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THE ELIAHOU-KERVAIRE RESOLUTION IS CELLULAR

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ABSTRACT. We construct a regular cell complex which supports the Eliahou-Kervaire resolution of a stable ideal.

1. Introduction. A central object in the study of a homogeneous ideal $I \subseteq S = k[x_1, \dots, x_n]$ is the minimal free resolution, which encodes much information about the homological and combinatorial structure of the ideal. While algorithms to compute minimal free resolutions are known, the problem of describing them explicitly has proven intractable, even for monomial ideals. Thus, there has been a lot of work in recent decades describing the minimal free resolutions of well-behaved classes of monomial ideals.

One of the most important results in this vein is the Eliahou-Kervaire resolution [**12**], which elegantly describes the minimal resolution of a *stable* ideal in terms of its monomial generators. The stable ideals are a large class of monomial ideals containing (in characteristic zero) the *Borel-fixed* ideals. These occur as generic initial ideals of arbitrary ideals [**3, 13**], and so arise in many contexts.

Another approach has been to study non-minimal free resolutions. These reveal slightly less information than do minimal free resolutions, but are often much easier to describe. For example, the Taylor resolution [**21**] is a very clean (but usually highly non-minimal) resolution for any monomial ideal.

One of the most exciting recent developments in the study of resolutions has been the idea of *simplicial resolutions* [**2**], resolutions which can be described completely in terms of a simplicial complex. The Taylor resolution is simplicial, as are the minimal resolutions of "generic" monomial ideals. This idea was extended by Bayer and Sturmfels [**7**] to regular cell complexes, and later by Jöllenbeck and Welker [**15**] to

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CW complexes. We say that a resolution is *cellular* (respectively *simplicial, CW*) if it can be encoded by a regular cell complex (respectively, a simplicial or CW complex). Velasco [**22**] uses this theory to construct families of monomial ideals whose minimal resolutions are characteristic-dependent, as well as monomial ideals whose minimal resolutions cannot be described by any CW complex. Batzies and Welker [**1**], using discrete Morse theory, show how to construct (not necessarily minimal) CW resolutions inside the Taylor resolution of any monomial ideal. There are techniques for using cellular resolutions to construct resolutions of new ideals; for example, Sinefakopoulos [**19**] builds the minimal resolutions of certain p-Borel-fixed ideals from a polytopal resolution of a power of the maximal ideal.

There are few examples of interesting resolutions which are cellular but not simplicial. Sinefakopoulos constructs in [**20**] a cellular complex supporting the minimal resolution of any Borel ideal generated in a single degree. Corso and Nagel [**9, 10**] describe cellular resolutions of Ferrers ideals of many graphs, including bipartite graphs and stable ideals generated by quadrics. Nagel and Reiner [**17**] extend this construction to a larger class of ideals, including Borel ideals generated in one degree. Even these examples are polytopal complexes (i.e., they can be embedded in $\mathbf{R}^n$ so that each cell is a polytope), however, so it was unclear that the full generality of the cellular case in [**7**] was necessary.

In Theorem 5.3 we show that the Eliahou-Kervaire resolution of any stable ideal is cellular. This is not a duplication of Sinefakopoulos's or of Nagel and Reiner's work, even in the case of a Borel ideal generated in one degree: their complexes have very different combinatorial structure (see Figures 4 and 5), and describe a different basis for the resolution than that given by Eliahou and Kervaire. As noted by the referee, it would be interesting to have a better understanding of the relationship between the constructions of [**17**] and [**10**]. For example, these complexes coincide for the square of the maximal ideal in four variables. The complex constructed in this paper is not polytopal in any obvious way. In the example above, the construction does seem to be closely connected to a CW complex described by Batzies and Welker [**4**], but it is not clear if the complex their construction produces is regular. Simultaneously with this project, Clark [**8**] has used the theory of poset resolutions ([**7**]) to prove the existence of a cell complex supporting the Eliahou-Kervaire resolution. It is possible to trace a construction through Clark's proof, and the resulting complex, while not embedded in $\mathbf{R}^n$ in any natural way, seems to be combinatorially equivalent to the complex given in Construction 4.1. This work and [**8**] approach the problem from very different directions: [**8**] begins with the resolution and builds a complex out of it, whereas we essentially build the complex from scratch and show that it supports the resolution, but the topological intuition appears to be the same.

Section 2 gives a quick overview of cellular resolutions and the Eliahou-Kervaire resolution. In Section 3, we recall a well-known cell complex that supports the Eliahou-Kervaire resolution of a power of the maximal ideal in $k[x_1, x_2, x_3]$, and in Sections 4 and 5 we generalize this construction to arbitrary dimension.

2. Background and notation. Let $S = k[x_1, \dots, x_n]$ be the polynomial ring in n variables. We impose a grading and multigrading on S by setting $\deg x_i = 1$ and $\operatorname{mdeg} x_i = x_i$.

A *monomial* of S is an element of the form $m = x_1^{a_1} \cdots x_n^{a_n}$. The *exponent vector* of m is $\mathbf{a} = (a_1, \dots, a_n)$. For convenience, we will frequently write $m = \mathbf{x}^{\mathbf{a}}$. The monomial $\mathbf{x}^{\mathbf{a}}$ has degree $|\mathbf{a}| = \sum a_i$ and multidegree $\mathbf{x}^{\mathbf{a}}$. By abuse of notation, we will routinely identify monomials (and multidegrees) with their exponent vectors.

A *monomial ideal* is an ideal M which is generated by monomials. All ideals appearing in this paper will be monomial ideals. Every monomial ideal has a unique minimal generating set of monomials $\operatorname{gens}(M)$; we call the elements of this set the generators of M.

Definition 2.1. For a monomial $m = \mathbf{x}^{\mathbf{a}}$, we set $\max(m) = \max\{i : a_i \neq 0\}$, the largest index with a positive exponent in m. Since the monomial 1 is the empty product, we set $\max(1) = 0$. The variable $x_{\max(m)}$ is thus the minimal variable dividing m in any of the natural term orders; we will attempt to avoid this source of confusion by discussing monomial orders as little as possible.

Definition 2.2. We say that a monomial ideal M is *stable* if it satisfies the condition:

Let $m \in M$ be a monomial, and suppose $i < \max(m)$. Then $m(x_i/x_{\max(m)}) \in M$ as well.

Stable ideals were introduced by Eliahou and Kervaire [**12**] as a class of ideals minimally resolved by the Eliahou-Kervaire resolution. The class of stable ideals includes Borel ideals, which occur as generic initial ideals in characteristic zero [**3, 13**].

Proposition 2.3. *Let M be a stable ideal and $m \in M$ a monomial. Then there exists a unique generator g and monomial h such that $m = gh$ and, for every x_i dividing h, we have $i \geq \max(g)$.*

Definition 2.4. Let M, m, g and h be as in Proposition 2.3. Then g and h are called the *beginning* and *end* of m, respectively, and we write $\operatorname{beg}(m) = g$ and $\operatorname{end}(m) = h$.

Proof of Proposition 2.3. If m is a generator of M, set $g = m$ and $h = 1$. Otherwise, set $m' = m/x_{\max(m)}$. Since m is not a generator, there exists some x_i dividing m such that $m/x_i \in M$. Since M is stable, it follows that $m' = (m/x_i)(x_i/x_{\max(m)}) \in M$ as well. By induction on the degree of m, we may write m' uniquely in the form $m' = g'h'$; set $g = g'$ and $h = h'x_{\max(m)}$. The uniqueness of this decomposition is immediate since $x_{\max(m)}$ must divide h. □

A *free resolution* of an ideal M is an exact sequence

$$\mathbf{F} : \cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

with each F_i a free S-module. When M is a monomial ideal, $\mathbf{F}$ inherits a natural multigraded structure if we require that the maps $\phi_i : F_i \to F_{i-1}$ preserve multidegree. The resolution F is *minimal* if each F_i has minimum possible rank, or, equivalently, if every entry in the matrices associated to the maps ϕ_i is contained in the homogeneous maximal ideal.

Stable ideals are minimally resolved by the Eliahou-Kervaire resolution [**12**], defined below.

Definition 2.5. Let M be a monomial ideal. An *EK-symbol* for M is a pair of the form $[f, \alpha]$, where $f \in \operatorname{gens}(M)$ is a minimal generator of

M and α is a squarefree monomial satisfying $\max(\alpha) \lneqq \max(m)$. The EK-symbol $[f, \alpha]$ has multidegree $f\alpha$ and homological degree $\deg(\alpha)$.

Definition 2.6. If α is a squarefree monomial andx_i divides α, put $\operatorname{sgn}(x_i, \alpha) = 1$ if the cardinality of the set $\{x_j : x_j \text{ divides } \alpha \text{ and } j \leq i\}$ is odd, and $\operatorname{sgn}(x_i, \alpha) = -1$ if it is even.

For an EK-symbol $[f, \alpha]$, the differential is given by

$$d(f, \alpha) = \sum_{x_i \text{ divides } \alpha} \operatorname{sgn}(x_i, \alpha) x_i \left[f, \frac{\alpha}{x_i}\right] - \sum_{x_i \text{ divides } \alpha} \operatorname{sgn}(x_i, \alpha) \operatorname{end}(x_i f) \left[\operatorname{beg}(x_i f), \frac{\alpha}{x_i}\right],$$

where we treat a pair $[f', \alpha']$ as zero if it is not an EK-symbol (i.e., if $\max(\alpha') \geq \max(f')$).

The proof that the Eliahou-Kervaire resolution is in fact a minimal resolution usually uses mapping cones. A nice treatment is given by Peeva and Stillman in [**18**].

Definition 2.7. For the formal definition of a regular cell complex, see [**6**, Chapter 6.2] or [**5**, Chapter 4.7]. For our purposes, a regular cell complex Δ is a finite collection of closed d-balls Δ_d (called d-cells) for every dimension d, such that the boundary of each d-cell is a union of $(d-1)$-cells. There is an *orientation* or *incidence function* $\varepsilon : \Delta \times \Delta \to \{-1, 0, 1\}$ which satisfies:

- $\varepsilon(F, G) = 0$ unless $F \in W_d$ and $G \in \Delta_{d-1}$ for some d.
- For all F and H, $\sum_G \varepsilon(F, G)\varepsilon(G, H) = 0$.

$\varepsilon(F, G)$ indicates whether G appears with positive or negative orientation in the boundary of F.

We say that a cell complex Δ is *simplicial* if each cell is a simplex, and *polytopal* if it can be embedded into some $\mathbf{R}^n$ in such a way that each cell is a polytope.

Intuitively, we say that a resolution $\mathbf{F}$ is supported on a cell complex Δ if the vertices of Δ can be labeled with monomials in a way that allows us to read off the maps of F from the incidence function ε. We formalize this as follows.

Definition 2.8. Let Δ be a regular cell complex and $\mathbf{F}$ be a resolution such that each free module F_d has a basis $\{f_G\}$ indexed by the d-cells of Δ. We say that $\mathbf{F}$ is *supported on* Δ if it is possible to label the cells of Δ with monomials such that:

- Each cell is labeled by the least common multiple of its vertices,

$$\text{label}\,(G) = \text{lcm}_{v \in G}(\text{label}\,(v)).$$

- The differential maps of $\mathbf{F}$ are given by

$$\phi(f_F) = \sum_G \varepsilon(F, G) \frac{\text{label}\,(F)}{\text{label}\,(G)}.$$

Implicit in this definition is the requirement that the number of d-dimensional cells in Δ be equal to the rank of the free module F_d.

Example 2.9. In this example, we show that the Taylor resolution (which non-minimally resolves every monomial ideal) is supported on a simplicial complex. For a monomial ideal $M = (g_0, \dots, g_m)$, the module F_s is the free module with basis consisting of the formal symbols $[g_{i_0}, \dots, g_{i_s}]$ with $0 \leq i_0 < i_1 < \dots < i_s \leq m$. The symbol $[g_{i_0}, \dots, g_{i_s}]$ has multidegree lcm $(g_{i_0}, \dots, g_{i_s})$ and differential

$$\phi_s([g_{i_0}, \dots, g_{i_s}]) = \sum_{j=0}^{s} (-1)^j \frac{\text{lcm}\,(g_{i_0}, \dots, g_{i_s})}{\text{lcm}\,(g_{i_0}, \dots, \widehat{g_{i_j}}, \dots, g_{i_s})} [g_{i_0}, \dots, \widehat{g_{i_j}}, \dots, g_{i_s}].$$

For example, if $M = (x^2, xy, y^3)$, the Taylor resolution of M is given by

$$0 \longrightarrow S[x^2, xy, y^3] \xrightarrow{\begin{pmatrix} x \\ 1 \\ y^2 \end{pmatrix}} \begin{matrix} S[xy, y^3] \\ \oplus \\ S[x^2, y^3] \\ \oplus \\ S[x^2, xy] \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -y^3 & -y \\ -y^2 & 0 & x \\ x & x^2 & 0 \end{pmatrix}} \begin{matrix} S[x^2] \\ \oplus \\ S[xy] \\ \oplus \\ S[y^3] \end{matrix} \xrightarrow{(x^2 \ xy \ y^3)} M \longrightarrow 0.$$

Now let Δ be the simplex on m vertices labeled $g_0, \ldots, g_m$, and label each face of Δ by label$(F) = \operatorname{lcm}\{g_i : g_i \in F\}$. The d-faces of F are indexed by ordered tuples $[g_{i_0}, \ldots, g_{i_d}]$ with $0 \le i_0 < \cdots < i_d \le m$, and the simplicial boundary maps are given by

$$d([g_{i_0}, \ldots, g_{i_s}]) = \sum_{j=0}^{s} (-1)^s [g_{i_0}, \ldots, \widehat{g_{i_j}}, \ldots, g_{i,s}],$$

which differ from the Taylor boundary maps only by the absence of the monomials, which can be recovered as label$([g_{i_0}, \ldots, g_{i_s}])$/label$([g_{i_0}, \ldots, \widehat{g_{i_j}}, \ldots, g_{i_s}])$. Thus, we say that the resolution of M is supported on the (simplicial) complex Δ.

In the example $M = (x^2, xy, y^3)$, the labeled simplex is as in Figure 1.

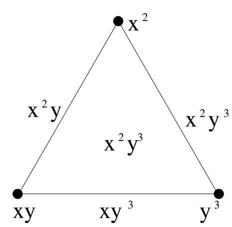

FIGURE 1. The Taylor resolution of (x^2, xy, y^3).

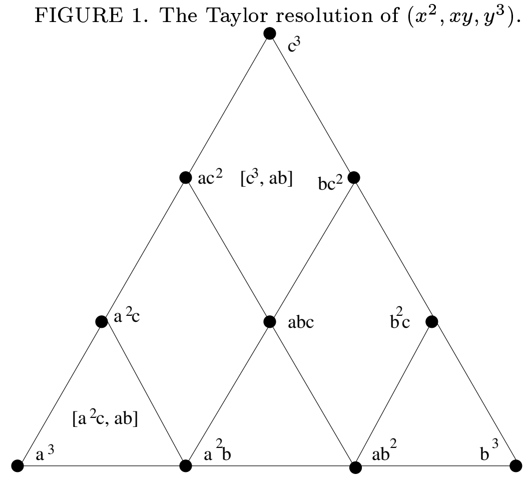

FIGURE 2. The Eliahou-Kervaire resolution of $(a, b, c)^3$.

3. Powers of the maximal ideal in three variables. In this section, we recall the well-known cell complex which supports the resolution of a power of the maximal ideal of $R = k[a, b, c]$. I am not sure where, if anywhere, the picture in Figure 2 has been published; I first saw it in a class taught by Irena Peeva in 2005.

We construct the complex supporting the resolution of $(a, b, c)^d$ as follows: First, we intersect the first orthant of $\mathbf{R}^3$ with the hyperplane $x_1 + x_2 + x_3 = d$, and take the lattice points as vertices. We label the vertices in the natural way (so that $(d, 0, 0)$ is labeled by a^d, etc.), and draw edges as follows:

For every vertex $m = nc$ divisible by c, add oriented edges pointing from m to nb and na (these edges will have labels nbc and nac, and correspond to the EK symbols $[nc, b]$ and $[nc, a]$, respectively), and for every vertex $m = nb$ divisible by b but not by c, add an oriented edge pointing from m to na (this will be labeled nab and correspond to the EK-symbol $[nb, a]$).

The faces consist of squares with vertices nc^2, nbc, nab, nac for every monomial n of degree $d-2$ (corresponding to the EK-symbol $[nc^2, ab]$), and triangles with vertices $a^r b^s c, a^r b^{s+1}, a^{r+1} b^s$ for $r + s = d - 1$ (corresponding to the EK-symbol $[a^r b^s c; ab]$); we orient them clockwise.

It is straightforward to verify that the complex constructed above supports the Eliahou-Kervaire resolution; it is much less obvious how it can be generalized for more variables. Our strategy is to break the cells down as simplicial complexes.

We observe the following:

Remark.

- Each of the rectangular cells in Figure 2 has unique top and bottom vertices. These are its last and first vertices, respectively, in the lex order.
- The edges at the boundary of each rectangular cell describe two oriented paths of length two from the top vertex to the bottom vertex.
- If we define the top and bottom vertices of a triangular cell to be its lexicographically least and greatest vertices, then the edges again trace out two oriented paths from the top to the bottom. One of these paths has length one; we will see later that this path is degenerate.

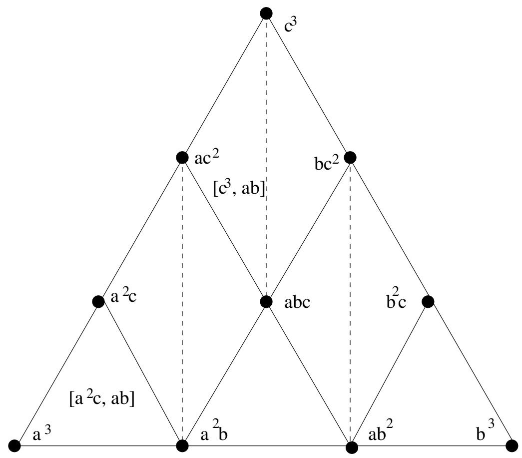

FIGURE 3. Decomposing the cells in Figure 2.

• If the edge $[m, x_i]$ points from m to n, then $n = m(x_i/x_{\max(m)})$. We have $x_i = b$ if the edge points from left to right, and $x_i = a$ if it points from right to left. Also, $x_{\max(m)} = c$ if the edge points down, and $x_{\max(m)} = b$ if it is horizontal.

Example 3.1. The cell named $[c^3, ab]$ has top vertex c^3 and bottom vertex abc. There are two paths from c^3 to abc, namely, (c^3, bc^2, abc) and (c^3, ac^2, abc). The cell named $[a^2c, ab]$ has top vertex a^2c and bottom vertex a^3. The two paths from a^2c to a^3 are (a^2c, a^2b, a^3) and (a^2c, a^3). The second path is a subset of the first.

Each of the (maximal) paths described above has three vertices; these vertices define a triangle. These triangles are bounded by the dotted lines in Figure 3. Note that each of the rectangular faces is the union of the triangles defined by its two paths, and each of the triangular faces is the triangle defined by its path.

When there are more than three variables, we will generalize this observation, defining the faces of the Eliahou-Kervaire resolution as unions of simplices.

Recall that the set of monomials m in a stable ideal is closed under multiplication by $x_i/x_{\max(m)}$ whenever $i < \max(m)$. This inspires the following notation.

Notation 3.2. Let m be any monomial, and let x_i be any variable.

(i) Set

$$m \to x_i = \begin{cases} m\frac{x_i}{x_{\max(m)}} & \text{if } i < \max(m) \\ m & \text{if } i \geq \max(m). \end{cases}$$

(ii) Let n be another monomial. Then we inductively define $m \to x_i n = (m \to x_i) \to n$.

In order for the second notation above to be well-defined, we need the following lemma.

Lemma 3.3. *Let m be any monomial, and let x_i and x_j be variables. Then $(m \to x_i) \to x_j = (m \to x_j) \to x_i$.*

Proof. We may assume that $i \leq j$, and we can write $m = nx_kx_\ell$, with $\max(n) \leq k \leq \ell$. There are then six cases to check, depending on the ordering of i, j, k and ℓ. For example, if $k \leq i \leq j \leq \ell$, then $(m \to x_i) \to x_j = (m \to x_j) \to x_i = nx_kx_i$. The other cases are unenlightening, and are left as an exercise. □

4. Powers of the maximal ideal. Throughout this section, fix positive integers n and d. Denote by I the ideal $(x_1, \dots, x_n)^d$, and by Δ the simplex in $\mathbf{R}^n$ obtained by intersecting the first orthant with the degree d hyperplane $z_1 + \cdots + z_n = d$. We will construct a regular cellular subdivision of Δ which supports the Eliahou-Kervaire resolution of I.

As in the previous section, we identify lattice points in the first orthant of $\mathbf{R}^n$ with monomials via the exponent vector. Thus, for example, the monomial $x_1^2x_2x_4$ is identified with the vector $(2, 1, 0, 1)$. By abuse of notation, we will treat monomials and vectors as interchangeable. (Thus, for a vector v, $\max(v)$ is the index of its last nonzero entry, and $v \to x_i$ is defined as for the corresponding monomial.)

Construction 4.1. Let m be any monomial of degree d, let $\alpha = x_{i_1} \dots x_{i_j}$ be a squarefree monomial with $\max(\alpha) < \max(m)$, and let $\sigma = (\sigma_1, \dots, \sigma_j)$ be any permutation of $(i_1, \dots, i_j)$. We denote by $\operatorname{ch}(m, \alpha, \sigma)$ the convex hull of the points $\{m, m \to x_{\sigma_1}, m \to x_{\sigma_1} x_{\sigma_2}, \dots, m \to \alpha\}$. We say that $\operatorname{ch}(m, \alpha, \sigma)$ is *nondegenerate* if it has dimension j.

We set the cell $U(m, \alpha)$ equal to the union over all σ of the $\operatorname{ch}(m, \alpha, \sigma)$.

A few observations are immediate.

Lemma 4.2. *Let m, α, σ be given. Then:*

(i) $\operatorname{ch}(m, \alpha, \sigma) \subset \Delta$.

(ii) $\operatorname{ch}(m, \alpha, \sigma)$ *is a simplex.*

(iii) $\operatorname{ch}(m, \alpha, \sigma)$ *is nondegenerate if and only if the $j+1$ monomials $m, m \to x_{\sigma_1}, \dots, m \to \alpha$ are distinct.*

(iv) *If σ is the unique decreasing permutation (i.e., $\sigma_1 > \dots > \sigma_j$), then* $\operatorname{ch}(m, \alpha, \sigma)$ *is nondegenerate.*

(v) *If* $\operatorname{ch}(m, \alpha, \sigma)$ *is degenerate, then there exists a permutation σ' such that* $\operatorname{ch}(m, \alpha, \sigma')$ *is nondegenerate and* $\operatorname{ch}(m, \alpha, \sigma)$ *is a face of* $\operatorname{ch}(m, \alpha, \sigma')$.

Thus, in particular, we can view $U(m, \alpha)$ as the union of the nondegenerate $\operatorname{ch}(m, \alpha, \sigma)$.

Proof. We prove (v). Suppose that $\operatorname{ch}(m, \alpha, \sigma)$ is degenerate. Then for some k, we have $m \to (x_{\sigma_1} \dots x_{\sigma_k}) = m \to (x_{\sigma_1} \dots x_{\sigma_{k+1}})$; choose the minimal such k. It follows that $\sigma_{k+1} > \sigma_k$. Let $\overline{\sigma} = (\sigma_1, \dots, \sigma_{k+1}, \sigma_k, \dots, \sigma_j)$ be the permutation obtained from σ by swapping the k^{th} and $(k+1)^{\text{th}}$ terms. It is immediate that $\operatorname{ch}(m, \alpha, \sigma)$ is a (not necessarily proper) face of $\operatorname{ch}(m, \alpha, \overline{\sigma})$. By induction, it is a face of some $\operatorname{ch}(m, \alpha, \sigma')$. □

Example 4.3. We return to the cells in Example 3.1. The top cell is $U(c^3, ab)$; it is divided into two triangles: the left triangle is ch $(c^3, ab, (a, b))$, and the right triangle is ch $(c^3, ab, (b, a))$. The bottom left cell is $U(a^2c, ab)$. It consists of a single triangle, which is ch $(a^2c, ab, (b, a))$. The degenerate ch $(a^2c, ab, (a, b))$ is the left edge. The other cells decompose similarly.

Now we study the geometry of the ch (m, α, σ).

Lemma 4.4. *Suppose that deg$(\alpha) = n - 1$ and* ch (m, α, σ) *is nondegenerate. Then the vertices of* ch (m, α, σ) *form a basis for* $\mathbf{R}^n$.

The next technical lemma is obvious after unwrapping a lot of notation.

Lemma 4.5. *Suppose that* $\deg(\alpha) = n - 1$ *and* ch (m, α, σ) *is nondegenerate. Let* $v_o = (v_{0,1}, v_{0,2}, \dots, v_{0,n})$ *be the exponent vector of* m, $v_1 = (v_{1,1}, v_{1,2}, \dots, v_{1,n})$ *the exponent vector of* $m \to x_{\sigma_1}$, *etc. Let* $k = \max(m \to \alpha)$. *Then*:

(i) *For all* i, *we have* $v_i - v_{i-1} = e_{\sigma_i} - e_{\max(m \to x_{\sigma_1} \cdots x_{\sigma_{i-1}})}$ *(where the* e_i *are the usual standard basis vectors).*

(ii) *If* $q < k$, *then* $v_{n-1,q} \geq v_{j,q}$ *for all* j.

(iii) *If* $q > k$, *then* $0 = v_{n-1,q} \leq v_{j,q}$ *for all* j.

(iv) *If* $\sigma_{n-1} = k$, *then* $v_{n-1,k} \geq v_{j,k}$ *for all* j.

(v) *If* $\sigma_{n-1} \neq k$, *then* $v_{n-1,k} \leq v_{j,k}$ *for all* k.

(vi) *If* q *is such that* $v_{n-1,q} \geq v_{j,q}$ *for all* j, *and* $\sigma_\ell = q$, *then* $v_{j,q} = v_{n-1,q}$ *for* $j \geq \ell$ *and* $v_{j,q} = v_{n-1,q} - 1$ *for* $j < \ell$.

The following lemma will allow us to recover ch (m, α, σ) given a point in its interior.

Lemma 4.6. *Let* $z = (z_1, \dots, z_n) \in$ ch (m, α, σ). *Observe that* $\sum z_i = d$. *Denote by* $\lceil z_i \rceil$ *the least integer greater than or equal to* z_i. *Using the same notation as in Lemma 4.5, we can recover* v_{n-1} *as follows.*

Step 1: *Set* $a = 0$, *and start with* $i = 1$.

Step 2: *Set* $v_{n-1,i} = \min(d-a, \lceil z_i \rceil)$.

Step 3: *Add* $v_{n-1,i}$ *to* a*, increment* i*, and return to Step* 2.

The coefficient c_{n-1} *is given by* $\min_{z_i < v_{n-1,i}} \{\mathrm{frac}\,(z_i)\}$. (*Here,* $\mathrm{frac}\,(z_i)$ *represents the fractional part of* z_i.)

Example 4.7. Suppose that $z = (.3, .45, .05, 1.15, .05)$. We compute $d = 2$ and set the counter a equal to 0. We have $\lceil .3 \rceil = 1 < 2 - 0$, so $v_{4,1} = 1$. We increase a to 1. We have $\lceil .45 \rceil = 1 = 2 - 1$, so $v_{4,2} = 1$. We increase a to 2. Now $2 - 2 = 0 < \lceil .05 \rceil$, so $v_{4,3} = 0$ and a is unchanged. Similarly, $v_{4,4} = v_{4,5} = 0$.

Finally, $c_4 = \min(.3, .45) = .3$. (In fact, $z = .05(0,0,0,1,1) + 0.5(0,0,0,2,0) + 0.1(0,0,1,1,0) + 0.05(0,1,1,0,0) + 0.3(1,1,0,0,0) \in \mathrm{ch}\,(x_4x_5, x_1x_2x_3x_4, (4,2,3,1))$.)

Proof. For each i, one of the following holds:

(i) $i < \max(v_{n-1})$, in which case, by Lemma 4.5 (ii), (vi), we have $v_{n-1,i} = \lceil z_i \rceil$ (and $d - a > v_{n-1,i}$ by induction on i).

(ii) $i = \max(v_{n-1})$, in which case we have $v_{n-1,i} = d - a$ by induction on i (and $v_{n-1,i} \le \lceil z_i \rceil$ by Lemma 4.5 (iv), (v), (vi)).

(iii) $i > \max(v_{n-1})$, in which case we have $v_{n-1,i} = 0 = d - a$ by induction on i.

By Lemma 4.5 (vi), whenever j is such that $v_{n-1,j} > z_j$, we have $\mathrm{frac}\,(z_j) = \sum_{q \ge \ell} c_q$, where ℓ is such that $\sigma_\ell = j$. Since all the c_q are positive, this is minimized for $j = \sigma_{n-1}$. □

Lemma 4.8. *Let* $z = (z_0, \dots, z_n)$ *be any vector in* Δ*. Then* z *may be written uniquely in the form* $z = \sum c_i v_i$ *for positive coefficients* c_i *such that* $\sum c_i = 1$ *and such that there exists some* $\mathrm{ch}\,(m, \alpha, \sigma)$ *having the* v_i *among its vertices.*

Note the requirement that the coefficients be nonzero; this means that the expansion may have fewer than n terms, and, as such, the choice of $\mathrm{ch}\,(m, \alpha, \sigma)$ may be nonunique. However, if z lies on the interior of any $\mathrm{ch}\,(m, \alpha, \sigma)$, the expansion must contain all n vertices and so is unique.

Proof. Lemma 4.6 tells us v_{n-1}, c_{n-1} and σ_{n-1}. Set $z' = z + c_{n-1}/(1-c_{n-1})(z - v_{n-1})$, so that $z = c_{n-1}v_{n-1} + (1 - c_{n-1})z'$. Let $y \in \mathbf{R}^{n-1}$ be the vector obtained by removing the $(\sigma_{n-1})^{\text{th}}$ entry from z'. By induction on n, y may be written in the form $\sum d_i v'_i$; reinserting the removed entry to each v'_i gives us the expression $z = c_{n-1}v_{n-1} + (1 - c_{n-1})(\sum d_i v_i)$. It remains to show that $v_{n-1} = v_{n-2} \to x_{\sigma_{n-1}}$.

Set $k = \sigma_{n-1}$, and write $z' = (z'_1, \dots, z'_n)$. We compute $z'_k = v_{n-1,k} - 1$, and $\lceil z'_j \rceil = v_{n-1,j}$ for all $j \neq k$ such that $z_j < v_{n-1,j}$. Let ℓ be minimal such that $z_\ell > v_{n-1,\ell}$. Then the algorithm in Lemma 4.6, applied to y, gives us $v_{n-2,\ell} = v_{n-1,\ell} + 1$ and $\ell = \max(v_{n-2})$. Thus, $v_{n-1} - v_{n-2} = e_k - e_\ell$, so, by Lemma 4.5 (i), we have $v_{n-1} = v_{n-2} \to \sigma_{n-1}$ as desired. □

We have proved the following:

Proposition 4.9. *The union of all the* $\operatorname{ch}(m, \alpha, \sigma)$ *is the simplex* Δ*. The intersection of two simplices* $\operatorname{ch}(m, \alpha, \sigma)$ *and* $\operatorname{ch}(m', \alpha', \sigma')$ *is a common face.*

Notation 4.10. Set $\boldsymbol{\alpha} = x_1 \cdots x_{n-1}$. For a collection of points $v_1, \dots, v_s$, let $\langle v_1, \dots, v_s \rangle$ represent their convex hull.

Definition 4.11. Let F be any facet of $\operatorname{ch}(m, \boldsymbol{\alpha}, \sigma)$. We say that F is *interior* if it also a facet of some $\operatorname{ch}(m', \boldsymbol{\alpha}, \sigma') \neq \operatorname{ch}(m, \boldsymbol{\alpha}, \sigma)$. Otherwise, we say that F is *exterior*.

Lemma 4.12. *Let* F *be a facet of some* $\operatorname{ch}(m, \boldsymbol{\alpha}, \sigma)$*, and write* $F = \langle v_1, \dots, v_{n-1} \rangle$*, with the* v_i *increasing in the lex order. For each* $i \geq 2$*, if* $v_i = v_{i-1} \to x_j$ *for some* j*, set* $\tau_i = j$*. If no such* j *exists, set* $\tau_i = 0$*.*

Then all the τ_i *are distinct, and exactly one of the following holds:*

(i) *There exists a unique* i *such that* $\tau_i = 0$*. We have* $v_{i+1} = v_i \to x_j x_k$ *where* $1 \leq j, k \leq n-1$ *are the two indices not occurring as any* τ_j*.* F *is interior.*

(ii) *None of the τ_i are equal to zero. Let $j < n$ be the unique index not occurring as any τ_i. F is exterior if $v_{1,j} = 0$ or if $v_{1,n} = 0$, and interior otherwise.*

Proof. Let m and σ be such that F is a facet of $\mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma)$, and write $\mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma) = \langle w_0, \dots, w_{n-1} \rangle$. Let w_ℓ be the vertex which is missing from F; we have $v_i = w_{i-1}$ for $i \leq \ell$ and $v_i = w_i$ for $i > \ell$. Also, $\tau_i = \sigma_{i-1}$ for $i \leq \ell$ and $\tau_i = \sigma_i$ for $i \geq \ell + 2$. Finally (if $\ell \neq 0, d-1$), we have $w_{\ell+1} = w_{\ell-1} \to x_{\sigma_\ell} x_{\sigma_{\ell+1}}$. If this is not equal to $w_{\ell-1} \to x_{\sigma_{\ell+1}}$ we are in case (i); otherwise, we are in case (ii).

In case (i), let σ' be the permutation obtained from σ by swapping σ_ℓ and $\sigma_{\ell+1}$. Then F is a facet of both $\mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma)$ and $\mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma')$.

In case (ii), return to the notation in the statement of the lemma. Suppose first that $v_{1,n} = 0$. It follows that $v_i \to x_{n-1} = v_i$ for all i and that $j = n-1$. Set $v_0 = v_1 + e_n - e_{n-1}$, $\sigma_1 = n-1$, and $\sigma_i = \tau_i$ for all $i \geq 2$. Then F is a facet only of $\mathrm{ch}\,(v_0, \boldsymbol{\alpha}, \sigma)$ and is exterior. Otherwise, let k be minimal such that $v_k \to x_j \neq v_k$. Let $\sigma = (\tau_1, \dots, \tau_k, j, \tau_{k+1}, \dots, \tau_{n-2})$, $\sigma' = (j, \tau_1, \dots, \tau_{n-2})$, and $v_0 = v_1 - e_j + e_n$. We have that F is a facet of $\mathrm{ch}\,(v_1, \boldsymbol{\alpha}, \sigma)$, and is a facet of $\mathrm{ch}\,(v_0, \boldsymbol{\alpha}, \sigma')$ provided that v_0 has nonnegative entries (i.e., $v_{1,j} \neq 0$). Thus, F is interior if $v_{1,j} \neq 0$ and exterior otherwise. □

Now we are in position to describe the orientations of the $\mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma)$. Since the $\mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma)$ form a subdivision of the big simplex Δ, there is a unique orientation function inherited from Δ. This assigns an orientation of $+1$ to the simplex $\mathrm{ch}\,(x_1^{d-1} x_n, \boldsymbol{\alpha}, (x_{n-1}, x_{n-2}, \dots, x_1))$, and satisfies:

($*$) Let G be an interior facet common to $F = \mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma)$ and $F' = \mathrm{ch}\,(m', \boldsymbol{\alpha}, \sigma')$. If G occurs with opposite signs in the simplicial boundaries of F and F', then F and F' have the same orientation. If G occurs with the same sign in both boundaries, then F and F' have opposite orientations.

The condition ($*$) is because $d(\Delta) = d(\sum_{F=\mathrm{ch}\,(m,\boldsymbol{\alpha},\sigma)} o(F)F)$ is supported on the boundary of Δ, i.e., on the exterior facets.

Because any two simplices $F = \mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma)$ and $F' = \mathrm{ch}\,(m', \boldsymbol{\alpha}, \sigma')$ are connected by a chain $F = F_0, \dots, F_r = F'$ such that F_i and F_{i+1}

share a facet, there is a unique solution to $(*)$. If we use the simplicial boundary $d(\langle v_0, \dots , v_n\rangle) = \langle v_0, \dots , v_{n-1}\rangle - \langle v_0, \dots , v_{n-2}, v_n\rangle + \cdots + (-1)^{n-i}\langle v_1, \dots , \widehat{v_i}, \dots , v_n\rangle + \cdots + (-1)^n\langle v_1, \dots , v_n\rangle$, it is straightforward to verify that the solution is as follows:

Proposition 4.13. *Let $F = \mathrm{ch}\,(m, \boldsymbol{\alpha}, \sigma)$. Then F has positive orientation if σ differs by an even permutation from the decreasing permutation $\boldsymbol{\sigma} = (n-1, \dots , 1)$, and negative orientation otherwise.*

Remark. The simplicial boundary map chosen above seems unorthodox. However, if we reorder the vertices in the lexicographic order, as is standard practice, and write $\langle v_0, \dots , v_n\rangle = \langle w_0, \dots , w_n\rangle$ with $w_0 = v_n, w_1 = v_{n-1}$, etc., we get $d(\langle w_0, \dots , w_n\rangle) = \langle w_1, \dots , w_n\rangle - \cdots + (-1)^i\langle w_0, \dots , \widehat{w_i}, \dots , w_n\rangle + \cdots + (-1)^n\langle w_0, \dots , w_{n-1}\rangle$, which is the usual simplicial differential.

Note that the orientation depends only on σ and not on m. We extend this observation to orient the lower-dimensional cells.

Notation 4.14. Let σ be a permutation of some subset $T \subset \{1, \dots , n-1\}$. We say that σ is *positive* if σ is an even permutation of the decreasing permutation on T, and that σ is *negative* otherwise. If $F = \mathrm{ch}\,(m, \alpha, \sigma)$ and σ is positive, we say that F has *positive orientation* and write $o(F) = o(\sigma) = 1$. If σ is negative, we say that F has *negative orientation* and write $o(F) = o(\sigma) = -1$.

Remark. If we view the Taylor resolution as being generated by symbols of the form $[g_1, \dots , g_s]$ (where $[g_1, \dots , g_s] = [h_1, \dots , h_s]$ if the s-tuples differ by an even permutation, and $[g_1, \dots , g_s] = -[h_1, \dots , h_s]$ if they disagree by an odd permutation), then $\mathrm{ch}\,(m, \alpha, \sigma)$ is the Taylor symbol $[m \to \sigma, \dots , m \to x_{\sigma_1}, m]$. We will see that the Eliahou-Kervaire resolution sits nicely inside the Taylor resolution.

Our next goal is to show that the topological differentials of the cells $U(m, \alpha)$ agree with the differentials in the Eliahou-Kervaire resolution.

Fix m and α; we will compute the topological differential of $U(m, \alpha)$.

(Essentially, we are analyzing the interior and exterior facets of the simplicial fan $U(m, \alpha)$ as we did with Δ above. The analysis is almost the same, so we omit many proofs.)

$U(m, \alpha)$ is oriented as

$$U(m, \alpha) = \sum_{\substack{\sigma: F=\mathrm{ch}\,(m,\alpha,\sigma) \\ \text{nondegenerate}}} o(F)F.$$

Set $p = \deg(\alpha)$. For ease of notation, we may assume that $\alpha = x_1 \cdots x_p$.

Choose a nondegenerate $F = \mathrm{ch}\,(m, \alpha, \sigma)$ and write $F = \langle m, m \to x_{\sigma_1}, \ldots, m \to \alpha \rangle$. The differential of F contributes three types of terms to the differential of U.

(1) Removing the first vertex gives the face $(-1)^p(-1)^{p-\sigma_1}\langle m \to x_{\sigma_1}, \ldots, m \to \alpha\rangle = (-1)^{\sigma_1}\mathrm{ch}\,(m \to x_{\sigma_1}, (\alpha/x_{\sigma_1}), \sigma')$, where $\sigma' = (\sigma_2, \ldots, \sigma_p)$. This face cannot arise from the differential of any other F.

(2) Removing the last vertex gives the face $\langle m, \ldots, m \to (\alpha/\sigma_p)\rangle = (-1)^{1+\sigma_p}\mathrm{ch}\,(m, (\alpha/\sigma_p), \sigma')$, where $\sigma' = (\sigma_1, \ldots, \sigma_{p-1})$. This face cannot arise from the differential of any other F.

(3) If we remove the i^{th} vertex, we are left with the face $(-1)^{p-i}\langle m, \ldots, m \to \widehat{x_{\sigma_1} \cdots x_{\sigma_i}}, \ldots, m \to \alpha\rangle$. This is canceled out by removing the i^{th} vertex from $F' = \mathrm{ch}\,(m, \alpha, \overline{\sigma})$ (where $\overline{\sigma}$ is obtained from σ by swapping the i^{th} and $(i+1)^{\text{th}}$ entries), unless $m \to x_{\sigma_1} \cdots x_{\sigma_{i-1}} x_{\sigma_{i+1}} = m \to x_{\sigma_1} \cdots x_{\sigma_{i+1}}$, in which case F' is degenerate and the face is $(-1)^{p-i}(-1)^{1+\sigma_i}\mathrm{ch}\,(m, (\alpha/x_{\sigma_i}), \sigma')$, where $\sigma' = (\sigma_1, \ldots, \widehat{\sigma}_i, \ldots, \sigma_{d-1})$.

Taking the sum $\sum d(o(F)F)$ over all F, (and omitting some tedious work), we are left with

(1) The sum of all $(-1)^i o(\sigma')\mathrm{ch}\,(m \to x_i, (\alpha/x_i), \sigma')$ (taken over all x_i dividing α, and all σ' such that the resulting simplex is nondegenerate).

(2) The sum of all $(-1)^{1+i} o(\sigma')\mathrm{ch}\,(m, (\alpha/x_i), \sigma')$ (taken over all x_i dividing α, such that $m \to (\alpha/x_i) \neq m \to \alpha$, and all σ' such that the resulting simplex is nondegenerate).

(3) The sum of all $(-1)^{1+i} o(\sigma')\mathrm{ch}\,(m, (\alpha/x_i), \sigma')$ (taken over all x_i dividing α, such that $m \to (\alpha/x_i) = m \to \alpha$, and all σ' such that the resulting simplex is nondegenerate).

The sum in (1) is simply $-\sum_i (-1)^{i+1} U(m \to x_i, (\alpha/x_i))$, and the sum of (2) and (3) is $\sum_i (-1)^i U(m, (\alpha/x_i))$.

Thus, $d(U(m,\alpha)) = \sum_i (-1)^i U(m, (\alpha/x_i)) - \sum_i (-1)^i U(m \to x_i, (\alpha/x_i))$.

This agrees (up to monomial coefficients) with the Eliahou-Kervaire differential, $\phi([m,\alpha]) = \sum_i (-1)^i x_i [m, (\alpha/x_i)] - \sum_i (-1)^i \operatorname{end}(m x_i)[m \to x_i, (\alpha/x_i)]$. Thus, we have proved the following:

Proposition 4.15. *The complex described in Construction* 4.1 *supports the Eliahou-Kervaire resolution.*

It remains to show that this is a regular cellular complex, i.e., that the cells $U(m,\alpha)$ are topological balls.

Lemma 4.16. *Let m and α be given, and suppose that F is an interior facet of $U(m,\alpha)$. Then F is a facet of at most two nondegenerate* $\operatorname{ch}(m,\alpha,\sigma)$.

Proof. This statement is actually immediate from the embedding in Δ, which is homeomorphic to $\mathbf{R}^{n-1}$. However, we will need it again in the next section, where we will not have the luxury of any ambient space, so we give a more involved proof here.

Without loss of generality, we may suppose $\alpha = x_1 \cdots x_p$. Following the notation of Lemma 4.12, we write $F = \langle v_1, \dots, v_{p-1} \rangle$, with the v_i increasing in the lex order, and, for each $i \geq 2$, if $v_i = v_{i-1} \to x_j$ for some j, we set $\tau_i = j$. If no such j exists, we set $\tau_i = 0$.

Suppose first that $\tau_i = 0$ for some i. Then it must be the case that $v_i = v_{i-1} \to x_k x_\ell$, where $k, \ell < p$ are the two indices not appearing in τ. If $\sigma = (\tau_2, \dots, \tau_{i-1}, k, \ell, \tau_i, \dots, \tau_{p-1})$ or $(\tau_2, \dots, \tau_{i-1}, \ell, k, \tau_i, \dots, \tau_{p-1})$, then it is clear that F is a face of $\operatorname{ch}(m,\alpha,\sigma)$. On the other hand, if $\sigma_j = k$ with $j < i-1$, then $\operatorname{ch}(m,\alpha,\sigma)$ does not contain v_{i-1}, and if $\sigma_j = k$ with $k > i$, then $\operatorname{ch}(m,\alpha,\sigma)$ does not contain v_i.

Now suppose instead that no $\tau_i = 0$; let k be the missing index from τ. If v_1 is the exponent vector of m, and let j be the maximal index such that $v_j \to x_k \neq v_j$. If $\sigma = (\tau_2, \dots, \tau_i, k, \tau_{i+1}, \dots, \tau_{p-1})$, then F is not a face of ch (m, α, σ) if $i < j$ (since this simplex does not contain v_j) or if $i > j$ (since this simplex is degenerate). If v_1 is not the exponent vector of m, then since ch (m, α, σ) containing F must contain m, we must have $\sigma = (k, \tau_2, \dots, \tau_p)$. □

Construction 4.17. Fix m and α. Let P be the set of all monomials that can be written in the form $m \to \beta$ for some β dividing α. Partially order the set P by $(m \to \beta) \leq_P (m \to \gamma)$ whenever β divides γ.

Observe that the maximal chains in P are in correspondence with the simplices ch (m, α, σ). (The simplex ch (m, α, σ) corresponds to the chain $m <_P m \to \sigma_1 <_P \cdots <_P m \to \alpha$.) Thus, $U(m, \alpha)$ is the order complex of P. We can label the Hasse diagram of P by labeling the edge from $m \to \beta$ to $m \to x_i\beta$ with x_i. This is an EL-labeling (see for example [**23**], so, applying [**23**, Theorem 3.2.2], we have:

Lemma 4.18. *$U(m, \alpha)$ is a shellable simplicial complex.*

Proposition 4.19. *$U(m, \alpha)$ is a ball.*

Proof. We have observed that $U(m, \alpha)$ is a pure p-dimensional shellable simplicial complex, and that each of its $(p-1)$-faces is contained in at most two p-faces. Thus $U(m, \alpha)$ satisfies the hypotheses of [**11**, Proposition 1.2], and so is a p-ball as desired. □

Putting everything together, we have shown the following:

Theorem 4.20. *The cells $U(m, \alpha)$ form a cellular subdivision of the $(n-1)$-simplex which supports the Eliahou-Kervaire resolution of any power of the maximal ideal of $k[x_1, \dots, x_n]$.*

The resolution of $(a, b, c, d)^2$ is pictured in Figure 4. It is isomorphic to the complex constructed by Batzies and Welker [**1**] using discrete Morse theory. I have been unable to determine whether these complexes continue to coincide with more than four variables. Even if they are the

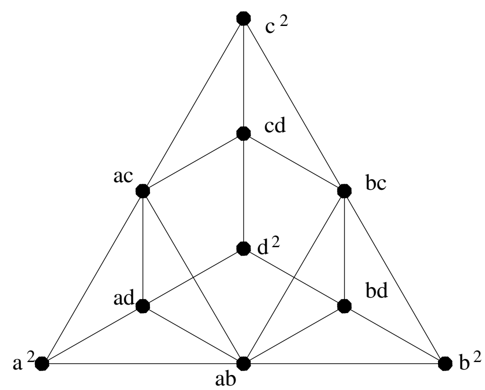

FIGURE 4. The Eliahou-Kervaire resolution of $(a, b, c, d)^2$.

same, the constructions are very different. Where we have constructed the cells explicitly, Batzies and Welker were demonstrating a special case of a more general construction, building down from the Taylor resolution. Batzies and Welker show that their complex is CW, but make no attempt to prove or disprove that the cells are regular.

Our construction is not polytopal. For example, the cell $U(cd, ab)$ contains the points ac and bc but none of the segment connecting them. It is unclear whether or not the complex could be deformed somehow to become polytopal. On the other hand, Sinefakopoulos [**19, 20**] gives an elegant inductive construction of a polytopal subdivision of the $(n-1)$-simplex which supports a minimal resolution of a power of the maximal ideal.

The combinatorial structure of the Sinefakopoulos resolution is very different from that of the Eliahou-Kervaire resolution, corresponding to their different embeddings in the Taylor resolution. Although they are isomorphic as algebraic chain complexes, I think these resolutions are nonetheless worthy of further study as distinct objects.

5. Stable ideals. Our final task is to exhibit a regular cell complex supporting the Eliahou-Kervaire resolution of any stable ideal.

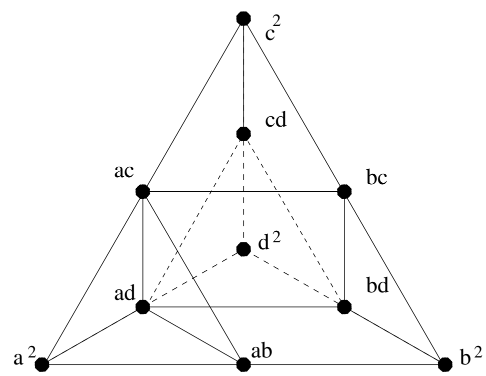

FIGURE 5. The Sinefakopoulos resolution of $(a,b,c,d)^2$.

Let B be a stable ideal, minimally generated by $g_1, \dots, g_r$.

If B is generated entirely in degree d, the cells $U(g_s,\alpha)$ form a subcomplex of the resolution of $(x_1,\dots,x_n)^d$ constructed in the previous section; this subcomplex supports the Eliahou-Kervaire resolution of B.

If B is not generated in a single degree, the situation is essentially the same, but some tweaking is required. Namely, we need to modify the operation $\to x_i$ to make sense in the new setting.

Definition 5.1. Let B be a stable ideal, $m \in B$ a monomial, and α a squarefree monomial. We set $m \to_B \alpha = \operatorname{beg}(m\alpha)$, the beginning of $m\alpha$.

Remark. If $B = (x_1,\dots,x_n)^d$ and m has degree d, then the operations $\to x_i$ and $\to_B x_i$ are the same.

Construction 5.2. For a generator g, squarefree monomial α, and permutation σ, let $\operatorname{ch}(g,\alpha,\sigma)$ be the simplex on vertices named $\langle g, g \to_B x_{\sigma_i}, \dots, g \to_B \alpha\rangle$. Define $U(g,\alpha)$ to be the union of the nondegenerate $\operatorname{ch}(g,\alpha,\sigma)$.

Treating the $\mathrm{ch}\,(g,\alpha,\sigma)$ and $U(g,\alpha)$ as abstract objects, we can repeat our arguments from the previous section to show that the $\{U(g,\alpha)\}$ support the Eliahou-Kervaire resolution, and that each cell $U(g,\alpha)$ is a pure shellable simplicial ball.

Thus, the $\{U(g,\alpha)\}$ form a regular cell complex supporting the Eliahou-Kervaire resolution of B, as desired. This proves:

Theorem 5.3. *Let B be any stable ideal of S. Then there is a regular cell complex which supports the Eliahou-Kervaire resolution of B.*

6. Consequences and further research. Now that we know the Eliahou-Kervaire resolution is cellular, there are techniques given in [**4**] to produce minimal cellular resolutions of new ideals. However, Borel ideals are sufficiently important that those resolutions are already well-known; the only new information is that those resolutions are also cellular. For example, the following result about "Borel-with-holes" ideals is due to Gasharov, Hibi and Peeva [**14**].

Corollary 6.1 [**14**]. *Fix exponents $e_1, \dots, e_n$, and a Borel ideal B. Let B' be the "Borel-with-holes" ideal generated by those monomials of B which are not divisible by any $x_i^{e_i}$. Then B' is minimally resolved by the subcomplex of the Eliahou-Kervaire resolution generated by the symbols $[m,\alpha]$ such that $m\alpha$ does not divide any $x_i^{e_i}$.*

6.1. Generalizing the construction. The Eliahou-Kervaire resolution of a stable ideal I is classically built from a mapping cone, relying on I having linear quotients with special structure. Can Construction 4.1 be generalized to describe the minimal resolution of any ideal with linear quotients?

Is it possible to describe a cellular structure on any (non-minimal) mapping cone, as the Taylor resolution puts a simplicial structure on a non-minimal resolution of any ideal?

6.2. Different minimal resolutions. Recall that the simplex $\mathrm{ch}\,(m,\alpha,\sigma)$ corresponds to the Taylor symbol $[m \to \sigma, \dots, m]$. It follows that the Eliahou-Kervaire resolution is embedded in the Taylor resolution, with the EK-symbol $[m, \alpha]$ corresponding to the sum

$$\sum_{\substack{\sigma \\ \text{nondegenerate}}} o(\sigma)\mathrm{ch}\,(m, \alpha, \sigma).$$

The Sinefakopoulos resolution [**19, 20**] also produces an embedding of the minimal resolution of $(x_1, \dots, x_n)$ inside the Taylor resolution.

We now know of two different cellular structures (due to Eliahou-Kervaire [**12**] and Sinefakopoulos [**19, 20**]) on the minimal resolution of $\mathbf{m}^n$, and possibly two more (the Morse theory construction of Batzies and Welker [**4**] and the "complex of boxes" of Nagel and Reiner [**17**]), each corresponding to a sparse basis for the minimal resolution inside the Taylor resolution.

What other ideals have multiple interesting realizations for their minimal free resolutions?

If $\mathbf{F}_\bullet$ and $\mathbf{G}_\bullet$ are two different minimal free resolutions of I, it is reasonable to wonder how they interact. What can be said about their sum or intersection inside the Taylor resolution? For example, it is known that the intersection of all (non-minimal) simplicial resolutions of an ideal is its Scarf complex (see [**16**, Chapter 6.2]). Do any other complexes arise in this way? Can an isomorphism from $\mathbf{F}_\bullet$ to $\mathbf{G}_\bullet$ be extended to an automorphism of the Taylor resolution?

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