a map of R-modules is surjective if and only if the corresponding maps on a finite affine cover of $\operatorname{Spec} R$ are surjective. $\square$

5. Cases where the two definitions agree. In this section, we prove that the two definitions agree in the cases (i) through (iv) mentioned in the introduction. We first do conditions (iii) and (iv).

Proposition 5.1. *Suppose that* $(R, \Delta, \mathfrak{a}^t)$ *is a triple. Further suppose that either*:

(a) $\mathfrak{a}$ *is a principal ideal, or*

(b) $\operatorname{Image}(\pi_{\Delta,e})$ *is a free* R^{1/p^e}*-module for some* $e > 0$.

Then $(R, \Delta, \mathfrak{a}^t)$ *is strongly* F*-regular* (*respectively, sharply* F*-pure*) *if and only if it is locally strongly* F*-regular* (*respectively, locally sharply* F*-pure*).

Proof. First assume we are in case (b). We claim that if $\operatorname{Image}(\pi_{\Delta,e})$ is cyclic as an R^{1/p^e}-module for some $e > 0$, then $\operatorname{Image}(\pi_{\Delta,ne})$ is also cyclic as an $R^{1/p^{ne}}$-module for all $n > 0$. In the case that $\Delta = 0$, this is essentially an exercise in applying the adjointness of $\otimes$ and Hom, see [**15**, Lemma 3.9]. In the case that $\Delta \neq 0$, R is normal and so one can reduce to the one dimensional case and argue in essentially the same way, see [**15**, Corollary 3.10]. Therefore, Remark 4.2 allows us to assume that we can find $e > 0$ so that condition (b) holds and, for that same e, we may find a surjective map in $\phi \in \mathrm{Image}\,(\pi_{\Delta,e}) \cdot (d\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$.

Now, in either case (a) or (b), every element of

$$\mathrm{Image}\ (\pi_{\Delta,e}) \cdot \left(d\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right)^{1/p^e}$$

can be written as a map of the form $\phi((da)^{1/p^e}\_\_)$ for some $\phi \in \mathrm{Image}\,(\pi_{\Delta,e})$ and some $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$. In the sharply F-pure case, set $d = 1$. The result then follows. $\square$

We now note that the two definitions are the same in the case that R is local.

Proposition 5.2. *Suppose that $(R, \Delta, \mathfrak{a}^t)$ is a triple. Further suppose that $(R, \mathfrak{m})$ is local. Then $(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular (respectively, sharply F-pure) if and only if it is locally strongly F-regular (respectively, locally sharply F-pure).*

Proof. Note that elements of the form $\phi((da)^{1/p^e}\_\_)$, for some $\phi \in \mathrm{Image}\,(\pi_{\Delta,e})$ and some $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$, generate $\mathrm{Image}\,(\pi_{\Delta,e}) \cdot (d\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ even as an R-module. Therefore, if all these elements are sent, by evaluation at 1, into the maximal ideal $\mathfrak{m}$, then so are any of their linear combinations. For the proof in the sharply F-pure case, set $d = 1$. $\square$

Finally, we verify the graded case at least assuming that $\Delta = 0$. One can do similar things when $\Delta \neq 0$ but the statements become more complicated.

Proposition 5.3. *Suppose that $(R, \mathfrak{a}^t)$ is a pair. Further suppose that $R = \oplus_{i\geq 0} R_i$ is an $\mathbf{N}$-graded ring and $\mathfrak{a}$ is a graded ideal. Then $(R, \mathfrak{a}^t)$ is strongly F-regular (respectively, sharply F-pure) if and only if it is locally strongly F-regular (respectively, locally sharply F-pure).*

Proof. Suppose first that R is locally strongly F-regular. It is sufficient to show that $(R, \mathfrak{a}^t)$ is strongly F-regular in the usual sense (the case of sharply F-pure rings is similar). We view R^{1/p^e} as a $\mathbf{Z}[1/p^e]$-graded R-module. Using standard techniques related to strong F-regularity, it is not difficult to see that it is sufficient to verify the statements of Lemma 3.3 for homogenous $d \in R^\circ$. Note that $\mathrm{Hom}_R(R^{1/p^e}, R)$ is generated by graded (degree-shifting) homomorphisms since R is F-finite. In particular, the image of the natural map

$$\text{Image } (\pi_{\Delta,e}) \cdot \left(d\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right)^{1/p^e} \longrightarrow R,$$

which evaluates an element ϕ at 1, is a graded submodule of R. Therefore, the map is surjective if and only if the image is not contained in R_+. One then argues exactly the same as in the local case. $\square$

One should note that most of the work done with the previous definition of strongly F-regular pairs was in the local setting, see for example [**12, 16, 18, 19**].

Corollary 5.4. *A triple $(R, \Delta, \mathfrak{a}^t)$ is locally strongly F-regular (respectively, locally sharply F-pure) if and only if $(R_\mathfrak{m}, \Delta|_{\operatorname{Spec} R_\mathfrak{m}}, \mathfrak{a}^t_\mathfrak{m})$ is strongly F-regular (respectively, sharply F-pure) for every maximal $\mathfrak{m} \in \operatorname{Spec} R$.*

We now recall the definition of the test ideal.

Definition 5.5 [**9, 13, 17**]. Let $X = \operatorname{Spec} R$ be an F-finite normal integral affine scheme. Further suppose that Δ is an effective $\mathbf{Q}$-divisor on X, $\mathfrak{a} \neq (0)$ is an ideal of R and $t \geq 0$ is a real number. We define the *big test ideal* $\tau_b(R; \Delta, \mathfrak{a}^t)$ of the triple $(R, \Delta, \mathfrak{a}^t)$ to be the unique smallest non-zero ideal J of R such that

$$\phi\left((\mathfrak{a}^{\lceil t(p^e-1)\rceil} J)^{1/p^e}\right) \subseteq J \tag{5.5.1}$$

for all $e \geq 0$ and all

$$\phi \in \text{Image}\left(\mathrm{Hom}_R\left((R(\lceil (p^e-1)\Delta\rceil))^{1/p^e}, R\right) \longrightarrow \mathrm{Hom}_R(R^{1/p^e}, R)\right).$$

This ideal always exists in the context described.

The big test ideal is often called the non-finitistic test ideal and is often denoted by $\widetilde{\tau}(R;\Delta,\mathfrak{a}^t)$.

Remark 5.6. Assume that $0 \neq c \in \tau_b(R;\Delta,\mathfrak{a}^t)$ (in other words, c is a *big sharp test element*). Then

$$\tau_b(R;\Delta,\mathfrak{a}^t) = \sum_{e\geq 0}\sum_{\phi} \phi((c\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}),$$

where the inner sum is over

$$\phi \in \text{Image}\Big(\text{Hom}_R\left((R(\lceil (p^e-1)\Delta\rceil))^{1/p^e}, R\right) \longrightarrow \text{Hom}_R(R^{1/p^e}, R)\Big).$$

It is clear that the sum on the right satisfies the condition from equation 5.5.1. It is also easy to see that the sum on the right is non-zero (consider the case where $e=0$). Thus the containment $\subseteq$ is clear. But the containment $\supseteq$ is also easy since $cR \subseteq \tau_b(R;\Delta,\mathfrak{a}^t)$ and again using equation 5.5.1. Thus the statement is proven.

For more discussion on the big test ideal in this context, see [**2**, Section 3] or [**13**, Subsection 2.2] and compare with [**8, 11, 17**].

Corollary 5.7. *The big test ideal $\tau_b(R;\Delta,\mathfrak{a}^t)$ is equal to R if and only if the triple $(R,\Delta,\mathfrak{a}^t)$ is locally strongly F-regular.*

Proof. Since the formation of the big test ideal commutes with localization, see [**2**, Section 3] and [**6**, Lemma 2.1], it is sufficient to prove Corollary 5.7 at each maximal ideal. Therefore, we can reduce to the case of a local ring $(R,\mathfrak{m})$. Then the proof is exactly the same as in [**17**, Lemma 2.3], or [**3**, Proposition 2.1]. We sketch another approach where we avoid Matlis duality and instead use a version of [**2**, Lemma 2.1] generalized to triples $(R,\Delta,\mathfrak{a}^t)$.

Now, still assuming $(R,\mathfrak{m})$ is local, $R = \tau_b(R;\Delta,\mathfrak{a}^t)$ if and only if for each $d \in R^\circ$ we have

$$1 \in \sum_{e\geq 0}\left(\sum_{\phi_e \in \text{Image}\,(\pi_{\Delta,e})} \phi_e\left((d\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}\right)\right),$$

see Remark 5.6, [**2**, Lemma 2.1], [**18**, Lemma 3.5] and [**14**, Lemma 2.20]. But then the statement is obvious since 1 is in the sum if and only if there are terms in the sum not contained in $\mathfrak{m}$. □

6. F-pure thresholds, test ideals, and uniformly F-compatible ideals. In this section we discuss how these new notions of strong F-regularity and sharp F-purity fit into the existing theory. We also correct a small error of the current author, in the paper [**13**].

6.1. The F-pure threshold. Recall that the *F-pure threshold* of a pair $(R, \mathfrak{a})$, where R is a reduced F-finite F-pure (not necessarily local) ring is defined to be

$$\begin{aligned} c(\mathfrak{a}) &= \sup\{s \in \mathbf{R}_{\geq 0} \mid \text{the pair } (R, \mathfrak{a}^s) \text{ is } F\text{-pure}\} \\ &= \sup\{s \in \mathbf{R}_{\geq 0} \mid \text{the pair } (R, \mathfrak{a}^s) \text{ is sharply } F\text{-pure}\}, \end{aligned}$$

see [**19**, Definition 2.1] and [**14**, Proposition 5.3]. This definition was stated originally for non-local rings, but we expect that a better definition would require that $(R, \mathfrak{a}^s)$ is locally sharply F-pure. Note, most results about the F-pure threshold were shown in the case that R is local. However, there is the following notable exception:

Remark 6.1. The rationality result for the F-pure threshold found in [**1**, Theorem 3.1] or [**2**] (at least when R is strongly F-regular), is a rationality result for the *locally-F-pure threshold.* Note that in [**1**, Theorem 3.1], it is assumed that R is regular, but it is not assumed that $\operatorname{Hom}_R(R^{1/p^e}, R)$ is free as an R^{1/p^e}-module (though it is locally free).

6.2. The test ideal of a locally sharply F-pure pair. We turn our attention again to test ideals. One nice fact about sharply F-pure pairs is that the associated generalized test ideal (of [**8**]) is a radical ideal. We now show directly that this also holds for the a-priori weaker condition of local sharp F-purity.

Proposition 6.2 [**14**, Corollary 3.15]. *If $(R, \mathfrak{a}^t)$ is locally sharply F-pure, then the test ideal $\tau(R, \mathfrak{a}^t)$ (as defined in [**8**]) is a radical ideal.*

Proof. The proof is identical to the proof of [**14**, Corollary 3.15] once one has Lemma 6.4, which we prove below. □

First we recall the following definition.

Definition 6.3. Given an ideal $I \subseteq R$, we define the $\mathfrak{a}^t$*-sharp Frobenius closure of* I, denoted $I^{F^\sharp \mathfrak{a}^t}$ as follows. The ideal $I^{F^\sharp \mathfrak{a}^t}$ is defined to be the set of elements $z \in R$ such that $\mathfrak{a}^{\lceil t(p^e-1)\rceil} z^{p^e} \subseteq I^{[p^e]}$ for all $e \gg 0$.

Lemma 6.4 [**14**, Remark 3.11]. *If* $(R, \mathfrak{a}^t)$ *is locally sharply* F*-pure, then* $I^{F^\sharp \mathfrak{a}^t} = I$ *for all ideals* I.

Proof. Suppose that $z \in I^{F^\sharp \mathfrak{a}^t}$. Thus there exists an $e_0 > 0$ such that $\mathfrak{a}^{\lceil t(p^e-1)\rceil} z^{p^e} \subseteq I^{[p^e]}$ for all $e \geq e_0$. Since $(R, \mathfrak{a}^t)$ is locally sharply F-pure, there exists a $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$, for some $e \geq e_0$, such that $\phi(1) = 1$ (we can increase e due to Lemma 4.1). We then write $\phi = \phi_1 \cdot a_1^{1/p^e} + \cdots + \phi_m \cdot a_m^{1/p^e}$ for $\phi_i \in \mathrm{Hom}_R(R^{1/p^e}, R)$ and $a_i \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$. Then for that same $e \geq e_0$,

$$
\begin{aligned}
z &= \phi((z^{p^e})^{1/p^e}) \\
&= \sum_{i=1}^{m} \phi_i \left((a_i z^{p^e})^{1/p^e} \right) \\
&\subseteq \sum_{i=1}^{m} \phi_i \left(\left(\mathfrak{a}^{\lceil t(p^e-1)\rceil} z^{p^e} \right)^{1/p^e} \right) \\
&\subseteq \sum_{i=1}^{m} \phi_i \left(\left(I^{[p^e]} \right)^{1/p^e} \right) \\
&\subseteq I. \qquad \square
\end{aligned}
$$

6.3. Uniformly F-compatible ideals and centers of F-purity. We begin by recalling the definition of a uniformly F-compatible ideal.

Definition 6.5. Suppose that $(R, \Delta, \mathfrak{a}^t)$ is a triple. Recall that an ideal $J \subseteq R$ is called *uniformly* $(\Delta, \mathfrak{a}^t, F)$*-compatible* if for all

$\phi \in \mathrm{Image}\,(\pi_{\Delta,e}) \subseteq \mathrm{Hom}_R(R^{1/p^e}, R)$ and all $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$, we have that

$$\phi((aJ)^{1/p^e}) \subseteq J.$$

A prime uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible ideal is called a *center of F-purity for* $(R, \Delta, \mathfrak{a}^t)$.

Remark 6.6. In [**14**], the author actually dealt with triples of the form $(R, \Delta, \mathfrak{a}_\bullet)$ where $\mathfrak{a}_\bullet$ is a graded system of ideals (that is, $\mathfrak{a}_i \cdot \mathfrak{a}_j \subseteq \mathfrak{a}_{i+j}$). For simplicity, we won't work with graded systems of ideals, although none of the results are more difficult in that generality.

Remark 6.7. An ideal J is uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible if and only if for all

$$\phi \in \mathrm{Image}\ (\pi_{\Delta,e}) \cdot \left(\mathfrak{a}^{\lceil t(p^e-1)\rceil}\right)^{1/p^e},$$

we have that

$$\phi(J^{1/p^e}) \subseteq J.$$

This can be seen because maps of the form $\phi \cdot a^{1/p^e}$ for $\phi \in \mathrm{Image}\,(\pi_{\Delta,e})$ and $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$ generate $\mathrm{Image}\,(\pi_{\Delta,e}) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ even as an R-module. In other words, the definition of uniformly F-compatible ideals is the same under the paradigm of "*local* strong F-regularity/*local* sharp F-purity" as it is under the paradigm of "strong F-regularity/sharp F-purity."

Remark 6.8. We now point out a misstatement in the current author's paper [**14**, Corollary 4.6]. In that paper, the author claimed that in the case of an F-finite normal domain R, a triple $(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular if and only if $(R, \Delta, \mathfrak{a}^t)$ has no proper nontrivial centers of F-purity. To prove this, the author showed (correctly) that a minimal prime of the non-strongly F-regular locus was a center of F-purity.

The non-strongly F-regular locus still makes sense using the old definition of a strongly F-regular pair or triple. It is the set of primes $Q \in \mathrm{Spec}\, R$ such that $(R_Q, \Delta|_{\mathrm{Spec} R_Q}, \mathfrak{a}_Q^t)$ is not strongly F-regular. This is a closed set because if $(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular at $Q \in \mathrm{Spec}\, R$, then it is also strongly F-regular (and locally strongly F-regular) in a neighborhood of Q.

Unfortunately, the non-strongly F-regular locus being empty is equivalent to a triple $(R, \Delta, \mathfrak{a}^t)$ being *locally* strongly F-regular (which we expect is a different condition than being strongly F-regular). Therefore, a correct statement, in the case of an F-finite normal domain, would be one of the following.

(1) A triple $(R, \Delta, \mathfrak{a}^t)$ has empty non-strongly F-regular locus if and only if $(R, \Delta, \mathfrak{a}^t)$ has no proper non-trivial centers of F-purity.

(2) A triple $(R, \Delta, \mathfrak{a}^t)$ is locally strongly F-regular if and only if $(R, \Delta, \mathfrak{a}^t)$ has no proper non-trivial centers of F-purity.

(3) The big test ideal $\tau_b(R; \Delta, \mathfrak{a}^t) = R$ if and only if $(R, \Delta, \mathfrak{a}^t)$ has no proper non-trivial centers of F-purity.

On the other hand, the only place where [**14**, Corollary 4.6] was applied in the paper [**14**], was in a case where $\Delta = 0$ and $\mathfrak{a} = R$, see [**14**, Corollary 7.8].

Perhaps more importantly, all the results of [**14**] for sharply F-pure triples extend to "locally sharply F-pure triples." For the most part, these generalizations can be accomplished by reducing to the local case where the two notions of strong F-regularity (respectively, sharp F-purity) agree. However, the most fundamental such result is the following and we prove it directly:

Proposition 6.9 [**14**, Corollary 3.3]. *If $(R, \Delta, \mathfrak{a}^t)$ is locally sharply F-pure and I is uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible, then R/I is also F-pure.*

Proof. Choose $\phi \in \operatorname{Image}(\pi_{\Delta,e}) \cdot (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ such that $\phi(1) = 1$. Consider the diagram:

$$\begin{tikzcd} I^{1/p^e} \arrow[r, "\phi|_{I^{1/p^e}}"] \arrow[d, hook] & I \arrow[d, hook] \\ R^{1/p^e} \arrow[r, "\phi"] \arrow[d, two heads] & R \arrow[d, two heads] \\ (R/I)^{1/p^e} \arrow[r, "\overline{\phi}"] & R/I. \end{tikzcd}$$

Since $\phi(1) = 1$, we also see that $\overline{\phi}(1) = 1$. This completes the proof. ◻

6.4. Links with Kawamata log terminal singularities. We finally note that the new definition of strongly F-regular, provided in this paper, is the notion that corresponds to Kawamata log terminal singularities when R is not necessarily local.

Proposition 6.10 [**4**], [**17**, Theorem 3.2], [**8**, Theorem 6.8], [**18**, Theorem 2.5]. *Let $(R, \Delta, \mathfrak{a}^t)$ be a triple that was reduced to characteristic $p \gg 0$ from a characteristic zero Kawamata log terminal triple (reduced with a log resolution, etc). Then $(R, \Delta, \mathfrak{a}^t)$ is locally strongly F-regular.*

Proof. Note that this implies that $K_{R_0} + \Delta_0$ was $\mathbf{Q}$-Cartier in characteristic zero. This statement has typically been stated in the case that R is a local ring (for example, it follows from [**18**, Theorem 2.5], also see [**17**, Theorem 3.2] and [**8**]). A priori, if you change the local ring, you might need to also change the particular characteristic $p \gg 0$ you are working in. However, the key injectivity needed to prove these results, holds at every local ring of a ring reduced to characteristic $p \gg 0$, see [**4**, subsections 4.4, 4.5]. In particular, if $(R, \Delta, \mathfrak{a}^t)$ is as stated above, then after localizing at each prime $Q \in \operatorname{Spec} R$, $(R, \Delta|_{\operatorname{Spec} R_Q}, \mathfrak{a}_Q^t)$ is strongly F-regular. Proposition 6.10 follows. $\square$

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