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SEQUENTIALLY COHEN-MACAULAYNESS VERSUS PARAMETRIC DECOMPOSITION OF POWERS OF PARAMETER IDEALS

SHIRO GOTO, YOSUKE HORIUCHI AND HIDETO SAKURAI

ABSTRACT. This paper proves that parametric decomposition of powers of parameter ideals characterizes, under some additional conditions, sequentially Cohen-Macaulayness in modules together with a certain *good* property of corresponding systems of parameters.

1. Introduction. Let A be a commutative ring, and let $\underline{x} = x_1, x_2, \dots, x_d \in A$ $(d > 0)$ be a system of elements in A. Let M be an A-module. For each integer $n \geq 1$ we put

$$\Lambda_{d,n} = \Big\{ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbf{Z}^d \mid \alpha_i \geq 1 \text{ for } 1 \leq \forall i \leq d,\ \sum_{i=1}^{d} \alpha_i = d + n - 1 \Big\}.$$

Let $Q = (x_1, x_2, \dots, x_d)$ and $Q(\alpha) = (x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_d^{\alpha_d})$ for each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \Lambda_{d,n}$. Then we say that the system $\underline{x}$ satisfies condition (PD) for M, or all the powers of Q have parametric decomposition in M, if the equality

$$Q^n M = \bigcap_{\alpha \in \Lambda_{d,n}} Q(\alpha) M$$

holds true for all $n \geq 1$. Here we notice that $Q^n M \subseteq Q(\alpha) M$ for every $\alpha \in \Lambda_{d,n}$ and so, condition (PD) requires the inclusion $Q^n M \supseteq \bigcap_{\alpha \in \Lambda_{d,n}} Q(\alpha) M$ only.

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The study on condition (PD) started from the paper [**7**] of Heinzer, Ratliff and Shah, in which they investigated the case where $M = A$ and the sequence $\underline{x} = x_1, x_2, \ldots, x_d$ is A-regular and eventually, showed that every A-regular sequence satisfies (PD). Subsequently, the first author and Shimoda [**4, 5**] focused their attention on the case where $\underline{x}$ is a system of elements in the maximal ideal $\mathfrak{m}$ in a Noetherian local ring $(A, \mathfrak{m})$ and proved several basic results on condition (PD) with respect to $\underline{x}$. Among them one can find that A is a Cohen-Macaulay local ring if and only if some (hence every) system $\underline{x} = x_1, x_2, \ldots, x_d$ of parameters of A satisfies condition (PD) and each x_i is, furthermore, a non-zerodivisor of A (cf. [**4**, Theorem (1.1)]). The latter condition in the *if* part is, unfortunately, not superfluous ([**4**, Example (3.6)]). They then tried to understand the reason why this additional condition is required for [**4**, Theorem (1.1)] to hold true, and explored in [**5**] the natural question of what happens on a Noetherian local ring $(A, \mathfrak{m})$, if every system of parameters for A satisfies (PD). Their answer [**5**, Theorem 1.1] shows that $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ (here $\mathrm{H}^0_{\mathfrak{m}}(A)$ denotes the $0^{\underline{th}}$ local cohomology module of A with respect to $\mathfrak{m}$) is a Cohen-Macaulay ring and $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(A) = (0)$ whence A is a very special kind of Buchsbaum local ring, if every system of parameters for A satisfies (PD), provided $d = \dim A \geq 2$. In [**5**, Proposition 2.2] they showed also, focusing their attention on some special systems of parameters, the parametric decomposition of powers of parameter ideals holds true in the case where A is, so-called, an approximately Cohen-Macaulay local ring (see [**3**] for the definition of approximately Cohen-Macaulay local rings).

Noetherian local rings whose systems of parameters always satisfy condition (PD) and approximately Cohen-Macaulay local rings are naturally *sequentially Cohen-Macaulay* rings (see Section 2 for a brief survey on sequentially Cohen-Macaulay rings). Therefore, the researches [**4, 5**] seem to suggest, overall, that there should be some relation between the parametric decomposition of powers of parameter ideals and the sequentially Cohen-Macaulayness in rings and modules. The final achievement is, however, recently performed by Cuong and Truong [**2**].

Let us note here their result.

Theorem 1.1 [**2**, Theorem 1.1]. *Let A be a Noetherian local ring and M a finitely generated A-module with $d = \dim_A M > 0$. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a system of parameters for M, and assume that*

$\underline{x}$ is a good system of parameters, that is,

$$(x_j \mid q < j \le d)M \cap L = (0)$$

for every A-submodule L $(\neq (0))$ of M with $\dim_A L = q$. Then the following two conditions are equivalent to each other.

(1) *M is a sequentially Cohen-Macaulay A-module, that is M admits a filtration $\mathcal{M} = \{M_i\}_{0 \le i \le t}$ $(t > 0)$ of A-submodules*

$$M_0 = (0) \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_t = M$$

such that $\dim_A M_{i-1} < \dim_A M_i$ and M_i/M_{i-1} is a Cohen-Macaulay A-module for all $1 \le i \le t$, where $\dim_A M_0 = -\infty$.

(2) *The system $\underline{x}$ satisfies condition* (PD) *for M, that is*

$$Q^n M = \bigcap_{\alpha \in \Lambda_{d,n}} Q(\alpha)M$$

for all $n \ge 1$, where $Q = (x_1, x_2, \ldots, x_d)$.

This remarkable theorem contains [**5**, Proposition 2.2] as a special case and one obtains [**5**, Theorem (1,1)] as a fairly direct consequence of it. At this moment it gives the most general assertion on the parametric decomposition of powers of parameter ideals in sequentially Cohen-Macaulay rings and modules.

The theorem provides, however, no practical way to check whether a given system $\underline{x}$ of parameters is *good* or not, and furthermore, says nothing about the relation between condition (PD) and the condition that $\underline{x}$ is a good system of parameters. This observation has argued the present research and we are now eager to report a different approach for the analysis of condition (PD).

Our result is summarized into the following Theorem 1.2, where our contribution is the implication (2) $\Rightarrow$ (1). Conditions (2) (ii) and (iii) are automatically satisfied, when $\underline{x}$ is a good system of parameters for M ([**2**, Lemma 2.1]).

Theorem 1.2. *Let A be a Noetherian local ring and M a finitely generated A-module with $d = \dim_A M > 0$. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be* *a system of parameters for M. Then the following two conditions are equivalent to each other.*

(1) *M is a sequentially Cohen-Macaulay A-module and $\underline{x}$ is a good system of parameters for M.*

(2) *The following three conditions are satisfied.*

(i) $Q^n M = \cap_{\alpha \in \Lambda_{d,n}} Q(\alpha) M$ *for all* $n \geq 1$, *where* $Q = (x_1, x_2, \ldots, x_d)$.

(ii) $[(0) :_M x_i^n] = [(0) :_M x_i]$ *for all* $n \geq 1$ *and* $1 \leq i \leq d$.

(iii) $[(0) :_M x_i] \subseteq [(0) :_M x_{i+1}]$ *for all* $1 \leq i < d$.

Although conditions (2) (ii) and (iii) play very important roles in Theorem 1.2, the theorem may still have some significance in the further analysis of parametric decomposition of powers of parameter ideals, indicating that condition (PD) involves, more or less, the sequentially Cohen-Macaulayness in modules and the *good* property of systems $\underline{x}$ of parameters as well.

Let us explain how this paper is organized. We shall prove Theorem 1.2 in Section 3. The proof of the implication (1) $\Rightarrow$ (2) (i) given in [**2**] is essentially the same as that of [**5**, Proposition 2.2]. We will briefly note the proof for the sake of completeness. In order to prove the implication (2) $\Rightarrow$ (1) in Theorem 1.2, we need some basic theory of dimension filtrations, which we will summarize in Section 2. As is suggested in [**2**], if Q is an ideal of A generated by a *good* system of parameters for a sequentially Cohen-Macaulay module M, the associated graded module $\mathrm{G}_Q(M) = \oplus_{n \geq 0} Q^n M / Q^{n+1} M$ over the graded ring $\mathrm{G}_Q(A) = \oplus_{n \geq 0} Q^n / Q^{n+1}$ is again sequentially Cohen-Macaulay with the dimension filtration $\{\mathrm{G}_Q(D_i) = \oplus_{n \geq 0} Q^n D_i / Q^{n+1} D_i\}_{0 \leq i \leq \ell}$, where $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ denotes the dimension filtration of M. In Section 4 we shall give a brief proof of this result.

Unless otherwise specified, throughout this paper let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Let M be a finitely generated A-module, and let $\underline{x} = x_1, x_2, \ldots, x_d$ $(d > 0)$ be a system of elements in $\mathfrak{m}$. We put $Q = (x_1, x_2, \ldots, x_d)$ and $Q_i = (x_1, x_2, \ldots, x_i)$ for $0 \leq i \leq d$.

Let $Q(\alpha) = (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_d^{\alpha_d})$ for each $\alpha \in \Lambda_{d,n}$, where

$$\Lambda_{d,n} = \Big\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbf{Z}^d \mid \alpha_i \geq 1 \quad \text{for } 1 \leq \forall i \leq d,\ \sum_{i=1}^{d} \alpha_i = d + n - 1\Big\}.$$

Let $\dim_A M = -\infty$, when $M = (0)$.

2. A brief survey on dimension filtrations. In this section we shall give a brief survey on dimension filtrations of rings and modules, which we will need later in this paper.

Let A be a Noetherian ring, which is not assumed to be a local ring. Let M $(\neq (0))$ be a finitely generated A-module with finite Krull dimension, say $d = \dim_A M$. We put

$$\mathrm{Assh}_A M = \{\mathfrak{p} \in \mathrm{Supp}_A M \mid \dim A/\mathfrak{p} = d\}.$$

Then

$$\mathrm{Assh}_A M \subseteq \mathrm{Min}_A M \subseteq \mathrm{Ass}_A M.$$

Let $S = \{\dim_A L \mid L \text{ is an } A\text{-submodule of } M, L \neq (0)\}$. We then have

$$S = \{\dim A/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_A M\}.$$

Let $\ell = \sharp S$. We number the elements $\{d_i\}_{1 \leq i \leq \ell}$ of S with $d_1 < d_2 < \cdots < d_\ell$ and put $d_0 = 0$. Then, because the base ring A is Noetherian, for each $1 \leq i \leq \ell$ the A-module M contains the largest A-submodule D_i with $\dim_A D_i = d_i$. Therefore, letting $D_0 = (0)$, we have the filtration $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$

$$D_0 = (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\ell = M$$

of A-submodules of M, which we call the dimension filtration of M. The notion of dimension filtration was firstly given by Schenzel [**8**], where he gave several characterizations of dimension filtrations. Our notion of dimension filtration is somewhat different from that of [**1, 8**], but throughout our paper let us utilize the above definition. It is then standard to check that $\{D_j\}_{0 \le j \le i}$ (respectively $\{D_j/D_i\}_{i \le j \le \ell}$) is the dimension filtration of D_i (respectively M/D_i) for every $1 \le i \le \ell$.

We put $C_i = D_i/D_{i-1}$ for $1 \le i \le \ell$.

Definition 2.1 [**8, 9**]. We say that M is a sequentially Cohen-Macaulay A-module, if C_i is a Cohen-Macaulay A-module for all $1 \le i \le \ell$.

Let us note here two characterizations of the dimension filtration $\mathcal{D} = \{D_i\}_{0 \le i \le \ell}$ of M. Let

$$(0) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_A M} M(\mathfrak{p})$$

be a primary decomposition of (0) in M, where $M(\mathfrak{p})$ is an A-submodule of M with $\mathrm{Ass}_A M/M(\mathfrak{p}) = \{\mathfrak{p}\}$ for each $\mathfrak{p} \in \mathrm{Ass}_A M$. With this notation we have the following.

Proposition 2.2 [**8**, Proposition 2.2, Corollary 2.3]. *The following assertions hold true.*

(1) $D_i = \cap_{\mathfrak{p} \in \mathrm{Ass}_A M,\ \dim A/\mathfrak{p} \ge d_{i+1}} M(\mathfrak{p})$ *for all* $0 \le i < \ell$.

(2) *Let* $1 \le i \le \ell$. *Then* $\mathrm{Ass}_A C_i = \{\mathfrak{p} \in \mathrm{Ass}_A M \mid \dim A/\mathfrak{p} = d_i\}$ *and* $\mathrm{Ass}_A D_i = \{\mathfrak{p} \in \mathrm{Ass}_A M \mid \dim A/\mathfrak{p} \le d_i\}$.

(3) $\mathrm{Ass}_A M/D_i = \{\mathfrak{p} \in \mathrm{Ass}_A M \mid \dim A/\mathfrak{p} \ge d_{i+1}\}$ *for all* $1 \le i < \ell$.

The second characterization of the dimension filtration is the following, which plays an important role in our proof of Theorem 1.2. The result might be known but let us include a brief proof.

Theorem 2.3. *Let* $\mathcal{M} = \{M_i\}_{0 \le i \le t}$ $(t > 0)$ *be a family of* A-*submodules of* M *such that*

(i) $M_0 = (0) \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_t = M$ *and*

(ii) $\dim_A M_{i-1} < \dim_A M_i$ *for all* $1 \le i \le t$.

Assume that $\mathrm{Ass}_A M_i/M_{i-1} = \mathrm{Assh}_A M_i/M_{i-1}$ *for all* $1 \le i \le t$. *Then* $t = \ell$ *and* $M_i = D_i$ *for every* $0 \le i \le \ell$.

Proof. We put $m_i = \dim_A M_i$ for $1 \le i \le t$. Then $\dim_A M_i/M_{i-1} = m_i$, thanks to condition (ii). Let $\mathfrak{p} \in \mathrm{Ass}_A M$. Then since

$$\mathrm{Ass}_A M \subseteq \bigcup_{i=1}^{t} \mathrm{Ass}_A M_i/M_{i-1},$$

we have $\mathfrak{p} \in \mathrm{Ass}_A M_i/M_{i-1}$ for some $1 \le i \le t$, whence $\dim A/\mathfrak{p} = m_i$ because $\mathrm{Ass}_A M_i/M_{i-1} = \mathrm{Assh}_A M_i/M_{i-1}$. Therefore $\dim A/\mathfrak{p} \in \{m_1, m_2, \dots, m_t\}$ for all $\mathfrak{p} \in \mathrm{Ass}_A M$, whence $S \subseteq \{m_1, m_2, \dots, m_t\}$ because $S = \{\dim A/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_A M\}$. Thus $t = \ell$ and $m_i = d_i$ for all $1 \le i \le \ell$, so that $M_i \subseteq D_i$ for all $0 \le i \le \ell$ by the definition of D_i. We want to show $M_i = D_i$. To see this, since $\{D_j\}_{0 \le j \le i}$ is the dimension filtration of D_i, without loss of generality we may assume that $i = \ell - 1$. Suppose $D_{\ell-1}/M_{\ell-1} \ne (0)$ and choose $\mathfrak{p} \in \mathrm{Ass}_A D_{\ell-1}/M_{\ell-1}$. Then because $\mathfrak{p} \in \mathrm{Ass}_A M/M_{\ell-1}$, we have $\dim A/\mathfrak{p} = d$, so that $[D_{\ell-1}]_{\mathfrak{p}} = (0)$ since $\dim_A D_{\ell-1} < d$, which is impossible. Thus $D_{\ell-1} = M_{\ell-1}$ as is wanted. $\square$

Corollary 2.4 [**8**, Proposition 4.3]. *Suppose that A is a local ring. Then M is a sequentially Cohen-Macaulay A-module if and only if M admits a Cohen-Macaulay filtration, that is, a family $\mathcal{M} = \{M_i\}_{0 \le i \le t}$ $(t > 0)$ of A-submodules of M with*

$$M_0 = (0) \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_t = M$$

such that

(i) $\dim_A M_{i-1} < \dim_A M_i$ *and*

(ii) M_i/M_{i-1} *is a Cohen-Macaulay A-module for all* $1 \le i \le t$.

Remark 2.5. Let $R = \oplus_{n \in \mathbf{Z}} R_n$ be a Noetherian H-local graded ring with the unique H-maximal ideal $\mathfrak{M}$ ([**6**, Definition 1.1.6]). Assume that $\mathfrak{M}$ is a maximal ideal in R, and let M $(\ne (0))$ be a finitely generated graded R-module. Then M admits the dimension filtration $\mathcal{D} = \{D_i\}_{0 \le i \le \ell}$ consisting of graded R-submodules. Thanks to Theorem 2.3, Corollary 2.4 holds true also for the graded R-module M, because

$$\mathrm{Ass}_R M_i/M_{i-1} = \mathrm{Assh}_R M_i/M_{i-1}$$

for all $1 \le i \le t$, once M_i/M_{i-1} is Cohen-Macaulay.

Let $f : A \to B$ be a flat local homomorphism of Noetherian local rings, and assume that M is a finitely generated A-module which admits the dimension filtration $\mathcal{D} = \{D_i\}_{0 \le i \le \ell}$. Then the family

$$\mathcal{D} \otimes_A B = \{D_i \otimes_A B\}_{0 \le i \le \ell}$$

of B-submodules of $M \otimes_A B$ is a filtration

$$D_0 \otimes_A B = (0) \subsetneq D_1 \otimes_A B \subsetneq D_2 \otimes_A B \subsetneq \cdots \subsetneq D_\ell \otimes_A B = M \otimes_A B$$

of $M \otimes_A B$ with $[D_i \otimes_A B]/[D_{i-1} \otimes_A B] = C_i \otimes_A B$ and $\dim_B D_i \otimes_A B = d_i + \dim B/\mathfrak{m}B$, where $\mathfrak{m}$ denotes the maximal ideal of A. Hence

$$\dim_B D_{i-1} \otimes_A B < \dim_B D_i \otimes_A B$$

for all $1 \le i \le \ell$. Recall that $\mathrm{Ass}_B C_i \otimes_A B = \cup_{\mathfrak{p} \in \mathrm{Ass}_A C_i} \mathrm{Ass}_B B/\mathfrak{p}B$ for $1 \le i \le \ell$. We then furthermore have the following.

Proposition 2.6. *The following conditions are equivalent.*

(1) *$\mathcal{D} \otimes_A B$ is the dimension filtration of $M \otimes_A B$.*

(2) *$\mathrm{Ass}_B B/\mathfrak{p}B = \mathrm{Assh}_B B/\mathfrak{p}B$ for every $\mathfrak{p} \in \mathrm{Ass}_A M$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathfrak{p} \in \mathrm{Ass}_A M$, and let $\dim A/\mathfrak{p} = d_i$ with $1 \le i \le \ell$. Then $\mathfrak{p} \in \mathrm{Ass}_A C_i$ by Proposition 2.2 (2). Hence for each $\mathfrak{q} \in \mathrm{Ass}_B B/\mathfrak{p}B$ we have $\mathfrak{q} \in \mathrm{Ass}_B C_i \otimes_A B$, so that $\dim B/\mathfrak{q} = \dim_B D_i \otimes_A B$ again by Proposition 2.2 (2), because $\mathcal{D} \otimes_A B$ is the dimension filtration of $M \otimes_A B$.

(2) $\Rightarrow$ (1). Thanks to Theorem 2.3, we have only to show

$$\mathrm{Ass}_B C_i \otimes_A B = \mathrm{Assh}_B C_i \otimes_A B$$

for every $1 \le i \le \ell$. Let $\mathfrak{q} \in \mathrm{Ass}_B C_i \otimes_A B$ and choose $\mathfrak{p} \in \mathrm{Ass}_A C_i$ so that $\mathfrak{q} \in \mathrm{Ass}_B B/\mathfrak{p}B$. Hence $\mathfrak{p} \in \mathrm{Ass}_A M$ and $\dim A/\mathfrak{p} = \dim_A C_i$ by Proposition 2.2 (2). We then have $\mathrm{Ass}_B B/\mathfrak{p}B = \mathrm{Assh}_B B/\mathfrak{p}B$ by assumption (2), whence

$$\begin{aligned}\dim B/\mathfrak{q} &= \dim B/\mathfrak{p}B = \dim A/\mathfrak{p} + \dim B/\mathfrak{m}B \\ &= \dim_A C_i + \dim B/\mathfrak{m}B = \dim_B C_i \otimes_A B.\end{aligned}$$

Thus $\mathrm{Ass}_B C_i \otimes_A B = \mathrm{Assh}_B C_i \otimes_A B$. $\square$

Corollary 2.7 (cf. [**8**, Theorem 6.2]). *The following conditions are equivalent.*

(1) *M is a sequentially Cohen-Macaulay A-module and $B/\mathfrak{m}B$ is a Cohen-Macaulay ring.*

(2) *$M \otimes_A B$ is a sequentially Cohen-Macaulay B-module and*

$$\mathrm{Ass}_B B/\mathfrak{p}B = \mathrm{Assh}_B B/\mathfrak{p}B$$

for every $\mathfrak{p} \in \mathrm{Ass}_A M$.

Proof. (1) $\Rightarrow$ (2). The B-module $C_i \otimes_A B$ is Cohen-Macaulay for all $1 \leq i \leq \ell$. Hence $M \otimes_A B$ is, by Theorem 2.3, a sequentially Cohen-Macaulay B-module with $\mathcal{D} \otimes_A B$ the dimension filtration, so that $\mathrm{Ass}_B B/\mathfrak{p}B = \mathrm{Assh}_B B/\mathfrak{p}B$ for every $\mathfrak{p} \in \mathrm{Ass}_A M$ by Proposition 2.6.

(2) $\Rightarrow$ (1). By Proposition 2.6 $\mathcal{D} \otimes_A B$ is the dimension filtration of $M \otimes_A B$. Because $C_i \otimes_A B$ is a Cohen-Macaulay B-module, so is the A-module C_i for all $1 \leq i \leq \ell$, whence the ring $B/\mathfrak{m}B$ is Cohen-Macaulay and the A-module M is sequentially Cohen-Macaulay. □

Corollary 2.8. *Suppose that A is a local ring with the maximal ideal $\mathfrak{m}$. Then the following two conditions are equivalent.*

(1) *M is a sequentially Cohen-Macaulay A-module.*

(2) *$\widehat{M}$ is a sequentially Cohen-Macaulay $\widehat{A}$-module and for every $\mathfrak{p} \in \mathrm{Ass}_A M$ the ring $A/\mathfrak{p}$ is unmixed, that is, $\mathrm{Ass}_{\widehat{A}} \widehat{A}/\mathfrak{p}\widehat{A} = \mathrm{Assh}_{\widehat{A}} \widehat{A}/\mathfrak{p}\widehat{A}$, where $\widehat{*}$ denotes the $\mathfrak{m}$-adic completion.*

When this is the case, the dimension filtration of $\widehat{M}$ is given by the completion $\widehat{\mathcal{D}} = \{\widehat{D_i}\}_{0 \leq i \leq \ell}$ of the dimension filtration $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ of M.

Since $A/\mathfrak{p}$ is unmixed for every prime ideal $\mathfrak{p}$ in a Cohen-Macaulay local ring A, by Corollary 2.8 we readily get the following.

Corollary 2.9. *Suppose that A is a homomorphic image of a Cohen-Macaulay local ring. Then M is a sequentially Cohen-Macaulay A-module if and only if so is the $\widehat{A}$-module $\widehat{M}$.*

Remark 2.10. Unless A is a homomorphic image of a Cohen-Macaulay local ring, the A-module M is not necessarily a sequentially Cohen-Macaulay A-module, even though $\widehat{M}$ is a sequentially Cohen-Macaulay $\widehat{A}$-module. For instance, let us look at Nagata's bad example, that is, a non-regular Noetherian local integral domain $(A, \mathfrak{m})$ with $\dim A = 2$ and $\mathrm{e}_{\mathfrak{m}}^0(A) = 1$, where $\mathrm{e}_{\mathfrak{m}}^0(A)$ denotes the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Then A is certainly not a sequentially Cohen-Macaulay local ring, while so is the completion $\widehat{A}$, because $\widehat{A} \cong k[[X,Y,Z]]/((X) \cap (Y,Z))$ with $k[[X,Y,Z]]$ the formal power series ring over a field k.

Suppose that A is a local ring and $d = \dim_A M > 0$. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a system of parameters for M. Let $\mathcal{D} = \{D_i\}_{0 \le i \le \ell}$ be the dimension filtration of M.

Definition 2.11 [**1**, Definition 2.2]. We say that $\underline{x}$ is a *good* system of parameters for M, if

$$(x_j \mid q < j \le d)M \cap L = (0)$$

for every A-submodule L ($\neq (0)$) of M with $\dim_A L = q$.

Since every A-submodule L of M with $\dim_A L = q$ is contained in D_i such that $\dim_A D_i = q$, this condition is equivalent to saying that

$$(x_j \mid d_i < j \le d)M \cap D_i = (0)$$

for all $1 \le i < \ell$. Therefore, when A is a local ring and $d = \dim_A M > 0$, good systems of parameters for M do exist (see [**1**, Lemma 2.5]).

We close this section with the following.

Proposition 2.12 [**2**, Lemma 2.1]. *Suppose that A is a local ring and $d = \dim_A M > 0$. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a good system of parameters for M. Then $[(0) :_M x_j] = D_i$ for all $0 \le i < \ell$ and $1 \le j \le d$ such that $d_i < j \le d_{i+1}$.*

3. Proof of Theorem 1.2. The purpose of this section is to prove Theorem 1.2. For the purpose we need a few results of [**2, 4**]. Let us note them for the sake of the reader's convenience.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$, and let M be a finitely generated A-module. Let $\underline{x} = x_1, x_2, \ldots, x_d$ $(d > 0)$ be a system of elements in the maximal ideal $\mathfrak{m}$. We put $Q = (x_1, x_2, \ldots, x_d)$ and $Q_i = (x_1, x_2, \ldots, x_i)$ for $0 \leq i \leq d$.

We begin with the following, which are also summarized in [**2**, Lemma 3.3].

Lemma 3.1 [**4**, Lemma (3.1)]. *Let $a \in A$, and assume that $[(0) :_M a^n] \subseteq aM$ for all $n \geq 1$. Then a is a non-zerodivisor on M.*

Proposition 3.2 [**4**, Proof of Lemma (3.2)]. *Suppose that $\underline{x} = x_1, x_2, \ldots, x_d$ satisfies condition* (PD) *for M. Then the following assertions hold true.*

(i) $[Q_{d-1}M :_M x_d^n] \subseteq QM + [(0) :_M x_d^n]$ *for all $n \geq 1$.*

(ii) *For all $1 \leq i \leq d$ the system $x_1, x_2, \ldots, x_i$ satisfies condition* (PD) *for M.*

The following result is also summarized in [**2**, Lemma 3.4]. But let us include a brief proof for the sake of completeness.

Corollary 3.3. *Assume that $\underline{x}$ satisfies condition* (PD) *for M and that $[(0) :_M x_i^n] = [(0) :_M x_i]$ for all $n \geq 1$ and $1 \leq i \leq d$. Then for all integers $1 \leq i \leq j \leq d$, the element x_j is a non-zerodivisor on $M/(Q_{i-1}M + [(0) :_M x_j])$, and hence the equality*

$$[Q_{i-1}M :_M x_j^2] = Q_{i-1}M + [(0) :_M x_j]$$

holds true.

Proof. Since condition (PD) is independent of the order of the elements $x_1, x_2, \ldots, x_d$, without loss of generality we may assume that $i = j$. Let $1 \leq i \leq d$, and put $N = M/(Q_{i-1}M + [(0) :_M x_i])$. Let $\varphi \in M$, and assume that $x_i^n \overline{\varphi} = 0$ in N for some integer $n \geq 1$, where $\overline{\varphi}$ denotes the image of φ in N. Then by Proposition 3.2 (ii) the subsystem $\underline{x}' = x_1, x_2, \ldots, x_i$ of $\underline{x}$ also satisfies condition (PD) and so, because $x_i^{n+1}\varphi \in Q_{i-1}M$, applying Proposition 3.2 (i) to the system $\underline{x}'$, we get

$$\varphi \in [Q_{i-1}M :_M x_i^{n+1}] \subseteq Q_iM + [(0) :_M x_i^{n+1}] = Q_iM + [(0) :_M x_i].$$

Hence $\overline{\varphi} \in x_iN$, so that by Lemma 3.1 x_i is a non-zerodivisor on N. $\square$

We are now ready to prove Theorem 1.2. Let $d = \dim_A M > 0$ and assume that $\underline{x} = x_1, x_2, \ldots, x_d$ is a system of parameters for M. The proof of the implication (1) $\Rightarrow$ (2) (i) given in [**2**] is essentially the same as that of [**5**, Proposition 2.2]. Let us briefly include the proof for the sake of completeness.

Proof of Theorem 1.2. (1) $\Rightarrow$ (2). See Proposition 2.12 for assertions (ii) and (iii) (for assertion (ii) notice that $x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}$ is also a good system of parameters of M for all positive integers n_i's). We shall prove that $\underline{x}$ satisfies condition (PD). Let $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ be the dimension filtration of M. By induction on ℓ we will show that

$$Q^nM = \bigcap_{\alpha \in \Lambda_{d,n}} Q(\alpha)M$$

for all $n \geq 1$. If $\ell = 1$, then M is a Cohen-Macaulay A-module, whence the assertion follows from [**7**] (see [**5**, Lemma 2.1] also; recall that $x_1, x_2, \ldots, x_d$ is an M-regular sequence).

Suppose that $\ell > 1$ and that our assertion holds true for $\ell - 1$. Let $n \geq 1$ be an integer. Then because $N = M/D_{\ell-1}$ is a Cohen-Macaulay A-module with $\dim_A N = d$ and $\underline{x}$ is a system of parameters for N, we get

$$Q^nN = \bigcap_{\alpha \in \Lambda_{d,n}} Q(\alpha)N.$$

Let $x \in \cap_{\alpha \in \Lambda_{d,n}} Q(\alpha)M$. We then have $\overline{x} \in Q^nN$ where $\overline{x}$ denotes the image of x in N, whence

$$x \in Q^nM + D_{\ell-1}.$$

Let us write $x = y + z$ with $y \in Q^nM$ and $z \in D_{\ell-1}$. Then, because $x \in Q(\alpha)M$, we have $z \in Q(\alpha)M \cap D_{\ell-1}$ so that $z \in Q(\alpha)D_{\ell-1}$ for each $\alpha \in \Lambda_{d,n}$ (recall that $x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_d^{\alpha_d}$ forms a regular sequence on $N = M/D_{\ell-1}$). We put $q = \dim_A D_{\ell-1}$. If $q = 0$, then $QD_{\ell-1} = (0)$ because $\underline{x}$ is a good system of parameters for M, so that $z = 0$ whence $x = y \in Q^n M$. Suppose that $q > 0$. Let $\beta \in \Lambda_{q,n}$ and look at the element

$$\alpha = (\beta_1, \beta_2, \ldots, \beta_q, 1, \ldots, 1) \in \Lambda_{d,n}.$$

We then have

$$z \in Q(\alpha)D_{\ell-1} = (x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_q^{\beta_q})D_{\ell-1},$$

because $(x_j \mid q < j \leq d)D_{\ell-1} \subseteq (x_j \mid q < j \leq d)M \cap D_{\ell-1} = (0)$. Thus

$$z \in \bigcap_{\beta \in \Lambda_{q,n}} Q_q(\beta)D_{\ell-1},$$

whence, thanks to the hypothesis of induction on ℓ, we get $z \in Q_q^n D_{\ell-1}$, because $x_1, x_2, \ldots, x_q$ is also a good system of parameters for the sequentially Cohen-Macaulay A-module $D_{\ell-1}$. Hence $z \in Q^n M$ so that we have $x \in Q^n M$ as is claimed.

(2) $\Rightarrow$ (1). For each $0 \leq i \leq d+1$ let

$$L_i = \begin{cases} (0) & (i = 0), \\ [(0) :_M x_i] & (1 \leq i \leq d), \\ M & (i = d+1). \end{cases}$$

Then $\mathcal{L} = \{L_i\}_{0 \leq i \leq d+1}$ is a family of A-submodules of M such that

$$L_0 = (0) \subseteq L_1 \subseteq \cdots \subseteq L_d \subsetneq L_{d+1} = M.$$

We need the following.

Claim 3.4. *The sequence $x_1, x_2, \ldots, x_i$ is M/L_i-regular for all $1 \leq i \leq d$.*

Proof of Claim 3.4. Let $1 \leq k \leq i$ and $\varphi \in [(Q_{k-1}M + L_i) :_M x_k]$. Then

$$(x_i x_k)\varphi \in Q_{k-1}M,$$

whence

$$x_i\varphi \in [Q_{k-1}M :_M x_k] \subseteq [Q_{k-1}M :_M x_k^2] = Q_{k-1}M + [(0) :_M x_k] \\ \subseteq Q_{k-1}M + [(0) :_M x_i],$$

thanks to conditions (i), (ii) and (iii) together with Corollary 3.3. Hence $x_i^2\varphi \in Q_{k-1}M$, so that

$$\varphi \in [Q_{k-1}M :_M x_i^2] = Q_{k-1}M + [(0) :_M x_i]$$

again by Corollary 3.3. Thus x_k is $M/(Q_{k-1}M + L_i)$-regular, whence $x_1, x_2, \ldots, x_i$ is an M/L_i-regular sequence. □

We then have the following. Consequently the filtration $\mathcal{L} = \{L_i\}_{0 \le i \le d+1}$ of M involves the dimension filtration $\mathcal{D} = \{D_i\}_{0 \le i \le \ell}$ inside (cf. Theorem 2.3), so that M is a sequentially Cohen-Macaulay A-module.

Claim 3.5. *Let $1 \le i \le d$ and assume that $L_i/L_{i-1} \neq (0)$. Then*

(1) $\dim_A L_i = i - 1$.

(2) *L_i/L_{i-1} is a Cohen-Macaulay A-module with* $\dim_A L_i/L_{i-1} = i - 1$.

Proof of Claim 3.5. Apply the depth lemma to the exact sequence

$$0 \longrightarrow L_i/L_{i-1} \longrightarrow M/L_{i-1} \longrightarrow M/L_i \longrightarrow 0$$

and we get $\mathrm{depth}_A L_i/L_{i-1} \ge i-1$, because $\mathrm{depth}_A M/L_{i-1} \ge i-1$ and $\mathrm{depth}_A M/L_i \ge i$ by Claim 3.4. Hence

$$\dim_A L_i \ge \dim_A L_i/L_{i-1} \ge \mathrm{depth}_A L_i/L_{i-1} \ge i-1,$$

so that $\dim_A L_i = i-1$ and the A-module L_i/L_{i-1} is Cohen-Macaulay of dimension $i-1$, because $\dim_A L_i \le i-1$ (recall that $(x_j \mid i \le j \le d)L_i = (0)$ by condition (iii)). □

Let us now check that $\underline{x} = x_1, x_2, \ldots, x_d$ is a good system of parameters for M. To see this, it suffices to show that

$$(x_j \mid d_i < j \le d)M \cap D_i = (0)$$

for all $1 \leq i < \ell$. We have nothing to prove when $\ell = 1$. Suppose that $\ell > 1$ and that our assertion holds true for $\ell - 1$. Let

$$i = \min \{0 \leq i \leq d+1 \mid L_i = D_{\ell-1}\}.$$

Then $1 \leq i \leq d$ and $L_{i-1} \subsetneq L_i = D_{\ell-1}$. Hence

$$QD_{\ell-1} = (x_1, x_2, \dots, x_{i-1})D_{\ell-1},$$

because $D_{\ell-1} = L_i$ and $(x_j \mid i \leq j \leq d)L_i = (0)$, thanks to condition (iii). Therefore if $i = 1$ (hence $\ell = 2$), then $QD_{\ell-1} = (0)$, so that $QM \cap D_{\ell-1} = QD_{\ell-1} = (0)$ (recall that the sequence $x_1, x_2, \dots, x_d$ is regular on $M/D_{\ell-1}$). Hence $\underline{x}$ is a good system of parameters for M.

Suppose that $i > 1$. Then $x_1, x_2, \dots, x_{i-1}$ is a system of parameters for $D_{\ell-1}$, as $\dim_A D_{\ell-1} = i - 1$ by Claim 3.5. On the other hand, thanks to Proposition 3.2 (ii), letting $\mathfrak{q} = (x_1, x_2, \dots, x_{i-1})$, we get

$$\bigcap_{\alpha \in \Lambda_{i-1,n}} \mathfrak{q}(\alpha) D_{\ell-1} \subseteq \bigcap_{\alpha \in \Lambda_{i-1,n}} \mathfrak{q}(\alpha) M = \mathfrak{q}^n M,$$

whence

$$\bigcap_{\alpha \in \Lambda_{i-1,n}} \mathfrak{q}(\alpha) D_{\ell-1} = \mathfrak{q}^n D_{\ell-1}$$

for all $n \geq 1$, because $\mathfrak{q}^n M \cap D_{\ell-1} = \mathfrak{q}^n D_{\ell-1}$ (use the fact that $x_1, x_2, \dots, x_{i-1}$ is a regular sequence on $M/D_{\ell-1}$). Thus conditions (i), (ii) and (iii) of assertion (2) are satisfied for the system $x_1, x_2, \dots, x_{i-1}$ of parameters of the sequentially Cohen-Macaulay A-module $D_{\ell-1}$, whence the hypothesis of induction on ℓ yields that $x_1, x_2, \dots, x_{i-1}$ is a good system of parameters for $D_{\ell-1}$. Consequently the whole system $\underline{x}$ is a good system of parameters for M, because the A-module $D_{\ell-1}$ admits the dimension filtration $\{D_j\}_{0 \leq j \leq \ell-1}$ and

$$\begin{aligned}(x_k \mid d_j < k \leq d)M \cap D_{\ell-1} &= (x_k \mid d_j < k \leq d)D_{\ell-1} \\ &= (x_k \mid d_j < k \leq i-1)D_{\ell-1}\end{aligned}$$

for all $1 \leq j < \ell$. This completes the proof of Theorem 1.2. $\square$

4. Sequentially Cohen-Macaulayness in the graded modules associated to sequentially Cohen-Macaulay modules. In this section let us maintain the same notations as in Section 3. For each A-module L we put

$$\mathcal{R}'_Q(L) = \bigoplus_{n\in\mathbf{Z}} Q^nL,$$
$$\mathcal{R}_Q(L) = \bigoplus_{n\geq 0} Q^nL, \text{ and}$$
$$\mathrm{G}_Q(L) = \bigoplus_{n\geq 0} Q^nL/Q^{n+1}L,$$

where we regard graded modules over the graded rings

$$\mathcal{R}'_Q(A) = \bigoplus_{n\in\mathbf{Z}} Q^n, \ \mathcal{R}_Q(A) = \bigoplus_{n\geq 0} Q^n, \text{ and } \mathrm{G}_Q(A) = \bigoplus_{n\geq 0} Q^n/Q^{n+1}$$

respectively. The purpose is to prove the following, which fairly readily follows from the argument in [**2**, Section 4]. Let us note a quick proof.

Theorem 4.1. *Assume that M is a sequentially Cohen-Macaulay A-module and that $\underline{x} = x_1, x_2, \ldots, x_d$ is a good system of parameters for M. Then $\mathcal{R}'_Q(M)$, $\mathcal{R}_Q(M)$ and $\mathrm{G}_Q(M)$ are sequentially Cohen-Macaulay modules over the graded rings $\mathcal{R}'_Q(A)$, $\mathcal{R}_Q(A)$, and $\mathrm{G}_Q(A)$ respectively.*

Proof. Let $\mathcal{D} = \{D_i\}_{0\leq i\leq \ell}$ be the dimension filtration of M and $C_i = D_i/D_{i-1}$ for $1 \leq i \leq \ell$. Then, since $\underline{x}$ is a good system of parameters for the sequentially Cohen-Macaulay A-module M, by induction on the length ℓ of the dimension filtration we have the equality

$$Q^nD_i \cap D_{i-1} = Q^nD_{i-1} = Q^n_{d_{i-1}}D_{i-1} \text{ and}$$
$$Q^nM \cap D_i = Q^nD_i = Q^n_{d_i}D_i$$

for all integers $1 \leq i \leq \ell$ and $n \in \mathbf{Z}$. Let $1 \leq i \leq \ell$. We then have, thanks to the first equality, the canonical exact sequence

$$0 \longrightarrow \mathcal{R}'_Q(D_{i-1}) \longrightarrow \mathcal{R}'_Q(D_i) \longrightarrow \mathcal{R}'_Q(C_i) \longrightarrow 0$$

of graded $\mathcal{R}'_Q(A)$-modules, which involves the exact sequences

$$0 \to \mathcal{R}_Q(D_{i-1}) \to \mathcal{R}_Q(D_i) \to \mathcal{R}_Q(C_i) \to 0 \text{ and}$$

$$0 \to \mathrm{G}_Q(D_{i-1}) \to \mathrm{G}_Q(D_i) \to \mathrm{G}_Q(C_i) \to 0$$

of graded $\mathcal{R}_Q(A)$-modules and graded $\mathrm{G}_Q(A)$-modules respectively. We similarly get by the second equality the embeddings

$$0 \to \mathcal{R}'_Q(D_i) \to \mathcal{R}'_Q(M),$$
$$0 \to \mathcal{R}_Q(D_i) \to \mathcal{R}_Q(M), \text{ and}$$

$$0 \to \mathrm{G}_Q(D_i) \to \mathrm{G}_Q(M)$$

of graded modules. On the other hand, letting $\mathfrak{q} = (x_1, x_2, \ldots, x_{d_i})$, we see that $\mathcal{R}'_Q(C_i) = \mathcal{R}'_{\mathfrak{q}}(C_i)$ since $QC_i = \mathfrak{q}C_i$, so that $\mathcal{R}'_{\mathfrak{q}}(C_i)$ is a Cohen-Macaulay graded $\mathcal{R}'_{\mathfrak{q}}(A)$-module, because $\mathfrak{q}$ is a parameter ideal for the Cohen-Macaulay A-module C_i. Since $\mathcal{R}'_Q(A) = \mathcal{R}'_{\mathfrak{q}}(A)[x_j^* \mid d_i < j \leq d]$ where x_j^* denotes x_j considered to be the element of degree 1 in the graded ring $\mathcal{R}'_Q(A) = \oplus_{n \in \mathbf{Z}} Q^n$ and since the elements x_j^* $(d_i < j \leq d)$ kill $\mathcal{R}'_{\mathfrak{q}}(C_i)$, we then have that $\mathcal{R}'_Q(C_i)$ is a Cohen-Macaulay graded $\mathcal{R}'_Q(A)$-module of dimension $d_i + 1$. This argument works also to show that $\mathcal{R}_Q(C_i)$ is a Cohen-Macaulay graded $\mathcal{R}_Q(A)$-module of dimension $d_i + 1$ (respectively dimension 0) if $d_i \geq 1$ (respectively $d_i = 0$) and $\mathrm{G}_Q(C_i)$ is a graded $\mathrm{G}_Q(A)$-module of dimension d_i. Hence, thanks to the graded version of Corollary 2.4 (cf. Remark 2.5), all of them are sequentially Cohen-Macaulay modules with the dimension filtrations $\{\mathcal{R}'_Q(D_i)\}_{0 \leq i \leq \ell}$, $\{\mathcal{R}_Q(D_i)\}_{0 \leq i \leq \ell}$, and $\{\mathrm{G}_Q(D_i)\}_{0 \leq i \leq \ell}$ respectively. □

Remark 4.2. Even though $\mathrm{G}_Q(M)$ is a sequentially Cohen-Macaulay graded $\mathrm{G}_Q(A)$-module for some ideal Q generated by a good system of parameters for M, a finitely generated module M over a Noetherian local ring A is not necessarily sequentially Cohen-Macaulay. For instance, let us again look at Nagata's bad example, that is a non-regular Noetherian local integral domain $(A, \mathfrak{m})$ with $\dim A = 2$ and $\mathrm{e}^0_{\mathfrak{m}}(A) = 1$. The completion $\widehat{A}$ of A is a sequentially Cohen-Macaulay local ring. Let us choose an ideal $\mathfrak{q}$ of $\widehat{A}$ so that $\mathfrak{q}$ is generated by a good system of parameters for $\widehat{A}$. Let $Q = \mathfrak{q} \cap A$. Then $\mathfrak{q} = Q\widehat{A}$ and so, Q is a parameter ideal in A also generated by a good system of parameters; actually *every* system of parameters for A is good, because A is an integral domain. We have by Theorem 4.1 $\mathrm{G}_Q(A) = \mathrm{G}_{\mathfrak{q}}(\widehat{A})$ to be a sequentially Cohen-Macaulay graded ring, while A itself is never sequentially Cohen-Macaulay.

REFERENCES

1. N.T. Cuong and D.T. Cuong, *On sequentially Cohen-Macaulay modules*, Kodai Math. J. **30** (2007), 409–428.

2. N.T. Cuong and H.L. Truong, *Parametric decomposition of powers of parameter ideals and sequentially Cohen-Macaulay modules*, Proc. Amer. Math. Soc. **137** (2009), 19–26.

3. S. Goto, *Approximately Cohen-Macaulay rings*, J. Algebra **76** (1982), 214–225.

4. S. Goto and Y. Shimoda, *Parametric decomposition of powers of ideals versus regularity of sequences*, Proc. Amer. Math. Soc. **132** (2004), 929–933.

5. ———, *On the parametric decomposition of powers of parameter ideals in a Noetherian local ring*, Tokyo J. Math. **27** (2004), 125–135.

6. S. Goto and K.-i. Watanabe, *On graded rings*, II ($\mathbf{Z}^n$-graded rings), Tokyo J. Math. **1** (1978), 237–261.

7. W. Heinzer, L.J. Ratliff and K. Shah, *Parametric decomposition of monomial ideals* (I), Houston J. Math. **21** (1995), 29–52.

8. P. Schenzel, *On the dimension filtration and Cohen-Macaulay filtered modules*, *Commutative algebra and algebraic geometry*, Freddy Van Oystaeyen, ed., Lect. Notes Pure Appl. Math. **206** (1999), Marcel Dekker, New York, 245–264.

9. R.P. Stanley, *Conbinatorics and commutative algebra,* Second Ed., Birkhäuser, Boston, 1996.

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan
Email address: goto@math.meiji.ac.jp

Hiroo Gakuen Junior and Senior High School, 5-1-14 Minami-Azabu, Minato-ku, Tokyo 106-0047, Japan
Email address: horiuchi@hiroogakuen.ed.jp

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan
Email address: hsakurai@math.meiji.ac.jp