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COTORSION PAIRS INDUCED BY DUALITY PAIRS

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ABSTRACT. We introduce the notion of a duality pair and demonstrate how the left half of such a pair is "often" covering and preenveloping. As an application, we generalize a result by Enochs et al. on Auslander and Bass classes, and we prove that the class of Gorenstein injective modules, introduced by Enochs and Jenda, is covering when the ground ring has a dualizing complex.

1. Introduction. What is now known as semi-dualizing modules were studied more than 25 years ago under other names by, e.g., Foxby [**15**] (PG-modules of rank one), Golod [**17** (suitable modules) and Vasconcelos [**33**] (spherical modules). As a common generalization of the notion of a semidualizing module and that of a dualizing complex, in the sense of Hartshorne [**18**], Christensen [**7**] introduced in 2001 the notion of a semidualizing complex, cf. (1.5).

Avramov and Foxby [**1**] and Christensen [**7**] demonstrated how a semidualizing complex C over a commutative Noetherian ring R gives rise to two important classes of R-modules, namely the so-called Auslander class A_0^C and Bass class B_0^C, cf. (1.6). Semidualizing complexes and their Auslander and Bass classes have caught the attention of several authors, but this paper is motivated by a result of Enochs and Holm [**10**], for which we prove the following generalization in Theorem 3.2.

Theorem A. *Let R be a commutative Noetherian ring, and let C be a semidualizing complex of R-modules. Then the following conclusions hold*:

(a) $(\mathsf{A}_0^C,(\mathsf{A}_0^C)^{\perp})$ *is a perfect cotorsion pair, in particular, the class* A_0^C *is covering. Furthermore,* A_0^C *is preenveloping.*

(b) *The class* B_0^C *is covering and preenveloping.*

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Cotorsion pairs, introduced by Salce [**28**], covering classes and preenveloping classes are central notions in relative homological algebra. We refer the reader to (1.2) and (1.3), and further to the monograph [**12**] by Enochs and Jenda, for relevant details about these notions.

Theorem A extends the main result of [**10**] in two directions: In [**10**], C is assumed to be a semidualizing *module* (as opposed to a semidualizing *complex*), and furthermore the covering property of B_0^C is new.

To prove Theorem A, we first establish Theorem 3.1 and then combine it with the fact that A_0^C and B_0^C are parts of appropriate *duality pairs*. The latter notion is introduced in Definition 2.1. The technique used to prove Theorem A applies to show that several other classes of modules are covering and/or preenveloping; see, for example, Theorem B below.

Now, assume that R has a dualizing complex D in the sense of Hartshorne [**18**], and consider a semidualizing *R-module* C. Then $C^\dagger = \mathbf{R}\mathrm{Hom}_R(C,D)$ is a semidualizing complex for which the associated Auslander and Bass classes can be characterized in terms of two homological dimensions introduced in [**22**], and studied further by Sather-Wagstaff, Sharif and White [**29–32**]. More precisely, for any R-module M, one has the following equivalences:

$$M \in \mathsf{A}_0^{C^\dagger}(R) \iff C\text{-}\mathrm{Gfd}_R M \leqslant \dim R,$$
$$M \in \mathsf{B}_0^{C^\dagger}(R) \iff C\text{-}\mathrm{Gid}_R M \leqslant \dim R.$$

Here $C\text{-}\mathrm{Gfd}_R\, M$ and $C\text{-}\mathrm{Gid}_R\, M$ are the so-called C-Gorenstein flat and C-Gorenstein injective dimensions of M. Naturally, Theorem A applies to the semidualizing complex $C^\dagger$, but in view of the equivalences above, Theorem B below, which is a special case of Theorem 3.3, gives more information.

Theorem B. *Let R be a commutative Noetherian ring with a dualizing complex, and let $n \geqslant 0$ be an integer. Consider the following classes of R-modules*:

$$\mathsf{GF}_n^C = \{M \mid C\text{-}\mathrm{Gfd}_R\, M \leqslant n\},$$
$$\mathsf{GI}_n^C = \{M \mid C\text{-}\mathrm{Gid}_R\, M \leqslant n\}.$$

Then the next conclusions hold:

(a) $(\mathsf{GF}_n^C, (\mathsf{GF}_n^C)^\perp)$ *is a perfect cotorsion pair, in particular,* GF_n^C *is covering. Furthermore,* GF_n^C *is preenveloping.*

(b) *The class* GI_n^C *is covering and preenveloping.*

For $C = R$ and $n = 0$, Theorem B(a) asserts that the class of Gorenstein flat modules is the left half of a perfect cotorsion pair, and that it is preenveloping. The first of these results is proved by Enochs and López-Ramos [**14**, Corollary 2.11] and the other follows immediately by combining [**14**, Prop. 2.10, Theorem 2.5 and Remark 3] with [**6**, Theorem 5.7]. For $C = R$ and $n = 0$, the second part of Theorem B(b) asserts that the class of Gorenstein injective modules is preenveloping. This is proved in [**14**, Corollary 2.7]. Actually, Krause [**24**, Theorem 7.12] proves the existence of special Gorenstein injective preenvelopes.

1. Preliminaries. In this section we introduce our terminology and recall a few notions relevant for this paper.

(1.1) **Setup.** Throughout, R denotes a ring with identity, and R° its opposite ring. Unless otherwise specified, all modules under consideration are unitary left modules. Recall that a right R-module can be identified with a left R°-module. We write $\mathsf{Mod}(R)$ for the category of all (left) R-modules.

(1.2) **Covers and envelopes.** The following notions were coined by Enochs [**9**].

Let M be any class of R-modules. An M-*precover* of an R-module N is a homomorphism $\varphi: M \to N$, where M is in M, with the property that, for every homomorphism $\varphi': M' \to N$, where M' is in M, there exists a (not necessarily unique) homomorphism $\psi: M' \to M$ with $\varphi' = \varphi\psi$. An M-precover $\varphi: M \to N$ is an M-*cover* if every homomorphism $\psi: M \to M$ satisfying $\varphi = \varphi\psi$ is an automorphism. The class M is called *(pre)covering* if every R-module has an M-(pre)cover.

The notion of an M-*(pre)envelope* is categorically dual to that of an M-*(pre)cover*, and thus we will omit the definition here.

(1.3) **Cotorsion pairs.** For a class M of R-modules one defines:

$$\begin{aligned} {}^{\perp}\mathsf{M} &= \{X \in \mathsf{Mod}(R) \mid \operatorname{Ext}^1_R(X, M) = 0 \text{ for all } M \in \mathsf{M}\}, \\ \mathsf{M}^{\perp} &= \{Y \in \mathsf{Mod}(R) \mid \operatorname{Ext}^1_R(M, Y) = 0 \text{ for all } M \in \mathsf{M}\}. \end{aligned}$$

A *cotorsion pair* is a pair (M,N) of classes of R-modules with $\mathsf{M} = {}^{\perp}\mathsf{N}$ and $\mathsf{M}^{\perp} = \mathsf{N}$. A cotorsion pair (M,N) is called *perfect* if M is covering and N is enveloping. These notions go back to Salce [**28**].

(1.4) **The derived category.** We denote by $\mathsf{D}(R)$ the *derived category* of the Abelian category $\mathsf{Mod}(R)$. We write $\mathsf{D}_{\mathrm{b}}(R)$ for the full subcategory of $\mathsf{D}(R)$ whose objects have bounded homology. The right derived Hom functor and the left derived tensor product functor are denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $-\otimes^{\mathbf{L}}_R-$, respectively. The reader is referred to Weibel [**34**, Chapter 10] for further details.

(1.5) **Semidualizing complexes.** The following is from Christensen [**7**, Definition (2.1)].

Assume that R is commutative and Noetherian. A complex $C \in \mathsf{D}_{\mathrm{b}}(R)$ with degreewise finitely generated homology is *semidualizing* if the natural homothety morphism $R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in the derived category $\mathsf{D}(R)$.

(1.6) **Auslander and Bass classes.** Assume that R is commutative and Noetherian, and let C be a semidualizing R-complex. The following definitions are due to Avramov and Foxby [**1**, (3.1)] and Christensen [**7**, (4.1)].

The *Auslander class* $\mathsf{A}^C(R)$ consists of all $M \in \mathsf{D}_{\mathrm{b}}(R)$ such that $C\otimes^{\mathbf{L}}_R M \in \mathsf{D}_{\mathrm{b}}(R)$ and the canonical morphism $M \to \mathbf{R}\mathrm{Hom}_R(C, C\otimes^{\mathbf{L}}_R M)$ in $\mathsf{D}(R)$ is an isomorphism.

The *Bass class* $\mathsf{B}^C(R)$ consists of all $N \in \mathsf{D}_{\mathrm{b}}(R)$ such that $\mathbf{R}\mathrm{Hom}_R(C, N) \in \mathsf{D}_{\mathrm{b}}(R)$ and the canonical morphism $C\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, N) \to N$ in $\mathsf{D}(R)$ is an isomorphism.

We write $\mathsf{A}^C_0(R)$ and $\mathsf{B}^C_0(R)$, or simply A^C_0 and B^C_0 if the ground ring is understood, for the class of R-modules which, when considered as objects in $\mathsf{D}(R)$, belong to $\mathsf{A}^C(R)$ and $\mathsf{B}^C(R)$, respectively.

(1.7) **Remark.** If C is a semidualizing *R-module*, then it is possible to define $\mathsf{A}^C_0(R)$ and $\mathsf{B}^C_0(R)$ directly in $\mathsf{Mod}(R)$ without using $\mathsf{D}(R)$, cf. [**7**, Observation (4.10)].

(1.8) **Homological dimensions.** Let M be an arbitrary R-module.

We write $\mathrm{fd}_R M$ and $\mathrm{id}_R M$ for the *flat* and *injective* dimension of M. These classical notions go back to Cartan and Eilenberg [**4**].

We write $\mathrm{Gfd}_R M$ and $\mathrm{Gid}_R M$ for the *Gorenstein flat* and *Gorenstein injective* dimension of M. These notions were introduced by Enochs, Jenda et al. [**11, 13**] and have subsequently been studied by several authors.

When R is commutative and Noetherian, the definitions of Enochs, Jenda et al. mentioned above have been extended in [**22**]: For a semi-dualizing R-module C, cf. (1.5), [**22**] introduces a *C-Gorenstein flat* dimension C-$\mathrm{Gfd}_R M$ and a C-Gorenstein injective dimension C-$\mathrm{Gid}_R M$. For $C = R$ these invariants agree with $\mathrm{Gfd}_R M$ and $\mathrm{Gid}_R M$, respectively. The C-Gorenstein dimensions have been studied by, e.g., Sather-Wagstaff, Sharif, and White [**29–32**].

(1.9) **Depth and width.** Assume that $(R, \mathfrak{m}, k)$ is commutative Noetherian local. The *depth* of a finitely generated R-module $M \neq 0$, that is, the length of a maximal M-regular sequence, can be computed as

$$\mathrm{depth}_R M = \inf\{m \in \mathbf{Z} \mid \mathrm{Ext}_R^m(k, M) \neq 0\}.$$

Foxby [**16**] defines the depth of an arbitrary R-module M by the equality above, and Yassemi [**36**] studies the dual notion of *width*, which is defined by

$$\mathrm{width}_R M = \inf\{m \in \mathbf{Z} \mid \mathrm{Tor}^R_m(k, M) \neq 0\}.$$

Note that $\mathrm{depth}_R 0 = \mathrm{width}_R 0 = \infty$.

2. Duality pairs. In this section we define duality pairs and give several examples. In the next section we will prove how suitable duality pairs induce cotorsion pairs. For unexplained notions and notation, the reader is referred to Section 1.

(2.1) **Definition.** A *duality pair* over R is a pair (M, C), where M is a class of R-modules and C is a class of R°-modules, subject to the following conditions:

(1) For an R-module M, one has $M \in \mathsf{M}$ if and only if $\mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) \in \mathsf{C}$.

(2) C is closed under direct summands and finite direct sums.

A duality pair (M,C) is called *(co)product-closed* if the class M is closed under (co)products in the category of all R-modules.

A duality pair (M,C) is called *perfect* if it is coproduct-closed, if M is closed under extensions, and if R belongs to M.

(2.2) **Example.** Consider for each integer $n \geqslant 0$ the following module classes:

$$\mathsf{F}_n = \{M \in \mathsf{Mod}(R) \mid \mathrm{fd}_R M \leqslant n\},$$
$$\mathsf{I}_n = \{M \in \mathsf{Mod}(R^\circ) \mid \mathrm{id}_{R^\circ} M \leqslant n\}.$$

The following examples of duality pairs are well-known.

(a) $(\mathsf{F}_n, \mathsf{I}_n)$ is a perfect duality pair. If R is right coherent, then this pair is product-closed by a classical result of Chase [**5**, Theorem 2.1].

(b) If R is right Noetherian, then $(\mathsf{I}_n, \mathsf{F}_n)$ is a product- and coproduct-closed duality pair (over R°), cf. Xu [**35**, Lemma 3.1.4] and Bass [**2**, Theorem 1.1].

(2.3) **Example.** Let B be a class of finitely presented R-modules. Following Lenzing [**26**, Section 2] and [**19**, Definition 2.3], we consider the class M of modules with *support* in B, and the class C of modules with *cosupport* in B defined by:

$$\mathsf{M} = \varinjlim \mathsf{B},$$
$$\mathsf{C} = \mathrm{Prod}\,\{\mathrm{Hom}_{\mathbf{Z}}(B, \mathbf{Q}/\mathbf{Z}) \mid B \in \mathsf{B}\}.$$

Then (M,C) is a coproduct-closed duality pair by [**26**, Proposition 2.1] and [**19**, Theorem 1.4]. For example, if B is the class of all finitely generated projective R-modules, then $(\mathsf{M},\mathsf{C}) = (\mathsf{F}_0, \mathsf{I}_0)$ by a classical result of Lazard [**25**].

(2.4) **Proposition.** *Assume that R is commutative and Noetherian, and let C be a semidualizing R-complex. Then one has*:

(a) $(\mathsf{A}_0^C, \mathsf{B}_0^C)$ *is a perfect and product-closed duality pair.*

(b) $(\mathsf{B}_0^C, \mathsf{A}_0^C)$ *is a product- and coproduct-closed duality pair.*

Proof. That $(\mathsf{A}_0^C, \mathsf{B}_0^C)$ and $(\mathsf{B}_0^C, \mathsf{A}_0^C)$ are duality pairs follow from (the proof of) [**6**, Lemma (3.2.9)]. That A_0^C and B_0^C are closed under products and coproducts follow from (the proof of) [**8**, Lemma 5.6]. The class A_0^C clearly contains R, and it is closed under extensions by (the proof of) [**6**, Lemma (3.1.13)]. □

(2.5) **Lemma.** *Consider for each integer* $n \geqslant 0$ *the following module classes*:

$$\mathsf{GF}_n = \{M \in \mathsf{Mod}(R) \mid \mathrm{Gfd}_R M \leqslant n\},$$
$$\mathsf{GI}_n = \{M \in \mathsf{Mod}(R^\circ) \mid \mathrm{Gid}_{R^\circ} M \leqslant n\}.$$

Then the following conclusions hold:

(a) *If* R *is right coherent, then* $(\mathsf{GF}_n, \mathsf{GI}_n)$ *is a perfect duality pair. If* R *is commutative Noetherian with a dualizing complex, then this duality pair is product-closed.*

(b) *If* R *is commutative Noetherian with a dualizing complex, then* $(\mathsf{GI}_n, \mathsf{GF}_n)$ *is a product- and coproduct-closed duality pair* (*over* R°).

Proof. (a) Since R is right coherent, it follows by [**20**, Proposition 3.11] that the given pair is a duality pair. The class GF_n is closed under coproducts by [**20**, Proposition 3.13], and to see that GF_n is closed under extensions one applies [**20**, Theorems 3.14 and 3.15]. It is clear that R belongs to GF_n.

If R is commutative and Noetherian with a dualizing complex, then GF_n is closed under products by [**8**, Theorem 5.7].

(b) Since R is commutative with a dualizing complex, it follows by (the proof of) [**8**, Proposition 5.1] that the given pair is a duality pair. The class GI_n is closed under products by [**20**, Theorem 2.6], and it is closed under coproducts by [**8**, Theorem 6.9]. □

(2.6) **Proposition.** *Assume that* R *is commutative and Noetherian, and let* C *be a semidualizing* R*-module. Consider for each* $n \geqslant 0$ *the following module classes*:

$$\mathsf{GF}_n^C = \{M \in \mathsf{Mod}(R) \mid C\text{-}\mathrm{Gfd}_R M \leqslant n\},$$
$$\mathsf{GI}_n^C = \{M \in \mathsf{Mod}(R) \mid C\text{-}\mathrm{Gid}_R M \leqslant n\}.$$

Then one has the following conclusions:

(a) $(\mathsf{GF}_n^C, \mathsf{GI}_n^C)$ *is a perfect duality pair. If* R *has a dualizing complex then this duality pair is product-closed.*

(b) *If R has a dualizing complex, then $(\mathsf{GI}_n^C, \mathsf{GF}_n^C)$ is a product- and coproduct-closed duality pair.*

Proof. (a) We denote by $R \ltimes C$ the trivial extension of R by C, cf. [**3**, subsection 3.3]. For any R-module M, it follows by [**22**, Theorem 2.16] that

$$(\dagger) \qquad C\text{-}\mathrm{Gfd}_R M = \mathrm{Gfd}_{R\ltimes C} M \quad \text{and} \quad C\text{-}\mathrm{Gid}_R M = \mathrm{Gid}_{R\ltimes C} M.$$

Combining this with [**20**, Proposition 3.11], one gets that

$$\begin{aligned} C\text{-}\mathrm{Gid}_R \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) &= \mathrm{Gid}_{R\ltimes C} \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) \\ &= \mathrm{Gfd}_{R\ltimes C} M \\ &= C\text{-}\mathrm{Gfd}_R M, \end{aligned}$$

from which we conclude that $(\mathsf{GF}_n^C, \mathsf{GI}_n^C)$ is a duality pair. By Lemma 2.5(a), the class $\mathsf{GF}_n(R \ltimes C)$ is closed under coproducts and extensions; and combining this with the first equality in (†), it follows that GF_n^C is closed under coproducts and extensions as well. Also note that R belongs to GF_n^C by [**22**, Example 2.8 (c)].

If R has a dualizing complex, then so has $R \ltimes C$, since it is a module finite extension of R. Hence, $\mathsf{GF}_n(R \ltimes C)$ is closed under products by Lemma 2.5(a), and by the first equality in (†) we then conclude that GF_n^C is closed under products.

(b) Similar to the proof of part (a), but using the second equality in (†) instead of the first, and using that for any R-module M one has:

$$\begin{aligned} C\text{-}\mathrm{Gfd}_R \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) &= \mathrm{Gfd}_{R\ltimes C} \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) \\ &= \mathrm{Gid}_{R\ltimes C} M \\ &= C\text{-}\mathrm{Gid}_R M. \end{aligned}$$

The equalities above follow from (†) and (the proof of) [**8**, Proposition 5.1]. ◻

(2.7) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be commutative Noetherian local. Consider for each integer $n \geqslant 0$ the following module classes*:

$$\begin{aligned} \mathsf{D}_n &= \{M \in \mathsf{Mod}(R) \mid \mathrm{depth}_R M \geqslant n\}, \\ \mathsf{W}_n &= \{M \in \mathsf{Mod}(R) \mid \mathrm{width}_R M \geqslant n\}. \end{aligned}$$

Then the following conclusions hold:

(a) $(\mathsf{D}_n, \mathsf{W}_n)$ *is a product- and coproduct-closed duality pair. If* $n \leqslant \operatorname{depth} R$, *then this duality pair is perfect.*

(b) $(\mathsf{W}_n, \mathsf{D}_n)$ *is a product- and coproduct-closed duality pair.*

Proof. For every R-module M one has:

$$\operatorname{depth}_R \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) = \operatorname{width}_R M,$$
$$\operatorname{width}_R \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) = \operatorname{depth}_R M,$$

from which it follows that the pairs in (a) and (b) are duality pairs. It is trivial from the definitions of depth and width, cf. (1.9), that D_n and W_n are closed under products, coproducts, direct summands and extensions. □

We end this section by noting that the next easily proved result can be applied to construct new duality pairs from existing ones.

(2.8) **Proposition.** *Let* $(\mathsf{M}_\mu, \mathsf{C}_\mu)$ *be a family of duality pairs over* R. *Then their intersection* $(\cap \mathsf{M}_\mu, \cap \mathsf{C}_\mu)$ *is also a duality pair. Furthermore, the following hold*:

(a) *If each* $(\mathsf{M}_\mu, \mathsf{C}_\mu)$ *is* (*co*)*product-closed then so is* $(\cap \mathsf{M}_\mu, \cap \mathsf{C}_\mu)$.

(b) *If each* $(\mathsf{M}_\mu, \mathsf{C}_\mu)$ *is perfect, then so is* $(\cap \mathsf{M}_\mu, \cap \mathsf{C}_\mu)$.

3. Existence of preenvelopes, covers, and cotorsion pairs.

The main result of this section, Theorem 3.1, shows that the left half of a duality pair is "often" preenveloping and covering. We apply this result to a few of the duality pairs found in Section 2.

(3.1) **Theorem.** *Let* (M, C) *be a duality pair. Then* M *is closed under pure submodules, pure quotients, and pure extensions. Furthermore, the following hold*:

(a) *If* (M,C) *is product-closed, then* M *is preenveloping.*

(b) *If* (M,C) *is coproduct-closed, then* M *is covering.*

(c) *If* (M,C) *is perfect, then* $(\mathsf{M}, \mathsf{M}^\perp)$ *is a perfect cotorsion pair.*

Proof. First we prove that M is closed under pure submodules, pure quotients, and pure extensions, that is, given a pure exact sequence of R-modules,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

then M is in M if and only if M', M'' are in M. Applying $\mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Q}/\mathbf{Z})$ to the sequence above, we get by Jensen and Lenzing [**23**, Theorem 6.4] a split exact sequence,

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(M'', \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(M', \mathbf{Q}/\mathbf{Z}) \longrightarrow 0.$$

By (2.1)(2) it follows that $\mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ is in C if and only if $\mathrm{Hom}_{\mathbf{Z}}(M', \mathbf{Q}/\mathbf{Z})$ and $\mathrm{Hom}_{\mathbf{Z}}(M'', \mathbf{Q}/\mathbf{Z})$ both are in C. The desired conclusion now follows by (2.1)(1).

(a) We have proved that M is closed under pure submodules. Since M is also closed under products by assumption, it follows by Rada and Saorín [**27**, Corollary 3.5 (c)] that M is preenveloping.

(b) We have proved that M is closed under pure quotients. By assumption, M is also closed under coproducts, and therefore it follows by [**21**, Theorem 2.5] that M is covering.

(c) We have proved that M is closed under pure submodules and pure quotients. By assumption, M is also closed under coproducts and extensions, and R belongs to M. Thus [**21**, Theorem 3.4] implies that $(\mathsf{M}, \mathsf{M}^{\perp})$ is a perfect cotorsion pair. □

As mentioned in the introduction, in the case where C is a semidualizing *module* (as opposed to as semidualizing *complex*), the following result, except the first assertion in part (b), is proved by Enochs et al. [**10**].

(3.2) **Theorem.** *Assume that R is commutative and Noetherian, and let C be a semidualizing R-complex. Then the following conclusions hold*:

(a) $(\mathsf{A}_0^C, (\mathsf{A}_0^C)^{\perp})$ *is a perfect cotorsion pair; in particular, the class* A_0^C *is covering. Furthermore,* A_0^C *is preenveloping.*

(b) *The class* B_0^C *is covering and preenveloping.*

Proof. (a) By Proposition 2.4(a), the class A_0^C is the left half of a perfect and product-closed duality pair. Thus, the conclusions follow from Theorem 3.1 (c,a).

(b) By Proposition 2.4(b), the class B_0^C is the left half of a product- and coproduct-closed duality pair. The conclusions follow from Theorem 3.1(a,b). $\square$

(3.3) **Theorem.** *Assume that R is commutative and Noetherian, let C be a semidualizing R-module, and let $n \geqslant 0$ be an integer. Then one has:*

(a) $(\mathsf{GF}_n^C, (\mathsf{GF}_n^C)^{\perp})$ *is a perfect cotorsion pair; in particular, GF_n^C is covering. If, in addition, R has a dualizing complex, then GF_n^C is preenveloping.*

(b) *If R has a dualizing complex, then GI_n^C is covering and preenveloping.*

Proof. (a) By Proposition 2.6(a), the class GF_n^C is the left half of a perfect duality pair. Thus, the claimed perfect cotorsion pair exists by Theorem 3.1(c). Under the assumption of the existence of a dualizing R-complex, GF_n^C is also product-closed by Proposition 2.6(a), and therefore GF_n^C is preenveloping by Theorem 3.1(a).

(b) If R has a dualizing complex, Proposition 2.6(b) gives that GI_n^C is the left half of a product- and coproduct-closed duality pair. Thus, the assertions follow from Theorem 3.1(b,a). $\square$

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