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A CLASS OF LOCAL NOETHERIAN DOMAINS

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ABSTRACT. In this paper, we construct factorial domains with a given specific completion. The material is based on Nishimura's paper [**12**], which is based on the work of Rotthaus in [**17**], Ogoma in [**15**] and Heitmann in [**6**].

1. Introduction. In this paper we present a method of constructing local Noetherian rings which has been very fruitful over the past 25 years in the construction of various examples and counterexamples in commutative algebra. After being introduced first in 1979 this method has been simplified, modified, extended, and generalized by many authors resulting in numerous papers published and unpublished. Our goal here is to describe this method comprehensively so that the construction of additional examples is merely a matter of choosing the right ideal in a polynomial ring and plugging in the appropriate equations. The presentation is guided by unpublished notes of Jun-ichi Nishimura. We are grateful to Nishimura for allowing us to make use of his notes in this paper. Nishimura's notes also include a large number of new and previously unknown examples of Noetherian rings which are not included here.

Over the past 60 years important examples of Noetherian local rings have been constructed using so called non-standard methods. Non-standard methods are methods which go beyond the standard ways of constructing Noetherian rings like extensions of finite type, localization, completion, and Henselization. These non-standard methods can roughly be divided into three classes. The first class of examples, called here the Akizuki-Nagata method, goes back to at least Akizuki [**1**]. It was used and extended by Nagata [**10**] in his famous example of a normal local Noetherian domain whose completion is not a domain. The methods presented in this paper can be understood as a modification of the Akizuki-Nagata method. The local Noetherian rings

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constructed via the Akizuki-Nagata method are birational extensions of a polynomial ring in finitely many variables over a field.

In a 1993 paper, Heitmann [**7**] introduced a powerful new method, called Heitmann's method, of constructing local Noetherian rings. In particular, given a complete local ring T, he found in [**7**] necessary and sufficient conditions for T to be the completion of a factorial local Noetherian domain. Heitmann's method was further developed by Loepp [**8**] and others to construct a variety of counterexamples in commutative algebra. This method is based on a clever transfinite induction argument. Typically it is not known which elements are contained in the constructed example.

The third class of non-standard examples are those examples whose construction has not (yet) been generalized to produce a large class of different examples. These sporadic examples include the Ferrand-Raynaud example [**4**] and the example in [**18**].

Since, in this paper, the results are modifications of the Akizuki-Nagata method, we now give the idea on which that method is based.

Let K be a field, $x, z_1, \dots, z_t$ variables over K, and let $R = K[x, z_1, \dots, z_t]_{(x, z_1, \dots, z_t)}$ be the localized polynomial ring over K. Let $\rho_1, \dots, \rho_m \in (R, (x))^\wedge = \widetilde{R} = K[z_1, \dots, z_t]_{(z_1, \dots, z_t)}[[x]]$ be the power series in x which are algebraically independent over $K(x, z_1, \dots, z_t)$. By adjoining infinitely many elements of $\widetilde{R}$, which are related to $\rho_1, \dots, \rho_m$, a quasi local ring B is constructed whose (x)-adic completion is $\widetilde{R}$. Moreover, this ring B is a birational extension of $R[\rho_1, \dots, \rho_m]$. In many cases B is Noetherian and a homomorphic image of B provides a counterexample to a problem in commutative algebra, as for example, in Nagata's example of a normal local domain whose completion is analytically reducible. For a long time the proof that the constructed ring B is Noetherian remained notoriously difficult and the construction was restricted to examples of small dimension (≤ 3). Then in the 1990's Heinzer, Rotthaus, and Wiegand related the question of whether or not B is Noetherian to a flatness condition on algebras of finite type over K. This result allowed the construction of Noetherian examples of higher dimension.

In 1979 Rotthaus constructed a local Noetherian Nagata ring with a non-closed singular locus [**17**]. In order to obtain such an example the Akizuki-Nagata method had to be modified in the following way:

Instead of adjoining power series $\rho_1, \dots, \rho_m$ in the (x)-adic completion of R, the ρ's had to be chosen in any (p)-adic completion of R where p runs through the prime elements of R. To make such a construction work the field K needed to be countable and, in addition, a suitable enumeration of the prime elements of R had to be chosen. Under these modifications the ring B is contained in $R^* = \cap_p (R, (p))^\wedge$ where p runs through all prime elements of R. In particular, for every ideal $I \subseteq R^*$, all prime elements $p \in R$, and all $n \in \mathbf{N}$ the ideal $I + (p^n)$ is extended from R. The proof that B is Noetherian is, then, surprisingly simple.

In 1980 Ogoma constructed his celebrated counterexample to Nagata's chain conjecture [**14**]. In his paper Ogoma introduced yet another modification of the Akizuki-Nagata method which had far-reaching consequences for the construction of a wide variety of counterexamples. Before Ogama's modification, all examples constructed via the Akizuki-Nagata method (including the Rotthaus example) had been a homomorphic image of a regular local ring, namely B. Of course, Ogoma's example of a non-catenary ring could not possibly be such a homomorphic image. By a modest technical variation in the construction (namely, by adjoining so called 'front pieces' instead of 'end pieces') Ogoma constructed a local Noetherian domain A which is a birational extension of R and has completion $\widehat{A} = \widehat{R}/I$ for a specific ideal $I \subseteq \widehat{R}$.

In 1982 Heitmann [**6**] found a method (unrelated to Heitmann's method discussed earlier in this paper) to enumerate the prime elements of R which simplified the previous constructions. Shortly thereafter Brodmann and Rotthaus [**2**] used Ogoma's and Heitmann's ideas to show Theorem 10 of this paper. This theorem states that for an affine algebra $T = K[z_1, \dots, z_t]_{(z_1, \dots, z_t)}/I$ where K is a suitable countable field there is a local Noetherian domain A with completion $\widehat{A} \cong \widehat{T}[[x]]$. Moreover, there is an ideal in the generic formal fiber of A corresponding to the ideal $(z_1, \dots, z_t)$ in $\widehat{T}[[x]]$. This yields the construction of a large class of local Noetherian domains with bad formal fibers.

In 1982 Ogoma constructed a Cohen-Macaulay factorial domain which is not Gorenstein. In this construction he substantially modified his method in [**14**] to obtain a factorial domain. Nishimura, in his unpublished notes [**12, 13**], realized that by using Heitmann's enumeration lemma Ogoma's example could be generalized to large class of factorial local domains with bad formal fibers. This method basically obtains the same result as the Brodmann-Rotthaus theorem with three significant differences: (1) the constructed domains are in addition factorial, (2) it is no longer necessary to add an additional variable x, and (3) (as a disadvantage) the construction only works for a certain class of prime ideals I of the polynomial ring.

In this paper we present the Brodmann-Rotthaus theorem in Section 2 and the Ogoma-Nishimura result in Section 3. Throughout the paper we were guided by Nishimura's unpublished notes. The only change we made is in Section 3 in the definition of k-absolute prime ideals. By requiring that those prime ideals extend to prime ideals in the power series ring it is easier to understand that the prime ideals under consideration remain prime under a k-automorphism of the power series ring.

2. The Brodmann-Rotthaus theorem. In this section we present the basic construction which will provide examples of the following type: Let K be a field, $R = K[z_1, \dots , z_n]_{(z_1,\dots ,z_n)}$ the localized polynomial ring in n variables, and $I \subset R$ an ideal of R. Our goal is the construction of a local Noetherian ring A, which birationally dominates R and has completion isomorphic to $\widehat{R}/I\widehat{R}$. In order to make this construction work we need to impose some mild conditions on I concerning the zero divisors of the ring $\widehat{R}/I\widehat{R}$. We also need to require that the field K is countable and of countable transcendence degree over the prime field. Under these assumptions we can choose a suitable enumeration of a set of representatives of the height one prime ideals of R. Using this enumeration we then define a K-automorphism φ of the completion $\widehat{R} = K[[z_1, \dots , z_n]]$. Under the automorphism φ the ideal $I\widehat{R}$ is mapped into an ideal J of $\widehat{R}$ which has the property that the ring $\widehat{R}/J$ is R-torsion free. The desired example is the intersection ring $A = Q(R) \cap \widehat{R}/J$. In order to show that A is a Noetherian ring we need to describe A as a nested union of algebras essentially of finite type over K.

The exposition of this construction is divided as follows: In subsection 2.1 we introduce notation and choose a suitable enumeration (Heitmann's enumeration) of elements of the polynomial ring R. In subsection 2.2 we define the automorphism φ of the power series ring

$K[[z_1, \dots, z_n]]$ by defining the elements ζ_i. Then we introduce the ideal I and construct A, first, in subsection 2.3 as an intersection ring, and then in subsection 2.4 we show that A is equal to a nested union of rings which birationally dominate the localized polynomial ring $K[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$. Subsection 2.5 proves the Brodmann-Rotthaus theorem and subsection 2.6 is an historical note which describes how the early examples by Rotthaus [**17**] and Ogoma [**14**] can be derived from the Brodmann-Rotthaus result.

2.1. The enumeration. Let K_0 be a countable field, and let K be a purely transcendental extension field of K_0 of countable degree, with transcendence basis $\{a_{ik} \mid i = 1, \dots, m; k \in \mathbf{N}\}$ over K_0. We express K as

$$K = \bigcup_k K_k \text{ where } K_k = K_{k-1}(a_{1k}, \dots, a_{mk}) \text{ for } k \in \mathbf{N}.$$

Let $z_1, \dots, z_n$ with $m \leq n$ be variables over K, and let

$$\begin{aligned} S_0 &= K_0[z_1, \dots, z_n] && \text{with maximal ideal} \quad \mathfrak{N}_0 = (z_1, \dots, z_n)S_0 \\ S_k &= S_{k-1}[a_{1k}, \dots, a_{mk}] && \text{with prime ideal} \quad \mathfrak{N}_k = (z_1, \dots, z_n)S_k \\ S &= \bigcup_k S_k \\ &= K_0[\{a_{ik}\}_{i,k}][z_1, \dots, z_n] && \text{with prime ideal} \quad \mathfrak{N} = (z_1, \dots, z_n)S. \end{aligned}$$

Localize the polynomial rings at these prime ideals:

$$\begin{aligned} R_0 &= (S_0)_{\mathfrak{N}_0} = K_0[z_1, \dots, z_n]_{(z_1, \dots, z_n)} && \text{with } \mathfrak{n}_0 = (z_1, \dots, z_n)R_0, \\ R_k &= (S_k)_{\mathfrak{N}_k} = K_k[z_1, \dots, z_n]_{(z_1, \dots, z_n)} && \text{with } \mathfrak{n}_k = (z_1, \dots, z_n)R_k, \\ R &= S_{\mathfrak{N}} = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)} && \text{with } \mathfrak{n} = (z_1, \dots, z_n)R. \end{aligned}$$

We write $R_k = R_{k-1}(a_{1k}, \dots, a_{mk})$, which means R_k is a localization of $R_{k-1}[a_{1k}, \dots, a_{mk}]$ at the prime ideal $(z_1, \dots, z_n)$. Note that $(R, \mathfrak{n})$ is a countable regular local ring. Moreover, $R = \cup_k R_k$ is a nested union of smaller localized polynomial rings.

Suppose $\Gamma \subseteq \mathfrak{N} = (z_1, \dots, z_n)S$ is a set of elements so that

(α) $0 \notin \Gamma$

(β) $z_1 + \cdots + z_n \in \Gamma$

(γ) for all $\mathfrak{p} \in \operatorname{Spec}(R_k)$ with $\mathfrak{p} \neq (0)$ we have that $\mathfrak{p} \cap \Gamma \neq \varnothing$.

(δ) $S_k \cap \Gamma$ is (countably) infinite for all k.

Since S is countable, Γ is countable.

We fix an enumeration $\rho : \mathbf{N} \to \Gamma$ with the following properties:

(a) $\rho(1) = z_1 + \cdots + z_n$

(b) For all $k \geq 2$: $\rho(k) \in S_{k-2}$,

and set $p_k = \rho(k)$.

We frequently use the following observation:

$$R_{k-2} \cap S = (S_{k-2})_{\mathfrak{N}_{k-2}} \cap S = S_{k-2}.$$

This follows from the assumption that the a_{ij} for $j \geq k-1$ are algebraically independent over the quotient field $Q(S_{k-2})$.

Let $\varepsilon_1, \ldots, \varepsilon_k, \ldots$ be a strictly increasing sequence of positive integers, for example, $\varepsilon_k = k$ for all $k \in \mathbf{N} \setminus (0)$. Define:

$$\begin{aligned} p_1 &= z_1 + z_2 + \cdots + z_n \\ z_{i0} &= z_i \\ q_k &= p_1 \cdots p_k \\ z_{ik} &= z_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{ik} q_k^{\varepsilon_k} \text{ for } k \geq 1 \text{ and } i \leq m \\ \mathfrak{P}_k &= (z_{1k}, \ldots, z_{mk}) R \text{ where } m \leq n. \end{aligned}$$

Note that the elements $z_{1k}, \ldots, z_{mk}$ form part of a regular system of parameters of R and that $\mathfrak{P}_k$ is a prime ideal of R of height m for all $k \geq 0$.

At this stage the choice of an increasing sequence of positive integers $\{\varepsilon_i\}$ other than $\{\varepsilon_i = i\}$ seems an unnecessary complication. However, a proper choice of the ε_i's becomes crucial in the second part of this paper where factorial rings will be constructed.

Theorem 1 (Heitmann's numbering lemma). *Suppose that $m < n$. Then for all integers $k \geq 0$ and all positive integers $h \leq k+1$: $p_h \notin \mathfrak{P}_k$.*

In order to prove the theorem we will show first by induction on k that the ideals $(z_{1k}, \dots, z_{sk})S_k$ are prime for all $1 \leq s \leq m$. This requires some lemmas.

Lemma 2. *Let A be a Noetherian domain, t a variable over A, and q, w a regular sequence in A. Then $(qt - w)A[t]$ is a prime ideal in $A[t]$.*

Proof. Note that $(qt - w)$ is an ideal contained in the kernel of the A-algebra morphism:

$$\varphi : A[t] \longrightarrow A\left[\frac{w}{q}\right]$$

where t maps to w/q. Since $(qt-w)$ is a prime ideal of $A_q[t]$ and q is not a zero divisor of A, we have to show: if $f(t) \in A[t]$ with $q^r f(t) \in (qt-w)$ then $f(t) \in (qt - w)$. Suppose that

$$f(t) = \sum_{i=0}^{m} a_i t^i$$

and $q^r f(t) \in (qt - w)$. This implies that $q^r a_0 \in (w)A$. Since w, q is also a regular sequence of A we have that $a_0 \in (w)$. Consider the polynomial

$$g(t) = \sum_{i=1}^{m} b_i t^i$$

where $a_0 = c_1 w$, $b_1 = a_1 - c_1 q$, and $b_i = a_i$ for $i > 1$. Then $g(t) = f(t) - c_1(qt - w)$ and $f(t) \in (qt-w)$ if and only if $g(t) \in (qt-w)$. A similar argument shows that $b_1 \in (w)$ and we can replace $g(t)$ by a polynomial of the same degree as f which is divisible by t^2. The process stops either if we obtain the zero polynomial showing that $f(t) \in (qt - w)$ or with a polynomial of the form ct^m, in which case we must have that $c = 0$. $\square$

Lemma 3. *Let A be a Noetherian domain and suppose that $q, w_1, \dots, w_m$ is a regular sequence of A. Then $q, w_2, \dots, w_m$ is a regular sequence of $A[w_1/q]$.*

Proof. We know from the first lemma, that

$$A\left[\frac{w_1}{q}\right] \cong A[t]/(qt - w_1).$$

Thus it is enough to show that $qt - w_1, q, w_2, \dots, w_m$ is a regular sequence of $A[t]$. The proof of the first lemma shows that $qt - w_1, q$ is a regular sequence of $A[t]$. Since $(qt - w_1, q) = (q, w_1)A[t]$ the statement follows since $A[t]$ is flat over A. □

Lemma 4. *Let A be a Noetherian domain and $q, w_1, \dots, w_m$ a regular sequence in A. The ideal $(qt_1 - w_1, \dots, qt_m - w_m)$ is a prime ideal in the polynomial ring $A[t_1, \dots, t_m]$.*

Proof. From Lemma 2 and Lemma 3. □

Proof of Theorem 1. We want to show by induction on k:

(a) For all $1 \le s \le m$ the ideal $(z_{1k}, \dots, z_{sk})$ is a prime ideal in S_k.

(b) $q_h \notin \mathfrak{P}_k$ for all $h \le k + 1$.

The statement is clear for $k = 0$. Suppose that statements (a) and (b) are true for $k - 1$. Then q_k is not in $\mathfrak{P}_{k-1}$; in particular, q_k is not in $(z_{1(k-1)}, \dots, z_{m(k-1)})S_{k-1}$. By the induction hypothesis $(z_{1(k-1)}, \dots, z_{s(k-1)})S_{k-1}$ is a prime ideal in S_{k-1} yielding that $z_{1(k-1)}, \dots, z_{s(k-1)}, q_k$ is a regular sequence in S_{k-1} for all $1 \le s \le m$. An induction argument shows that the sequence $q_k, z_{1(k-1)}, \dots, z_{m(k-1)}$ is also regular and so is the sequence $q_k^{\varepsilon_k}, z_{1(k-1)}, \dots, z_{m(k-1)}$. Considering

$$z_{ik} = z_{i(k-1)} + a_{ik} q_k^{\varepsilon_k}$$

as linear polynomials in the variables a_{ik}, it follows from Lemma 4 that the ideals $(z_{1k}, \dots, z_{sk})$ are prime in S_k. This shows (a) for k.

In order to show (b) observe that the elements $a_{1k}, \dots, a_{nk}$ are variables over S_{k-1} and its field of quotients $L_{k-1} = Q(S_{k-1})$. Thus

$$(z_{1k}, \dots, z_{mk})L_{k-1}[a_{1k}, \dots, a_{mk}]$$

is a proper ideal in the polynomial ring $L_{k-1}[a_{1k}, \dots, a_{mk}]$. Obviously,

$$(z_{1k}, \dots, z_{mk})L_{k-1}[a_{1k}, \dots, a_{mk}] \cap L_{k-1} = (0).$$

This implies that:

$$(z_{1k}, \dots, z_{mk})S_{k-1}[a_{1k}, \dots, a_{mk}] \cap S_{k-1} = (z_{1k}, \dots, z_{mk})S_k \cap S_{k-1} = (0).$$

R_k is a localization of S_k and $(z_{1k}, \dots, z_{mk})$ is a prime ideal in S_k. Therefore:

$$\mathfrak{P}_k \cap S_k = (z_{1k}, \dots, z_{mk})R_k \cap S_k = (z_{1k}, \dots, z_{mk})S_k$$

and

$$\mathfrak{P}_k \cap R_{k-1} = (0).$$

Since $p_h \in R_{k-1}$ for all $h \leq k+1$, assertion (b) follows. □

2.2. The automorphism φ. Suppose that we have chosen an enumeration p_i of the elements of Γ so that $p_1 = z_1 + \dots + z_n$ and that for all $k \geq 2$: $p_k \in S_{k-2}$. Let ε_i be a sequence of strictly increasing positive integers. As before, we put for all $k \geq 1$:

$$q_k = p_1 \cdots p_k$$

and, for all $1 \leq i \leq m$ and for all $k \geq 1$:

$$z_{ik} = z_i + a_{i1}q_1^{\varepsilon_1} + a_{i2}q_2^{\varepsilon_2} + \dots + a_{ik}q_k^{\varepsilon_k}.$$

For $m+1 \leq j \leq n$ we set $z_{jk} = z_j$.

We let ζ_i be the power series which is the limit of the z_{ik} for $k \to \infty$, that is:

$$\zeta_i = z_i + \sum_{l=2}^{\infty} a_{il}q_l^{\varepsilon_l} \in K[[z_1, \dots, z_n]]$$

if $1 \leq i \leq m$ and $\zeta_j = z_j$ for $m+1 \leq j \leq n$. Note that the elements $\zeta_1, \dots, \zeta_m, z_{m+1}, \dots, z_n$ form a regular system of parameters of the power series ring $K[[z_1, \dots, z_n]] = K[[\zeta_1, \dots, \zeta_m, z_{m+1}, \dots, z_n]]$. The map

$$z_i \longmapsto \varphi(z_i) = \zeta_i \text{ if } 1 \leq i \leq m$$
$$z_i \longmapsto \varphi(z_i) = z_i \text{ if } m+1 \leq i \leq n$$

extends uniquely to an automorphism φ of the power series ring $K[[z_1, \dots, z_n]]$.

In this first section the automorphism φ changes the first m variables:

$$\varphi(z_i) = \zeta_i = z_i + \sum_{l=2}^{\infty} a_{il} q_l^{\varepsilon_l} \text{ for } 1 \leq i \leq m$$

while it leaves the remaining $n - m > 0$ variables $z_{m+1}, \dots, z_n$ fixed. In the next section on the construction of factorial domains we need to change all variables. Since $m < n$ is a necessary condition in Heitmann's lemma, this will require a modified enumeration of Γ.

2.3. The intersection ring A**.** Let $I \subseteq R = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$ be an ideal that satisfies the following condition:

$(*)$ For every associated prime ideal:

$$Q \in \operatorname{Ass}\left(K[[z_1, \dots, z_n]] / IK[[z_1, \dots, z_n]]\right)$$

we have that $Q \subseteq (z_1, \dots, z_m)K[[z_1, \dots, z_n]]$.

Under the automorphism φ of $K[[z_1, \dots, z_n]]$ (as defined in (2.2)) the ideal $IK[[z_1, \dots, z_n]]$ is mapped into an ideal J of $K[[z_1, \dots, z_n]]$ which then satisfies the following condition:

$(**)$ For every associated prime ideal:

$$Q' \in \operatorname{Ass}\left(K[[z_1, \dots, z_n]]/J\right)$$

we have that $Q' \subseteq (\zeta_1, \dots, \zeta_m)K[[z_1, \dots, z_n]]$. In particular, $J \subseteq (\zeta_1, \dots, \zeta_m)K[[z_1, \dots, z_n]]$.

We define $\widehat{P} = (\zeta_1, \dots, \zeta_m)K[[z_1, \dots, z_n]]$. Since the elements $\zeta_1, \dots, \zeta_m$ are part of a regular system of parameters of $\widehat{R} = K[[z_1, \dots, z_n]]$ we obtain that $\widehat{P}$ is a prime ideal of height m in $\widehat{R} = K[[z_1, \dots, z_n]]$.

Lemma 5. $\widehat{P} \cap R = (0)$.

Proof. Suppose that the intersection is not trivial. Since Γ has a nonempty intersection with every nonzero prime ideal of R, there is an integer $h \in \mathbf{N}$ such that $p_h \in \widehat{P} \cap R$. By construction of the ζ's this implies that

$$p_h, z_{1(h-1)}, \dots , z_{m(h-1)} \in \widehat{P} \cap R.$$

The elements $z_{1(h-1)}, \dots , z_{m(h-1)}$ form part of a regular sequence in R, which implies that the ideal $\mathfrak{P}_{h-1} = (z_{1(h-1)}, \dots , z_{m(h-1)})$ is a prime ideal of height m in R. By Heitmann's numbering lemma, $p_h \notin \mathfrak{P}_{h-1}$. This yields a contradiction since the intersection ideal $\widehat{P} \cap R$ is a prime ideal of height at most m. □

The lemma implies that $J \cap R = (0)$ and that the composition of natural maps:

$$R \xrightarrow{\delta} \widehat{R} \xrightarrow{\nu} \widehat{R}/J$$

gives an embedding $\pi = \nu\delta$ of R into $\widehat{R}/J$:

$$\pi : R \longrightarrow \widehat{R}/J.$$

By condition $(**)$ every nonzero element of R is mapped via π into a non-zerodivisor of $\widehat{R}/J$.

The ring of interest is the intersection ring:

$$A = Q(R) \cap (\widehat{R}/J) = Q(R) \cap (K[[z_1, \dots , z_n]]/J).$$

In the next section we show that A is a local Noetherian ring with completion $\widehat{R}/J$. As pointed out in [**5**] it is usually difficult to describe what the elements in the intersection ring A are. In this particular construction we are able to describe the intersection ring A in a different way. We first construct a ring B contained in A which is a nested union of essentially finitely generated algebras over K. B is a local Noetherian ring with completion $\widehat{R}/J$. Since $Q(A) = Q(B)$ by flatness it follows that $A = B$.

2.4. Construction of the nested union ring B. Let $I \subseteq R$ be an ideal which satisfies condition $(*)$ and let $F_1(z_1,\dots,z_n),\dots, F_r(z_1,\dots,z_n) \in K[z_1, \dots , z_n]$ be a generating set for I. The elements $f_j = \varphi(F_i(z_1, \dots , z_n)) = F_i(\zeta_1, \dots , \zeta_m, z_{m+1}, \dots , z_n)$ form a generating system of the ideal J in $K[[z_1,\dots,z_n]]$. With the notation of (2.1) and (2.2):

$$\zeta_i = z_i + \sum_{l=1}^{\infty} a_{il} q_l^{\varepsilon_l} \in K[[z_1,\dots,z_n]] \text{ for } 1 \le i \le m$$

$$z_{ik} = z_i + \sum_{l=1}^{k} a_{il} q_l^{\varepsilon_l} \in K[z_1,\dots,z_n] \text{ for } 1 \le i \le m \text{ and}$$

$$z_{ik} = z_i \text{ for } m+1 \le i \le n.$$

We consider for all $1 \le j \le m$ the following elements in the field of quotients of R:

$$\alpha_{jk} = \frac{1}{q_k^{\varepsilon_k}} F_j(z_{1k},\dots,z_{mk},z_{m+1},\dots,z_n) \in K(z_1,\dots,z_n).$$

A similar construction is used in [**5**] where accordingly the α's are called "front pieces." Consider:

$$\alpha_{j(k+1)} = \frac{1}{q_{k+1}^{\varepsilon_{k+1}}} F_j(z_{1(k+1)},\dots,z_{m(k+1)},z_{m+1},\dots,z_n)$$
$$= \frac{1}{q_{k+1}^{\varepsilon_{k+1}}} F_j(z_{1k}+a_{1(k+1)}q_{k+1}^{\varepsilon_{k+1}},\dots,z_{mk}+a_{m(k+1)}q_{k+1}^{\varepsilon_{k+1}},z_{m+1},\dots,z_n).$$

By Taylor's formula:

$$\alpha_{j(k+1)} = \frac{1}{q_{k+1}^{\varepsilon_{k+1}}} F_j(z_{1k},\dots,z_{mk},z_{m+1},\dots,z_n) - s_{jk}$$

with $s_{jk} \in R$. This yields a *recursion formula* which is essential for the remainder of construction:

$$\alpha_{jk} = \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\varepsilon_k}}[\alpha_{j(k+1)} + s_{jk}].$$

Note that

$$s_{jk} \in R \text{ and } \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\varepsilon_k}} \in R.$$

Lemma 6. *For all* $1 \leq j \leq r$ *and all* $k \in \mathbf{N}$:

$$\alpha_{jk} \in Q(R) \cap (\widehat{R}/J).$$

Proof. We have that

$$q_k^{\varepsilon_k} \alpha_{jk} = F_j(z_{1k}, \dots, z_{mk}, z_{m+1}, \dots, z_n)$$

which implies

$$\begin{aligned} f_j - q_k^{\varepsilon_k} \alpha_{jk} &= F_j(\zeta_1, \dots, \zeta_m, z_{m+1}, \dots, z_n) \\ &\quad - F_j(z_{1k}, \dots, z_{mk}, z_{m+1}, \dots, z_n) \\ &= q_{k+1}^{\varepsilon_{k+1}} \eta_{jk} \end{aligned}$$

for some element $\eta_{jk} \in \widehat{R}$. $f_j \in J$ for all $1 \leq j \leq r$ and therefore $f_j = 0$ in $\widehat{R}/J$. Thus:

$$q_k^{\varepsilon_k} \alpha_{jk} = -q_{k+1}^{\varepsilon_{k+1}} \eta_{jk} \text{ in } \widehat{R}/J.$$

Since $q_k^{\varepsilon_k}$ is regular on $\widehat{R}/J$, it follows that:

$$\alpha_{jk} = \frac{1}{q_k^{\varepsilon_k}} F_j(z_{1k}, \dots, z_{mk}, z_{m+1}, \dots, z_n) \in \widehat{R}/J. \qquad \square$$

As described below the subring B is obtained from R by essentially adjoining all the elements $\alpha_{jl} \in Q(R)$.

Note that the canonical injection:

$$\pi : R \longrightarrow \widehat{R}/J$$

extends to a commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\quad \pi \quad} & \widehat{R}/J \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \widehat{\tau}} \\ Q(R) & \xrightarrow{\quad \lambda \quad} & S^{-1}(\widehat{R}/J) \end{array}$$

where $S = R \setminus (0)$ and $\tau, \widehat{\tau}$ are the natural maps. By assumption $\widehat{R}/J$ is R-torsionfree and the map $\widehat{\tau}$ is injective. We have just shown that $\alpha_{ik} \in \widehat{R}/J$. Thus the restriction of λ yields for all $k \in \mathbf{N}$ injective morphisms:

$$\lambda_k : R[\alpha_{1k}, \dots , \alpha_{rk}] \longrightarrow \widehat{R}/J.$$

Let

$$\mathfrak{w}_k = \lambda_k^{-1}(\mathfrak{m}),$$

be the contraction of the maximal ideal $\mathfrak{m}$ of $\widehat{R}/J$. For all $k \in \mathbf{N}$ set:

$$B_k = R[\alpha_{1k}, \dots , \alpha_{rk}]_{\mathfrak{w}_k}.$$

Using the recursion formula for the α_{jk}'s:

$$\alpha_{jk} = \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\varepsilon_k}}[\alpha_{j(k+1)} + s_{jk}]$$

we see that there are canonical inclusions:

$$R[\alpha_{1k}, \dots , \alpha_{rk}] \subseteq R[\alpha_{1(k+1)}, \dots , \alpha_{r(k+1)}]$$

which yield inclusions of the local rings:

$$B_k \subseteq B_{k+1}$$

for all $k \in \mathbf{N}$. The desired ring B is obtained as the nested union:

$$B = \bigcup_{k \in \mathbf{N}} B_k.$$

It is obvious that B is a quasilocal ring which is contained in $A = Q(R) \cap \widehat{R}/J$. In the next section we show that B is a local Noetherian ring with completion $\widehat{B} = \widehat{R}/J$. This implies that $A = B$.

2.5. The main results.

Proposition 7. *B is a local Noetherian ring. Moreover, for every nonzero prime ideal $\mathfrak{q} \subseteq B$ the quotient ring $B/\mathfrak{q}$ is essentially of finite type over K.*

Proof. We want to show that every prime ideal of B is finitely generated. Let $\mathfrak{q} \subseteq B$ be a nonzero prime ideal. Since $B \subseteq Q(R)$, the intersection $\mathfrak{q} \cap R \neq (0)$ and there is an $l \in \mathbf{N}$ such that $p_l \in \Gamma \cap \mathfrak{q}$. We claim that the natural morphism:

$$\nu : R \longrightarrow B/p_l B$$

is surjective.

Let $\omega \in B$ be a nonzero element. By the definition of B there is an integer $k \in \mathbf{N}$ with $k \geq l$ so that $\omega \in B_k = R[\alpha_{1k}, \dots, \alpha_{rk}]_{\mathfrak{w}_k}$. Hence there are polynomials $h, g \in R[x_1, \dots, x_r]$ so that

$$\omega = \frac{h(\alpha_{1k}, \dots, \alpha_{rk})}{g(\alpha_{1k}, \dots, \alpha_{rk})}$$

where $g(\alpha_{1k}, \dots, \alpha_{rk})$ is a unit in $\widehat{R}/J$. Since all the elements α_{ik} are contained in the maximal ideal of $\widehat{R}/J$ the constant term g_0 of g is a unit in R. Let h_0 denote the constant term of h. We use the recursion formula again and write:

$$\alpha_{jk} = \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\varepsilon_k}}[\alpha_{j(k+1)} + s_{jk}].$$

Since $k \geq l$ and $\varepsilon_{k+1} > \varepsilon_k$ the element p_l divides α_{jk} in B_{k+1}. Thus we may write:

$$\omega = \frac{h_0 + p_l \rho}{g_0 + p_l \lambda}$$

where ρ and λ are elements of B_{k+1}. This shows that

$$\omega = \frac{h_0}{g_0} \text{ in } B/p_l B$$

where $h_0, g_0 \in R$ and g_0 a unit in R. Thus ν is surjective and $B/p_l B$ is a homomorphic image of R. □

Proposition 8. $\widehat{B} = \widehat{R}/J$.

Proof. The embeddings

$$R \hookrightarrow B \hookrightarrow \widehat{R}/J$$

induce morphisms on the completions:

$$\widehat{R} \xrightarrow{\iota} \widehat{B} \xrightarrow{\widehat{\mu}} \widehat{R}/J$$

with $\mu\iota = \widehat{\pi}$ the natural map from $\widehat{R}$ to $\widehat{R}/J$. Proposition 7 shows that the maximal ideal of B is generated by $z_1, \dots, z_n$. By [**6**, Theorem 8.4] the ring $\widehat{B}$ is a finitely generated $\widehat{R}$-module with generator $\iota(1) = 1$. Thus the induced morphism ι on the completions is surjective. It suffices to show that $\ker(\iota) = J$. Obviously $\ker(\iota) \subseteq J$.

Claim. $J \subseteq \ker(\iota)$.

We have shown earlier (proof of Lemma 6) that

$$\begin{aligned} f_j - q_k^{\varepsilon_k}\alpha_{jk} &= F_j(\zeta_1, \dots, \zeta_m, z_{m+1}, \dots, z_n) \\ &\quad - F_j(z_{1k}, \dots, z_{mk}, z_{m+1}, \dots, z_n) \\ &= q_{k+1}^{\varepsilon_{k+1}}\eta_{jk} \text{ for all } \quad 1 \le j \le r;\; k \in \mathbf{N} \end{aligned}$$

for some element $\eta_{jk} \in \widehat{R}$. Thus $\iota(f_j) \in q_k^{\varepsilon_k}\widehat{B}$ for all $k \in \mathbf{N}$ and $\iota(f_j) = 0$. □

Proposition 9. *The ideal* $\widehat{P}_0 = (\zeta_1, \dots, \zeta_m)\widehat{R}/J$ *is in the generic formal fiber of* B*, that is,*

$$B \cap (\zeta_1, \dots, \zeta_m)\widehat{B} = (0).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\mu} & \widehat{R}/J \\ \uparrow & & \uparrow \widehat{\pi} \\ R & \xrightarrow[=]{} & \widehat{R}. \end{array}$$

$\widehat{P}_0$ is the image of $\widehat{P}$ under $\widehat{\pi}$. By Lemma 5, $\widehat{P} \cap R = (0)$ and the assertion follows, since $J \subseteq \widehat{P}$. □

We summarize the results in the following theorem:

Theorem 10. *Let K be a countable field of infinite transcendence degree over a prime field, and let $R = K[z_1, \dots, z_n]_{(z_1,\dots,z_n)}$ be the localized polynomial ring in n variables. Let $m \in \mathbf{N}$ be an integer with $m < n$ and $I \subseteq R$ an ideal which satisfies the following condition*:

($*$) *For every associated prime ideal $\mathfrak{Q} \in \operatorname{Ass}(\widehat{R}/I\widehat{R})$ it holds that $\mathfrak{Q} \subseteq (z_1, \dots, z_m)\widehat{R}$ where $m < n$.*

Then there is an automorphism φ of the completion $\widehat{R} = K[[z_1, \dots, z_n]]$ and a local Noetherian domain B which birationally dominates R with the following properties:

(a) $\widehat{B} = \widehat{R}/\varphi(I\widehat{R}) \cong \widehat{R}/I\widehat{R} = K[[z_1, \dots, z_n]]/IK[[z_1, \dots, z_n]]$.

(b) *For every nonzero prime ideal $\mathfrak{q} \subseteq B$ the ring $B/\mathfrak{q}$ is essentially of finite type over K.*

(c) *The prime ideal $\widehat{P}_0 = (\varphi(z_1), \dots, \varphi(z_m))\widehat{B}$ is in the generic formal fiber of B.*

2.6. Historical note. The first example using a suitable enumeration of the prime elements of R was constructed in [**17**]. In this paper a local Nagata ring A is produced which has a non-open regular locus in Spec(A). The construction starts with a four-dimensional localized polynomial ring $R = K[z_1, z_2, z_3, z_4]_{(z_1,z_2,z_3,z_4)}$. Using an appropriate enumeration of the prime elements of R similar to subsection 2.2, two algebraically independent elements ζ_1 and ζ_2 are constructed which yield a transcendental element $\omega = \zeta_1\zeta_2$. The construction in [**17**] differs from the construction in Section 2 in the following way. In [**17**] so-called 'end-pieces' of ω (instead of 'front-pieces' as in Section 2) are adjoined to R in order to show that the intersection ring $A_0 = Q(R)(\omega) \cap \widehat{R}$ is a nested union of five-dimensional localized polynomial rings. This is used to show that A_0 is a regular local ring with completion $\widehat{A}_0 = \widehat{R} = K[[z_1, z_2, z_3, z_4]]$. Moreover, A_0 is a non excellent Nagata ring, $\mathfrak{p} = (\omega)$ is a prime ideal in A_0, and the ring $A = A_0/(\omega)$ has a non open regular locus. By setting $I = (z_1z_2)$, Theorem 10 in Section 2 recovers the ring A of [**17**].

In 1980 Ogoma provided his celebrated counterexample to Nagata's chain conjecture [**14**]. The chain conjecture states that the normalization of a Noetherian domain is universally catenary. Ogoma constructs a local Nagata ring A whose completion $\widehat{A}$ fails to be equidimensional implying that A is not universally catenary. Of course, the ring A is not a homomorphic image of a regular local ring. In [**14**] Ogoma realized that he needed to adjoin 'front-pieces' (instead of 'end-pieces' as in [**17**]) in order to directly produce a local Noetherian ring with completion a homomorphic image of $\widehat{R}$. Let $R = K[z_1, z_2, z_3, z_4]_{(z_1,z_2,z_3,z_4)}$ be the four-dimensional localized polynomial ring over a countable field K and $I = (z_1z_2, z_1z_3) \subseteq R$. The ring B of Theorem 10 is Ogoma's example.

Both examples in [**14, 17**] have in common that while it is easy to show that the constructed rings are Noetherian, it is rather tedious to find a suitable enumeration of the prime elements of R. In [**6**] Heitmann simplified Ogoma's construction by showing that any enumeration which satisfies conditions (a) and (b) of subsection 2.1 implies Theorem 1 (Heitmann's numbering lemma). Using Heitmann's numbering lemma in [**2**] Brodmann and Rotthaus showed essentially Theorem 10 (more precisely, they proved Theorem 10 if $m \leq n-1$).

3. The Ogoma-Nishimura result. In this section our objective is a modification of the construction in Section 2 in order to produce factorial rings B. Examples of this type were first introduced by Ogoma [**9**]. Our presentation here uses material from Nishimura's paper [**7**], in which Ogoma's construction has been simplified by incorporating Heitmann's enumeration lemma into the construction. A modification of the construction to produce factorial rings B is a nontrivial exercise that requires several additional considerations. First it is obvious that our choice of ideals $I \subseteq R$ is restricted. Ogoma introduced the notion of absolute prime ideals which impose the "right" conditions on I; however, putting more conditions on I is not enough. The major change in the construction is that we need to construct a certain sequence of prime elements $p_1, \dots, p_k, \dots$ in R which will play the role of Γ. In the sequence of the p_k's repetitions are allowed, that is, for some k it may happen that $p_k = p_{k+1}$. Therefore, strictly speaking, this sequence may not be an enumeration of some subset of R. The difficulty is that the sequence of the p_k's is constructed inductively along with the exponents ε_{k-1} which occur in the construction of the power series ζ_i. However, this construction is guided by an enumeration of the set Γ which will be defined as in Section 2. The reader should be aware that in this section I will be renamed Q representing a prime ideal.

In subsection 3.1 we first choose a suitable enumeration of all prime elements of R using Heitmann's lemma. In addition, k-absolute prime ideals are defined. Ogoma's construction is based on an inductive argument that constructs from the given enumeration of all prime elements of R an appropriate infinite subsequence. This process is described in subsections 3.2 and 3.3. In subsection 3.4 the desired ring is constructed by making use of this subsequence of prime elements of R and the Ogoma-Nishimura result is proven. The historical note, subsection 3.5, describes again how the first examples by Ogoma [**15**] and Weston [**19**] can be obtained from the Ogoma-Nishimura result.

3.1. The set up. We start with the same general assumptions as in the standard example. Let $K = \cup_k K_k$ be a countable field which is the nested union of fields $K_k = K_{k-1}(a_{1k}, \dots , a_{nk})$, where the elements $a_{1k}, \dots , a_{nk}$ are algebraically independent over K_{k-1}. We define the rings S_k, S, R_k and R as before, and take a subset $\Gamma \subset \mathfrak{N} = (z_1, \dots , z_n)S$ that satisfies the same conditions as in Section 2, namely:

(a) Every element of Γ is a prime element of R.

(b) For all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \neq (0)$ there is at least one element in $\mathfrak{p} \cap \Gamma$.

Next we take an enumeration

$$\rho : \mathbf{N} \longrightarrow \Gamma,$$

put $\rho(i) = \rho_i$, and assume that ρ satisfies the following conditions:

(Γ_a) $\rho_1 \in (z_1, \dots , z_n)k_0[z_1, \dots , z_n]$ is a prime element which is not contained in the prime ideal $(z_1, \dots , z_m)$ for some $m < n$. After introducing the ideal $Q \cong I$ we need to be more specific about the choice of ρ_1 and m. Indeed the construction only works for certain ideals Q together with a particular choice of the prime element ρ_1.

(Γ_b) For all $k \geq 2$: $\rho_k \in S_{k-2}$.

We want to construct inductively a sequence of elements in Γ:

$$p_1, \dots, p_k, \dots, \dots$$

If $p_1, \dots, p_n$ have been constructed we put $q_k = p_1 \dots p_k$ for all $1 \leq k \leq n$. Along with the inductive construction of p_k's we construct a sequence of strictly increasing integers $\varepsilon_i \in \mathbf{N}$. We define according to Section 2:

$$\begin{aligned} z_{i0} &= z_i \\ z_{ik} &= z_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{ik} q_k^{\varepsilon_k} \text{ for } k \geq 1, \end{aligned}$$

where i ranges from $1 \leq i \leq n$. Obviously, the elements $z_{1k}, \dots, z_{nk}$ form a regular system of parameters in R for every $k \geq 0$. Again we denote by $\mathfrak{P}_k$ the prime ideal $(z_{1k}, \dots, z_{mk})$ of R. In subsection 2.1 we used the condition $p_k \in S_{k-2}$ to prove Heitmann's lemma, namely, that $p_h \notin \mathfrak{P}_k$ for all $h \leq k+1$. Note that the proof of Heitmann's lemma requires that at least one variable z_{m+1} remain unchanged. In our new construction all variables $z_1, \dots, z_n$ need to be changed. The sequence $p_1, \dots, p_k, \dots$ is distinguished from the sequence $\rho_1, \dots, \rho_k, \dots$ in that we can no longer force $p_k \in S_{k-2}$ for all $k \in \mathbf{N}$. However, later we will use the fact that $\rho_k \in S_{k-2}$ to prove that

$$p_h \notin \mathfrak{P}_k \text{ for all } h \leq k+1.$$

To begin the construction we first define:

Definition. Let k be a field, $z_1, \dots, z_n$ variables over k, and $Q \subseteq (z_1, \dots, z_n)k[z_1, \dots, z_n]$ a prime ideal. We call Q a *k-absolute prime* if for all field extensions L of k the ideal $QL[[z_1, \dots, z_n]]$ is a prime ideal in the power series ring $L[[z_1, \dots, z_n]]$.

This definition of a k-absolute prime is a modification of Ogoma's definition of an absolute prime [**9**].

In the following let $Q_0 \subseteq (z_1, \dots, z_n)K_0[z_1, \dots, z_n]$ be a prime ideal which satisfies the following three conditions:

(I_a) Q_0 is a K_0-absolute prime ideal.

(I_b) There is an $m < n$ such that for every associated prime ideal $\mathfrak{Q} \in \operatorname{Ass}(\widehat{R}/Q_0\widehat{R})$ we have that $\mathfrak{Q} \subseteq (z_1, \ldots, z_m)\widehat{R}$.

(I_c) The element ρ_1 is a prime element in the domain $L[[z_1, \ldots, z_n]]/Q_0L[[z_1, \ldots, z_n]]$ for all field extensions L of K_0.

From now on we assume that the enumeration ρ of Γ is chosen in such a way that in condition (Γ_a) $\rho_1 \notin (z_1, \ldots, z_m)$ where $m < n$ is given by condition (I_b). Furthermore we assume that ρ_1 satisfies condition (I_c) from above.

Before continuing with a key observation for the induction step we fix generators for Q_0:

$$Q_0 = (f_1(z_1, \ldots, z_n), \ldots, f_r(z_1, \ldots, z_n)).$$

3.2. The key lemma. Let k be a field and $z_1, \ldots, z_n$ variables over k. We denote by $D = k[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)}$ the localized polynomial ring over k and by $\mathfrak{m} = (z_1, \ldots, z_n)$ the maximal ideal of D. Suppose that $Q_0 = (f_1, \ldots, f_r) \subseteq (z_1, \ldots, z_n)k[z_1, \ldots, z_n]$ is a k-absolute prime ideal of $k[z_1, \ldots, z_n]$, that is, for all field extensions L of k the ideal

$$Q_0L[[z_1, \ldots, z_n]]$$

is prime in $L[[z_1, \ldots, z_n]]$.

Let $a_1, \ldots, a_n$ be variables over D, and let $D(a)$ denote the local ring

$$D(a) = D[a_1, \ldots, a_n]_{\mathfrak{m}D[a]}.$$

$D(a)$ is a regular local ring with regular system of parameters $z_1, \ldots, z_n$. We denote by $k(a)$ the purely transcendental extension field $k(a_1, \ldots, a_n)$ of k.

Suppose that $p_1, \ldots, p_k \in (z_1, \ldots, z_n)k[z_1, \ldots, z_n]$ are prime elements in D and let $\varepsilon \in \mathbf{N}$ be an integer with $\varepsilon > 1$. Put

$$q = (p_1 \cdots p_k)^{\varepsilon}.$$

We first observe:

Proposition 11. *The ideal* $Q_1 = (f_1(z_1 + a_1q, \ldots, z_n + a_nq), \ldots, f_r(z_1 + a_1q, \ldots, z_n + a_nq)) \subseteq D(a)$ *is a* $k(a)$*-absolute prime ideal in* $D(a)$.

Proof. The elements $z_1 + a_1q, \ldots, z_n + a_nq$ form a regular system of parameters in $D(a)$ and its completion $k(a)[[z_1, \ldots, z_n]]$. Let L be an extension field of $k(a)$. There is an automorphism of $L[[z_1, \ldots, z_n]]$ defined by $z_i \mapsto z_i + a_iq$ which maps $Q_0L[[z_1, \ldots, z_n]]$ into $Q_1L[[z_1, \ldots, z_n]]$. Since $Q_0L[[z_1, \ldots, z_n]]$ is prime so is $Q_1L[[z_1, \ldots, z_n]]$. In particular, Q_1 is a prime ideal of $D(a)$. □

In addition we make the following assumptions on the prime elements p_i:

(II) For all $1 \le i \le k$ the ideal $Q_0 + (p_i)$ is a prime ideal in D with $Q_0 \ne Q_0 + (p_i)$.

Note that condition (II) implies that for all $1 \le i \le k$ the ideal $Q_1 + (p_i) = (Q_0 + (p_i))D(a) \subseteq D(a)$ is prime in $D(a)$ and that the elements $p_1, \ldots, p_k$ are prime elements of $D(a)/Q_1$.

Let $\sigma \in D$ be an element so that:

$$\sigma \equiv p_1^{e_1} \ldots p_k^{e_k} s \bmod Q_0$$

where $\varepsilon > \sum_{i=1}^k e_i$ and $s \notin \cup_{i=1}^k (Q_0 + (p_i))$.

Lemma 12 (The key lemma). *Under the above assumptions*:

(a) $\sigma \equiv p_1^{e_1} \cdots p_k^{e_k} t \bmod Q_1$ *where* $t \notin \cup_{i=1}^k (Q_1 + (p_i))D(a)$.

(b) s *is a unit in* D/Q_0 *if and only if* t *is a unit in* $D(a)/Q_1$.

(c) *If* σ *is a prime element in* D *and* t *is not a unit in* $D(a)/Q_1$, *then* t *is a prime element in* $D(a)/Q_1$.

Proof. (a) and (b). By assumption

$$\sigma = p_1^{e_1} \cdots p_k^{e_k} s + v$$

where $v \in Q_0$. Thus we can write:

$$v = \sum_{j=1}^{r} u_j f_j(z_i).$$

For all $1 \leq j \leq r$:

$$f_j(z_i + a_i q) = f_j(z_i) + qH_j$$

where $H_j \in D(a)$. This yields in $D(a)$:

$$\begin{aligned}\sigma &= p_1^{e_1} \cdots p_k^{e_k} s + \sum_{j=1}^{r} u_j (f_j(z_i + qa_i) - qH_j) \\ &= p_1^{e_1} \cdots p_k^{e_k} s - q \sum_{j=1}^{r} u_j H_j + \sum_{j=1}^{r} u_j f_j(z_i + a_i q),\end{aligned}$$

and therefore:

$$\sigma \equiv p_1^{e_1} \cdots p_k^{e_k} \Big(s - \delta \sum_{j=1}^{r} u_j H_j \Big) \bmod Q_1$$

where $\delta = q / p_1^{e_1} \cdots p_k^{e_k} \in \mathfrak{m}D$. We set:

$$t = s - \delta \sum_{j=1}^{r} u_j H_j \in D(a).$$

Since $\delta \in \cap_{i=1}^{k} (Q_1 + (p_i))$ and $s \notin \cup_{i=1}^{k} (Q_0 + (p_i)) D(a) = \cup_{i=1}^{k} (Q_1 + (p_i))$ assertions (a) and (b) follow.

(c) If t is not a unit the elements q, t form a regular sequence in $D(a)/Q_1$. This implies that $D(a)/(Q_1 + (t))$ is a domain if and only if the ring $(D(a)/(Q_1 + (t)))_q$ is a domain. Since $p_1, \ldots, p_k$ are units in D_q:

$$(D(a)/(Q_1 + (t)))_q = (D(a)/(Q_1 + (\sigma)))_q.$$

First observe that:

$$(D(a)/(Q_1 + (\sigma)))_q = T^{-1}(D_q[a_1, \ldots, a_n]/(Q_1 + (\sigma)))$$

where T is the multiplicative set $T = D[a_1, \ldots, a_n] - \mathfrak{m}D[a_1, \ldots, a_n]$. Therefore it suffices to show that $D_q[a_1, \ldots, a_n]/(Q_1 + (\sigma))$ is a domain. Note that:

$$D_q[a_1, \ldots, a_n] = D_q[z_1 + qa_1, \ldots, z_n + qa_n]$$

and put $y_i = z_i + qa_i$. Then:

$$\begin{aligned} D_q[a_1, \dots, a_n]/(Q_1 + (\sigma)) \\ &= (D/(\sigma))_q[y_1, \dots, y_n]/(f_1(y), \dots, f_r(y)) \\ &= (D/(\sigma))_q \otimes_k k[y_1, \dots, y_n]/(f_1(y), \dots, f_r(y)) \\ &\subseteq Q(D/(\sigma)) \otimes_k k[y_1, \dots, y_n]/(f_1(y), \dots, f_r(y)) \\ &= Q(D/(\sigma))[y_1, \dots, y_n]/(f_1(y), \dots, f_r(y)) \\ &\subseteq Q(D/(\sigma))[[y_1, \dots, y_n]]/(f_1(y), \dots, f_r(y)). \end{aligned}$$

Since $Q = (f_1(y), \dots, f_r(y))$ is a k-absolute prime ideal, the last ring is a domain and t is a prime element in $D(a)/Q$. □

3.3. The induction step. We want to define inductively prime elements $p_1, \dots, p_k$ and positive integers $\varepsilon_1, \dots, \varepsilon_{k-1}$. At any stage we define:

$$\begin{aligned} q_i &= p_1 \dots p_i \text{ for } 1 \le i \le k \\ z_{i0} &= z_i \text{ for } i \le n \\ z_{ik-1} &= z_i + a_{i1}q_1^{\varepsilon_1} + \dots + a_{ik-1}q_{k-1}^{\varepsilon_{k-1}} \text{ for } i \le n. \end{aligned}$$

Obviously, $z_{1k-1}, \dots, z_{nk-1}$ are a regular system of parameters of R_{k-1}. By a simple induction argument Proposition 11 shows that for all $k \ge 1$ the ideal

$$Q_{k-1} = (f_1(z_{1k-1}, \dots, z_{nk-1}), \dots, f_r(z_{1k-1}, \dots, z_{nk-1}))$$

is a K_{k-1}-absolute prime ideal of R_{k-1}. As in Section 2 we define the ideals $\mathfrak{P}_j = (z_{1j}, \dots, z_{mj}) \subseteq R$ for $1 \le j \le k-1$. We also denote by $\mathfrak{P}_j$ the ideal $\mathfrak{P}_j \cap R_{k-1}$ which is generated by $z_{1j}, \dots, z_{mj}$ in R_{k-1}. Since the elements $z_{1j}, \dots, z_{mj}$ are part of a regular sequence, the ideals $\mathfrak{P}_j$ are prime ideals in R and R_{k-1}, respectively.

At stage k we assume that $p_1, \dots, p_k$ and $\varepsilon_1, \dots, \varepsilon_{k-1}$ have been constructed so that the following conditions are satisfied:

(III_a) For all $1 \le i \le k$: $p_i \in R_{k-1}$ and $p_i \notin Q_{k-1}$.

(III_b) The ideals $Q_{k-1} + (p_i)$ are prime ideals in R_{k-1} for all $1 \le i \le k$.

(III_c) For all $1 \le i \le k$ the elements ρ_i split over R_{k-1}/Q_{k-1} into products $\rho_i = \gamma_i p_1^{e_{1i}} \cdots p_i^{e_{ii}}$ where γ_i is a unit in R_{k-1}/Q_{k-1}.

(III_d) For all $1 \leq i \leq k-1$: $\varepsilon_i > \sum_{j=1}^{i} e_{ji}$.

(III_e) For all $h \leq k$: $p_h \notin \mathfrak{P}_{k-1}$.

We start the induction by setting $p_1 = \rho_1$. Note that condition (I_c) implies (III_a), (III_b) and (III_c). Condition (III_e) follows from (Γ_a).

We now assume that $p_1, \ldots, p_k$ and $\varepsilon_1, \ldots, \varepsilon_{k-1}$ have been constructed so that conditions $(III_{(a-e)})$ are satisfied. In order to construct p_{k+1} and ε_k consider a decomposition of the prime element $\rho_{k+1} = \sigma$ of R_{k-1} in R_{k-1}/Q_{k-1}:

$$\sigma = \rho_{k+1} = p_1^{e_1} \cdots p_k^{e_k} s \in R_{k-1}/Q_{k-1}$$

where $s \notin \cup_{i=1}^{k}(Q_{k-1} + (p_i))$. Although the elements $p_1, \ldots, p_k$ are prime elements in R_{k-1}/Q_{k-1} we are not assuming that they generate different prime ideals. Therefore the decomposition of $\sigma = \rho_{k+1}$ may not be unique. However, the element s is uniquely determined. We choose ε_k so that the following conditions hold:

$$\varepsilon_k > \sum_{j=1}^{k} e_{jk} \quad (\text{with } e_{jk} \text{ as in } (III_c))$$
$$\varepsilon_k > \varepsilon_{k-1}$$

and define accordingly:

$$z_{ik} = z_{ik-1} + a_{ik} q_k^{\varepsilon_k}$$
$$Q_k = (f_1(z_{1k}, \ldots, z_{nk}), \ldots, f_r(z_{1k}, \ldots, z_{nk})).$$

We apply the key lemma to this situation by making the following substitutions:

$$D \simeq R_{k-1}$$
$$a_1, \ldots, a_n \simeq a_{1k}, \ldots, a_{nk}$$
$$D(a) \simeq R_k$$
$$\varepsilon \simeq \varepsilon_k$$
$$Q_0 \simeq Q_{k-1}$$
$$Q_1 \simeq Q_k.$$

By the key lemma the element $\sigma = \rho_{k+1}$ factors in R_k/Q_k as follows:

$$\sigma = \rho_{k+1} = p_1^{e_1} \cdots p_k^{e_k} t$$

where t is either a unit or a prime element in R_k/Q_k. If t is a unit we set $p_{k+1} = p_i$ for some $1 \le i \le k$ with $e_i \ge 1$. If t is not a unit by the key lemma the element t is a prime in R_k/Q_k since $\sigma = \rho_{k+1}$ is a prime element of R_k. In this case we choose an element $p_{k+1} \in \Gamma$ which maps onto an associate of t under the natural map $R_k \to R_k/Q_k$, that is, $t = p_{k+1}\gamma_{k+1} \in R_k/Q_k$ where γ_{k+1} is a unit in R_k/Q_k. By definition of Γ the element p_{k+1} is prime in R_k.

It remains to show that $p_1, \dots, p_{k+1}$ and $\varepsilon_1, \dots, \varepsilon_k$ satisfy conditions $(III_{(a-e)})$:

Lemma 13. (a) *For all* $1 \le i \le k+1$: $p_i \notin Q_k$.

(b) *The ideals* $Q_k + (p_i)$ *are prime ideals in* R_k *for all* $1 \le i \le k+1$.

(c) *For all* $1 \le i \le k+1$ *the elements* ρ_i *split over* R_k/Q_k *into products* $\rho_i = \gamma_i p_1^{e_{1i}} \cdots p_i^{e_{ii}}$ *where* γ_i *is a unit in* R_k/Q_k.

(d) *For all* $1 \le i \le k$: $\varepsilon_i > \sum_{j=1}^{i} e_{ji}$.

(e) *For all* $h \le k+1$: $p_h \notin \mathfrak{P}_k$.

Proof. We first show conditions (a)–(e) for $p_1, \dots, p_k$ and $\varepsilon_1, \dots, \varepsilon_{k-1}$. By assumption $p_i \notin \mathfrak{P}_{k-1}$ for $1 \le i \le k$ and $\mathfrak{P}_{k-1} + (p_i) = \mathfrak{P}_k + (p_i)$. Since $\mathfrak{P}_{k-1}$ and $\mathfrak{P}_k$ are both prime ideals of height m it follows that $p_i \notin \mathfrak{P}_k$ for all $1 \le i \le k$. By construction $Q_k \subseteq \mathfrak{P}_k$ and therefore $p_i \notin Q_k$ for $1 \le i \le k$. Since $Q_k + (p_i) = Q_{k-1} + (p_i)$, the ideals $Q_k + (p_i)$ are prime in R_k for $1 \le i \le k$. This shows conditions (a), (b), (d) and (e) for $p_1, \dots, p_k$ and $\varepsilon_1, \dots, \varepsilon_{k-1}$. Condition (c), the appropriate splitting of the ρ_i for $1 \le i \le k$, is a consequence of the key lemma.

It remains to show that p_{k+1} and ε_k satisfy conditions (a)–(e). (a), (c) and (d) follow immediately from the definition of p_{k+1} and ε_k. Note that p_{k+1} was chosen so that $Q_k + (p_{k+1})$ is a prime ideal in R_k and that γ_{k+1} is a unit in R_k/Q_k.

It remains to show that $p_{k+1} \notin \mathfrak{P}_k$. Since by assumption $\rho_{k+1} \in R_{k-1}$ and

$$\mathfrak{P}_k \cap R_{k-1} = (0)$$

we see that $\rho_{k+1} \notin \mathfrak{P}_k$. By construction:

$$\rho_{k+1} = p_1^{e_1} \cdots p_k^{e_k} t \in R_k/Q_k$$

where $t = p_{k+1}\gamma_{k+1}$ or t a unit in R and $p_{k+1} = p_i$ for some $1 \le i \le k$ with $e_i \ge 1$. Since $Q_k \subseteq \mathfrak{P}_k$ the assertion follows. □

3.4. The result. Using Lemma 13 we construct a sequence of prime elements of R, $p_1, \ldots, p_n, \ldots$ together with a sequence of positive integers $\varepsilon_1, \ldots, \varepsilon_n, \ldots$ so that for all $k \in \mathbf{N}$, $p_1, \ldots, p_k$ and $\varepsilon_1, \ldots, \varepsilon_{k-1}$ satisfy conditions (III_a)–(III_e). Similarly to Section 2 for all $k \in \mathbf{N}$ we set:

$$q_k = p_1 \cdots p_k$$

and define for all $1 \le i \le n$:

$$\zeta_i := z_i + \sum_{l=1}^{\infty} a_{il} q_l^{\varepsilon_l} \in K[[z_1, \ldots, z_n]].$$

The map $z_i \mapsto \zeta_i$ for $1 \le i \le n$ defines an automorphism φ on the power series ring $K[[z_1, \ldots, z_n]] = \widehat{R}$. With $J = \varphi(Q)\widehat{R}$ we set

$$A = Q(R) \cap \widehat{R}/J.$$

We want to show that A is a local Noetherian factorial domain with completion $\widehat{R}/J$.

Let $\widehat{P}$ denote the prime ideal $\widehat{P} = (\zeta_1, \ldots, \zeta_m)$ of $\widehat{R} = K[[z_1, \ldots, z_n]]$. As in the previous section we set for all $1 \le i \le n$,

$$z_{ik} = z_i + \sum_{l=1}^{k} a_{il} q_l^{\varepsilon_l} \in K[z_1, \ldots .z_n].$$

Again $\mathfrak{P}_k$ denotes the prime ideal $\mathfrak{P}_k = (z_{1k}, \ldots, z_{mk})$. We first show:

Lemma 14. $\widehat{P} \cap R = (0)$.

Proof. If $\widehat{P} \cap R \ne (0)$, then $\mathfrak{p} = \widehat{P} \cap R$ is a nonzero prime ideal and there is an element $\rho_k \in \Gamma$ with $\rho_k \in \mathfrak{p}$. For all $l \ge k$, $\rho_k \in R_l$ and by (III_c)

$$\rho_k = p_1^{e_{1k}} \cdots p_k^{e_{kk}} u_l + v_l,$$

where $u_l \in R_l$ a unit and $v_l \in Q_l$. Note that u_l and v_l may vary with l while by the key lemma the exponents e_{ik} are independent of l. Since for all $l \in \mathbf{N}$

$$\widehat{P} + (q_{l+1}^{\varepsilon_{l+1}}) = \mathfrak{P}_l + (q_{l+1}^{\varepsilon_{l+1}})$$

and $Q_l \subseteq \mathfrak{P}_l$, we obtain for all $l \geq k$ that

$$p_1^{e_{1k}} \cdots p_k^{e_{kk}} \in \mathfrak{P}_l + (q_{l+1}^{\varepsilon_{l+1}})$$

and therefore

$$p_1^{e_{1k}} \cdots p_k^{e_{kk}} \in \bigcap_{l \geq k} \mathfrak{P}_l + (q_{l+1}^{\varepsilon_{l+1}}) = \bigcap_{l \geq k} \widehat{P} + (q_{l+1}^{\varepsilon_{l+1}}) = \widehat{P}.$$

Thus $p_j \in \widehat{P}$ for some $1 \leq j \leq k$. Since $\widehat{P}$ and $\mathfrak{P}_j$ are both prime ideals of $\widehat{R}$ of height m with $\widehat{P} + (p_j) = \mathfrak{P}_{j-1} + (p_j)$, it follows that $p_j \in \mathfrak{P}_{j-1}$, a contradiction to condition (III_e). □

In order to show that $A = Q(R) \cap \widehat{R}/J$ is a Noetherian ring, similar to Section 2 we define a subring B of A which is a nested union of algebras essentially of finite type over K. We will show that B is a local Noetherian ring with $\widehat{B} = \widehat{R}/J$ implying $A = B$. For all $1 \leq j \leq n$ and all $k \in \mathbf{N}$, consider the following elements in the field of quotients of R:

$$\alpha_{jk} = \frac{1}{q_k^{\varepsilon_k}} f_j(z_{1k}, \ldots, z_{nk}) \in K(z_1, \ldots, z_n).$$

As in Section 2 we have a recursion formula:

$$\alpha_{jk} = \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\varepsilon_k}} [\alpha_{j(k+1)} + s_{jk}]$$

with $s_{jk}, (q_{k+1}^{\varepsilon_{k+1}}/q_k^{\varepsilon_k}) \in R$. This implies:

Lemma 15. *For all $1 \leq j \leq n$ and all $k \in \mathbf{N}$:*

$$\alpha_{jk} \in Q(R) \cap \widehat{R}/J.$$

Proof. The proof is identical with the proof of Lemma 6. □

Exactly the same proof as in Section 2 shows that for all $k \in \mathbf{N}$ there is an embedding:

$$\lambda_k : R[\alpha_{1k}, \dots, \alpha_{nk}] \longrightarrow \widehat{R}/J.$$

With $\mathfrak{w}_k = \lambda_k^{-1}(\mathfrak{m})$ we put again for all $k \in \mathbf{N}$:

$$B_k = R[\alpha_{1k}, \dots, \alpha_{nk}]_{\mathfrak{w}_k}.$$

Since $B_k \subseteq B_{k+1}$ we can now define B as the nested union:

$$B = \bigcup_k B_k.$$

Proposition 16. *B is a local Noetherian domain. Moreover, for every nonzero prime ideal $\mathfrak{p} \subseteq B$ the ring $B/\mathfrak{p}$ is essentially of finite type over K.*

Proof. Let $\mathfrak{p} \subseteq B$ be a nonzero prime ideal of B. Since $\mathfrak{p} \cap R \neq (0)$ there is an element $\rho_k \in \mathfrak{p} \cap \Gamma$. Choose an integer $l > k$. Then ρ_k is an element of R_l and by (III_c):

$$\rho_k = p_1^{e_{1k}} \cdots p_{k-1}^{e_{k-1k}} p_k^{e_{kk}} u_l + v_l$$

where $u_l \in R_l$ a unit, $v_l = \sum_{j=1}^r d_j f_{jl} \in Q_l$ where $f_{jl} = f_j(z_{1l}, \dots, z_{nl})$, and $\varepsilon_l > \sum_{i=1}^k e_{ik}$. By construction

$$q_l^{\varepsilon_l} \alpha_{jl} = f_{jl}$$

and therefore in B:

$$\rho_k = p_1^{e_{1k}} \cdots p_{k-1}^{e_{k-1k}} p_k^{e_{kk}} [u_k + \delta_k]$$

where δ_k is divisible by q_k and the element $u_k + \delta_k$ is a unit in B. Since $\mathfrak{p}$ is a prime ideal in B which contains ρ_k, $\mathfrak{p}$ contains one of the elements p_j for some $1 \leq j \leq k$.

Suppose that $p_l \in \mathfrak{p}$ for some $1 \leq l \leq k$. Exactly the same proof as for Proposition 7 of Section 2 yields that the natural morphism:

$$\nu : R \longrightarrow B/(p_j)$$

is surjective. This shows that B is a local Noetherian ring and that $B/\mathfrak{p}$ is essentially of finite type over K for every nonzero prime ideal $\mathfrak{p}$ of B. □

Proposition 17. $\widehat{B} = \widehat{R}/J$.

Proof. This is exactly the same proof as for Proposition 8 of Section 2. □

Proposition 18. *B is factorial.*

Proof. We have to show that every height one prime ideal of B is principal. Let $\mathfrak{p} \subseteq B$ be a height one prime ideal. By the proof of Proposition 16 there is an $h \in \mathbf{N}$ with $p_h \in \mathfrak{p}$. We claim that $\mathfrak{p} = p_h B$. Since B is a local Noetherian ring with completion $\widehat{R}/J$ we have by faithful flatness:

$$p_h B = (p_h \widehat{B}) \cap B.$$

Since $J + (p_h)\widehat{R} = Q_h \widehat{R} + (p_h)\widehat{R}$ again faithful flatness yields:

$$(J + (p_h)\widehat{R}) \cap R = (Q_h + (p_h))\widehat{R} \cap R = Q_h R + (p_h).$$

This implies:

$$p_h B \cap R = Q_h R + (p_h).$$

and:

$$B/(p_h) \cong R/(Q_h R + (p_h)).$$

By construction $R/(Q_h R + (p_h))$ is a domain and therefore $\mathfrak{p} = p_h B$. □

To summarize the result, let K be a countable field of countable infinite transcendence degree over a subfield K_0. Put $R = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$ and let $Q \subseteq K_0[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$ be an ideal that satisfies the following conditions:

(a) For every extension field L of K_0 the ring

$$L[[z_1, \dots, z_n]]/QL[[z_1, \dots, z_n]]$$

is a domain.

(b) There is an integer $m < n$ such that for every associated prime ideal $\mathfrak{Q} \in \operatorname{Ass}(\widehat{R}/Q\widehat{R})$ we have that $\mathfrak{Q} \subseteq (z_1, \dots, z_m)\widehat{R}$. Then:

Theorem 19. *Under the above assumptions there is an automorphism φ of the power series ring $\widehat{R} = K[[z_1, \dots, z_n]]$ and a local Noetherian factorial ring B which birationally dominates R so that*:

(a) $\widehat{B} = \widehat{R}/\varphi(Q)\widehat{R} \cong \widehat{R}/Q\widehat{R} = K[[z_1, \dots, z_n]]/QK[[z_1, \dots, z_n]]$.

(b) *For every nonzero prime ideal* $\mathfrak{p} \subseteq B$ *the ring* $B/\mathfrak{p}$ *is essentially of finite type over* K.

(c) *The prime ideal* $\widehat{P} = (\varphi(z_1), \dots, \varphi(z_m))\widehat{B}$ *is in the generic formal fiber of* B.

3.5. Historical note. The method in Section 3 of constructing a factorial local Noetherian domain with a given completion is due to Ogoma. Ogoma first applied his method to the construction of a Cohen-Macaulay factorial domain which fails to be Gorenstein [**15**]. In [**16**] he used his method again to produce an acceptable Cohen-Macaulay ring without a canonical module. His method was also adopted by Weston in [**19**] to construct a local Noetherian ring with a Gorenstein module of rank two but with no canonical module. Our presentation of Ogoma's method is due to Nishimura [**12**]. Nishimura modified and simplified Ogoma's method by incorporating Heitmann's enumeration lemma into the construction.

There is a significant difference between the constructions in Sections 2 and 3. Let K be a countable field, $I \subseteq K[z_1, \dots, z_n]$ an ideal in the polynomial ring over K, and $D = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$. If the generators of I involve all variables $z_1, \dots, z_n$, by Section 2 there is a local Noetherian domain A so that $\widehat{A} = \widehat{D}[[y]]$ where y is an additional variable. If I is a K-absolute prime, then by Section 3 there is a factorial domain B with completion $\widehat{B} = \widehat{D}$. While in Section 2 an additional variable is needed no additional variables are needed in Section 3. Therefore the second method can be used to construct examples of excellent rings (with isolated singularities). For example in [**12**], Nishimura used the method of Section 3 to construct an example of an excellent Cohen-Macaulay ring which fails to admit a Gorenstein module.

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