PROPERTIES OF CUT IDEALS
ASSOCIATED TO RING GRAPHS

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ABSTRACT. A cut ideal of a graph records the relations among the cuts of the graph. These toric ideals have been introduced by Sturmfels and Sullivant who also posed the problem of relating their properties to the combinatorial structure of the graph. We study the cut ideals of the family of ring graphs, which includes trees and cycles. We show that they have quadratic Gröbner bases and that their coordinate rings are Koszul, Hilbertian, and Cohen-Macaulay, but not Gorenstein in general.

1. Introduction. Let G be any finite (simple) graph with vertex set $V(G)$ and edge set $E(G)$. In [**15**] Sturmfels and Sullivant associate a projective variety X_G to G as follows. Let $A|B$ be an unordered partition of the vertex set of G. Each such partition defines a cut of the graph, denoted by $Cut(A|B)$, which is the set of edges $\{i, j\}$ such that $i \in A$, $j \in B$ or $j \in A$, $i \in B$. For each $A|B$, we can then assign variables to the edges according to whether they are in $Cut(A|B)$ or not. Thus, coordinates $q_{A|B}$ are indexed by unordered partitions $A|B$, while the variables encoding whether the edge $\{i, j\}$ is in the cut are s_{ij} and t_{ij} (for "separated" and "together"). The variety X_G, which we call *the cut variety of* G, is specified by the following homomorphism between polynomial rings:

$$\phi_G : K[q_{A|B} : A|B \text{ partition}] \to K[s_{ij}, t_{ij} : \{i, j\} \text{ edge of } G],$$

$$q_{A|B} \mapsto \prod_{\{i,j\} \in Cut(A|B)} s_{ij} \prod_{\{i,j\} \in E(G) \setminus Cut(A|B)} t_{ij}$$

The cut ideal I_G is the kernel of the map ϕ_G. It is a homogeneous toric ideal (note that $\deg \phi_G(q_{A|B}) = |E(G)|$). The variety X_G is defined by the cut ideal I_G.

Cut ideals generalize toric ideals arising in phylogenetics and the study of contingency tables. However, the algebraic properties of cut ideals are largely unknown. It is clear that the properties of the cut ideal depend on the combinatorics of the graph. Sturmfels and Sullivant pose the following conjecture.

Conjecture 1.1. ([**15**, Conjecture 3.7.]) *The semigroup algebra $K[q]/I_G$ is normal if and only if $K[q]/I_G$ is Cohen-Macaulay if and only if G is free of K_5 minors.*

We provide evidence for this conjecture by establishing it for a large class of such graphs. We refer to Section 6 for the definition of ring graphs.

Theorem 1.2. *The cut ideal of a ring graph admits a square-free quadratic Gröbner basis. Hence, its coordinate ring is Cohen-Macaulay.*

This and further results on cut ideals of ring graphs are established in Section 6.

The next section is a collection of necessary definitions and results that we use repeatedly. In particular, we recall the theorem of Sturmfels and Sullivant [**15**] about clique sums. Section 3 is devoted to cycles. Using the correspondence between cut ideals of cycles and certain phylogenetic ideals on claw trees, we provide a squarefree quadratic Gröbner bases for these cut ideals. Moreover, we find a formula for their number of minimal generators. In Section 4 we turn to cut ideals of trees. By [**15**], the algebraic properties of the cut ideal of a tree depend only on the number of edges. We determine its h-vector. Section 5 complements the study of clique sums by providing a generating set and a Gröbner basis for the cut ideal of a disjoint union of two graphs. The results about cut ideals of ring graphs are established in Section 6. In particular, we provide an estimate of their Castelnuovo-Mumford regularity.

2. Clique sums, Segre products, and Gröbner bases. We recall some concepts and results that we use later on. Throughout this paper all graphs are assumed to be finite and simple. The vertex and the edge set of such a graph G are denoted by $V(G)$ and $E(G)$, respectively. A clique of G is a subset of $V(G)$ such that the vertex-induced subgraph of G is complete, that is, there is an edge between any two vertices.

Let now $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2$ is a clique of both graphs. Then the *clique sum* of G_1 and G_2 is the graph $G = G_1 \# G_2$ with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. If the clique $V_1 \cap V_2$ consists of $k + 1$ vertices, then G is also called the *k-sum* of G_1 and G_2. If $0 \leq k \leq 2$, then Sturmfels and Sullivant [15] relate the graph-theoretic operation of forming clique sums to the algebraic operation of taking toric fiber products as defined in [16]. Defining two operations, Lift and Quad, they show that the generators of the cut ideal I_G can be obtained from the generators of the cut ideals I_{G_1} and I_{G_2}. More precisely, their result is:

Theorem 2.1. ([16, Theorem 2.1]) *Let G be a k-sum of G_1 and G_2 with $0 \leq k \leq 2$. Denote by $\mathbf{F_1}$ and $\mathbf{F_2}$ binomial generating sets for the smaller cut ideals I_{G_1} and I_{G_2}. Then*

$$\mathbf{M} = Lift(\mathbf{F_1}) \cup \mathbf{Lift}(\mathbf{F_2}) \cup \mathbf{Quad}(\mathbf{G_1}, \mathbf{G_2})$$

is a generating set for the cut ideal I_G. Furthermore, if $\mathbf{F_1}$ and $\mathbf{F_2}$ are Gröbner bases, then there exists a term order such that $\mathbf{M}$ is a Gröbner basis of I_G.

Let $X \subset \mathbb{P}^n$ be a projective subvariety over a field K. We denote its homogeneous coordinate ring by A_X. It is a standard graded K-algebra. The variety X is said to be arithmetically Cohen-Macaulay if A_X is a Cohen-Macaulay ring. The Hilbert function of X or A_X is defined by $h_X(j) = h_{A_X}(j) = \dim_K [A_X]_j$. If $j \gg 0$, then it becomes polynomial in j. This polynomial is the Hilbert polynomial $p_X = p_{A_X}$ of X or A_X. Following [1], the variety X (or its coordinate ring A_X) is said to be *Hilbertian* if its Hilbert function is polynomial in every non-negative degree, i.e., for every integer $j \geq 0$, $h_X(j) = p_X(j)$.

The cut ideal of a graph G on n vertices defines a projective variety $X_G \subset \mathbb{P}^{2^{n-1}-1}$. If G is the 0-sum of G_1 and G_2, then its cut variety X_G

is isomorphic to the Segre product of $X_{G_1} \times X_{G_2}$. Algebraically, this means that the coordinate ring A_{X_G} is the Segre product $A_{X_{G_1}} \boxtimes A_{X_{G_2}}$ of $A_{X_{G_1}}$ and $A_{X_{G_2}}$. The Segre product of Cohen-Macaulay rings is often not Cohen-Macaulay. The precise result is (see, e.g., [**12**], Theorem I.4.6):

Lemma 2.2. *Let A, B be two graded Cohen-Macaulay K-algebras that both have dimension at least two. Then their Segre product $A \boxtimes B$ is Cohen-Macaulay if and only if A and B are Hilbertian.*

Cut ideals are examples of toric ideals. In this note a toric ideal I is any ideal that defines an affine or projective variety X that is parametrized by a set of monomials. Thus, I can be generated by a set of binomials. The following result is well-known to specialists. We provide a proof for the convenience of the reader.

Lemma 2.3. *Let I be a homogeneous toric ideal of a polynomial ring S. If, for some monomial order on S, the initial ideal of I is squarefree, then S/I is Cohen-Macaulay and Hilbertian.*

Proof. The ideal I defines an affine toric variety Y and a projective toric variety X. X is projectively normal if and only if Y is normal. Theorem 13.15 in [**13**] says that, if for some term order $\prec$ the initial ideal $in_\prec(I)$ is squarefree, then X is projectively normal.

By assumption, S/I is isomorphic to a semigroup ring $K[B]$ for a semigroup $B \subset \mathbb{N}_0^d$. Thus, using Proposition 13.5 in [**13**], we conclude that the semigroup B is normal. Hence, by a theorem of Hochster [**8**], the semigroup ring $K[B]$ is Cohen-Macaulay. Furthermore, in non-negative degrees its Hilbert function is equal to the Ehrhart polynomial of the corresponding polytope (see, e.g., [**10**], Lemma 12.4), thus S/I is Hilbertian.   □

3. Cut ideals of cycles. The starting point of this section is the realization of the cut ideals associated to cycles as certain phylogenetic ideals. This will enable us to use the main result from [**3**].

Recall that the claw tree $K_{1,n}$ is the complete bipartite graph with n edges from one vertex (the root) to the other n vertices (the leaves). We denote by I_n the ideal of phylogenetic invariants for the general group-based model for the group $\mathbb{Z}_2$ on the claw tree $K_{1,n}$, as in [3]. This ideal is the kernel of the following homomorphism between polynomial rings (see [5]):

$$\varphi_n : R := K[q_{g_1,\dots,g_n} : g_1,\dots,g_n \in \mathbb{Z}_2] \to K[a_g^{(i)} :$$
$$g \in \mathbb{Z}_2, i = 1,\dots,n+1] =: R'$$
$$q_{g_1,\dots,g_n} \mapsto a_{g_1}^{(1)} a_{g_2}^{(2)} \cdots a_{g_n}^{(n)} a_{g_1+g_2+\cdots+g_n}^{(n+1)}.$$

The coordinate $q_{g_1,\dots,g_n}$ corresponds to observing the element g_1 at the first leaf of the tree, g_2 at the second, and so on, though here we are considering the phylogenetic ideals in Fourier coordinates instead of probability coordinates (cf. [14]).

Denote by C_{n+1} the $(n+1)$-cycle on the vertex set $[n+1] := \{1,\dots,n+1\}$. We are ready to state the announced comparison result.

Lemma 3.1. *The phylogenetic ideal I_n on a claw tree with n leaves and the cut ideal I_{C_n} of an $(n+1)$-cycle agree up to renaming variables.*

Proof. This follows from Theorem 5.5 in [15]. □

Recall that each toric ideal has a Gröbner basis consisting of binomials. We say that an ideal has a *squarefree Gröbner basis* if it has a Gröbner basis consisting of binomials, where each binomial is a difference of squarefree monomials.

Combining Lemma 3.1 and 3, Proposition 3, we obtain the following consequence.

Proposition 3.2. *For each integer $n \geq 4$, there is an order on the variables such that the cut ideal of the n-cycle has a quadratic squarefree Gröbner basis with respect to the resulting lexicographic order. In particular, the initial ideal of the cut ideal with respect to this order is squarefree.*

Proof. In case $K = \mathbb{C}$, Proposition 3 in [**3**] gives the analogous results for the phylogenetic ideal I_{k-1} on the claw tree with $k - 1$ edges. The arguments of the proof are valid over an arbitrary field. Hence, the claim follows by Lemma 3.1.   □

Remark 3.3. By [**15**], Corollary 2.4, the cut varieties defined by n-cycles are not smooth if $n \geq 4$.

Invoking Lemma 2.3, we obtain our first contribution to Conjecture 1.1.

Corollary 3.4. *The cut variety defined by any cycle is arithmetically Cohen-Macaulay.*

Remark 3.5. The ideal I_{C_4} is Gorenstein because it is a complete intersection cut out by three quadrics. However, the cut ideals I_{C_5} and I_{C_6} are not Gorenstein. We suspect that I_{C_n} is not Gorenstein whenever $n \geq 5$.

We conclude this section by determining the number of minimal generators of the cut ideals of cycles. As preparation, we establish a recursion. Denote by A_n the homogeneous coordinate ring of the variety defined by the phylogenetic ideal I_n, that is,

$$A_n := K[a_{g_1}^{(1)} a_{g_2}^{(2)} \cdots a_{g_n}^{(n)} a_{g_1+g_2+\cdots+g_n}^{(n+1)} : g_i \in \mathbb{Z}_2, i = 1, \ldots, n].$$

Assigning each variable degree one, as a K-algebra A_n is generated by 2^n monomials of degree $n + 1$. We claim:

Lemma 3.6. *If $n \geq 2$, then $\dim_K [A_n]_{2(n+1)} = \dim_K [A_{n-1}]_{2n} + 3^n - 2^{n-1}$.*

Proof. The following set of monomials is a K-basis of $[A_n]_{2(n+1)}$:

$$\mathcal{M} := \{a_{g_1}^{(1)} a_{h_1}^{(1)} \cdots a_{g_n}^{(n)} a_{h_n}^{(n)} a_{g_1+\cdots+g_n}^{(n+1)} a_{h_1+\cdots+h_n}^{(n+1)} : g_j, h_j \in \mathbb{Z}_2\}.$$

We are going to compare this basis with the set of monomials

$$\mathcal{N} := \{a_{g_1}^{(1)} a_{h_1}^{(1)} \cdots a_{g_n}^{(n)} a_{h_n}^{(n)} : g_j, h_j \in \mathbb{Z}_2\}.$$

Observing that, for the monomials in $\mathcal{M}$, the variables $a^{(n+1)}_{g_1+\cdots+g_n} a^{(n+1)}_{h_1+\cdots+h_n}$ are determined by the remaining variables, one is tempted to guess that there is a bijection between $\mathcal{M}$ and $\mathcal{N}$. However, this is not quite true. To illustrate this, consider the following example, where $n = 3$. By interchanging the factors $a^{(1)}_0$ and $a^{(1)}_1$ in

$$m_1 := a^{(1)}_0 a^{(2)}_0 a^{(3)}_0 a^{(1)}_1 a^{(2)}_1 a^{(3)}_0 = a^{(1)}_1 a^{(2)}_0 a^{(3)}_0 a^{(1)}_0 a^{(2)}_1 a^{(3)}_0 =: m_2$$

this monomial in $\mathcal{N}$ produces two different monomials in $\mathcal{M}$, namely

(E) $$m_1 a^{(n+1)}_{0+0+0} a^{(n+1)}_{1+1+0} = m_1 a^{(n+1)}_0 a^{(n+1)}_0 \in [A_3]_8,$$

(O) $$m_2 a^{(n+1)}_{1+0+0} a^{(n+1)}_{0+1+0} = m_2 a^{(n+1)}_1 a^{(n+1)}_1 \in [A_3]_8.$$

To keep track if a monomial in $\mathcal{N}$ gives rise to one or two monomials in $\mathcal{M}$ we use the following decomposition

$$\mathcal{N} = \mathcal{N}_1 \sqcup \mathcal{N}_2 \sqcup \mathcal{N}_3,$$

where

$\mathcal{N}_1 := \{n \in \mathcal{N} : g_1 + \cdots g_n + h_1 + \cdots + h_n = 1\}$

$\mathcal{N}_2 := \{n \in \mathcal{N} : n \text{ is a square}\}$

$\mathcal{N}_3 := \{n \in \mathcal{N} : g_1 + \cdots g_n + h_1 + \cdots + h_n = 0, n \text{ is not a square}\}.$

The monomials in $\mathcal{N}_1$ and $\mathcal{N}_2$ give rise to just one monomial in $\mathcal{M}$, whereas the monomials in $\mathcal{N}_3$ produce precisely two monomials in $\mathcal{M}$ by interchanging the factors $a^{(i)}_{g_i}$ and $a^{(i)}_{h_i}$, where $g_i \neq h_i$. This interchange alters the parity of $g_1 + \cdots + g_n = h_1 + \cdots + h_n$. It follows that

$$|\mathcal{M}| = |\mathcal{N}_1| + |\mathcal{N}_2| + 2|\mathcal{N}_3| = |\mathcal{N}| + |\mathcal{N}'_3|,$$

where

$$\mathcal{N}'_3 := \{n \in \mathcal{N} : g_1 + \cdots + g_n = h_1 + \cdots + h_n = 0, n \text{ is not a square}\}.$$

Each monomial in $\mathcal{N}$ is the product of n quadratic monomials of the form $a^{(i)}_{g_i} \cdot a^{(i)}_{h_i}$, where $g_i, h_i \in \mathbb{Z}_2$. For fixed i, there are three such monomials, thus we get $|\mathcal{N}| = 3^n$, hence

$$|\mathcal{M}| = 3^n + |\mathcal{N}'_3|.$$

Notice that the condition $g_1 + \cdots + g_n = h_1 + \cdots + h_n = 0$ is equivalent to $g_1 + \cdots + g_{n-1} = g_n$ and $h_1 + \cdots + h_{n-1} = h_n$. It follows that the dimension of $[A_{n-1}]_{2n}$ equals the sum of $|\mathcal{N}_3'|$ and the number of squares $n \in \mathcal{N}$ with $g_1 + \cdots + g_{n-1} = g_n$. Since, there are 2^{n-1} such squares, we obtain $\dim_K[A_n]_{2(n+1)} = |\mathcal{M}| = 3^n + \dim_K[A_{n-1}]_{2n} - 2^{n-1}$, as claimed.   □

Now we are ready to compute the number of minimal generators of a cut ideal associated to a cycle. Possibly, it is not too surprising that it has a nice combinatorial interpretation.

Recall that the Stirling number $S(n,k)$ of the second kind is the number of partitions of a set with n elements into k blocks (cf. [11], page 33). Note that $S(n,4) = \frac{1}{24}(4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4)$.

Proposition 3.7.   *If $n \geq 2$, then the cut ideal of an $(n+1)$-cycle is minimally generated by $3 \cdot S(n+1,4)$ quadratic binomials.*

Proof.   By Lemma 3.1, it is equivalent to compute the number of minimal generators of the phylogenetic ideal I_n corresponding to the claw tree $K_{1,n}$. Using the above notation, I_n is an ideal in the polynomial ring R with 2^n variables. We first compute the Hilbert function of the quotient ring R/I_n in degree 2, that is, $h_n(2) := \dim_K[R/I_n]_2$. Since $I_n = \ker\varphi_n$, we get $h_n(2) = \dim_K[A_n]_{2(n+1)}$. Hence, Lemma 3.6 gives if $n \geq 2$:

$$h_n(2) = h_{n-1}(2) + 3^n - 2^{n-1}.$$

Using $h_1(2) = 3$, it follows that

$$h_n(2) = h_1(2) + \sum_{i=2}^{n} 3^n - \sum_{i=1}^{n} 2^n = \frac{1}{2}[3^{n+1} - 1] - [2^n - 1] = \frac{3}{2}3^n - 2^n + \frac{1}{2}.$$

Since, by Proposition 3.2, the ideal I_n is generated in degree two, its number $\mu(I_n)$ of minimal generators is

$$\mu(I_n) = \dim_K[R]_2 - h_n(2) = \binom{2^n + 1}{2} - \frac{3}{2}3^n + 2^n - \frac{1}{2}$$
$$= \frac{1}{2}4^n - \frac{3}{2}3^n + \frac{3}{2}2^n - \frac{1}{2}.$$

It is easily checked that the last number equals $3 \cdot S(n+1, 4)$. The proof is complete. $\square$

4. Cut ideals of trees. As observed by Sturmfels and Sullivant in [**15**], the algebraic properties of the cut ideal associated to a tree only depend on the number of edges and not on the specific structure of the tree. Here we use this to determine the h-vector. It turns out that its entries admit combinatorial interpretations.

Our starting point is:

Theorem 4.1. *Let T be a tree with $n \geq 1$ edges, and let $X_T \subset \mathbb{P}^{2^n - 1}$ be the toric variety defined by the cut ideal I_T. Then X_T is arithmetically Gorenstein of dimension n and degree $n!$. More precisely, X_T is isomorphic to the Segre embedding of $(\mathbb{P}^1)^n$ into $\mathbb{P}^{2^n - 1}$ and its Hilbert function is*

$$h_{S_T/I_T}(i) = (i+1)^n \quad (i \geq 0).$$

Proof. Since each tree with $n \geq 2$ edges can be obtained as the zero sum of an edge and a tree with $n - 1$ edges, the cut variety X_T is isomorphic to the claimed Segre embedding as shown in [**15**, Example 2.3]. This also implies that X_T is arithmetically Gorenstein ([**7**], Corollary 3.3).

The Hilbert function of a Segre product is the product of the Hilbert functions of the factors. This gives the claim on the Hilbert function of X_T. $\square$

Again, we see that the number of minimal generators of a cut ideal grows rapidly when the number of edges increases.

Corollary 4.2. *If T is a tree with n edges, then its cut ideal is minimally generated by $2 \cdot 4^{n-1} + 2^{n-1} - 3^n$ quadrics.*

Proof. The cut ideal I_T lies in a polynomial ring S_n with 2^n variables. Using that, by the above theorem, $h_{S_n/I_T}(2) = 3^n$, the claim follows. $\square$

The cut ideal of a tree has a minimal generating set that is even a Gröbner basis. Note that the cut ideal of a tree with one edge is trivial.

Corollary 4.3. *If T is a tree with a least two edges, then there is a monomial order such that its cut ideal has a quadratic squarefree Gröbner basis. In particular, the corresponding initial ideal is squarefree.*

Proof. This follows by Theorem 2.1 as the Lift and Quad operations preserve the squarefree structure and degree of the binomials. (See also Corollary 18 of [**16**].)   □

Our next goal is to make the Hilbert series of the cut variety of a tree explicit. We will see that it admits a combinatorial interpretation.

Recall that the Hilbert series of a standard graded K-algebra B is the formal power series

$$H_B(t) = \sum_{i \geq 0} h_B(i) t^i.$$

It is a rational function that can be uniquely written as

$$H_B(t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d},$$

where d is the (Krull) dimension of B, $h_0 = 1, h_1, \ldots, h_s$ are integers, and $h_s \neq 0$. If B is Cohen-Macaulay, then all h_i are non-negative and $(h_0, \cdots, h_s)$ is called the h-vector of B. For a projective subscheme $X \subset \mathbb{P}^N$, these concepts are defined by using its homogeneous coordinate ring A_X. However, the *Castelnuovo-Mumford regularity* of X is defined as the regularity of its ideal sheaf. Thus, $\operatorname{reg} X = \operatorname{reg} A_X + 1$.

For a positive integer n, denote by $\mathcal{S}_n$ the symmetric group on n letters. The n-th *Eulerian polynomial* A_n is defined as

$$A_n(t) := \sum_{\sigma \in \mathcal{S}_n} t^{1+d(\sigma)},$$

where $d(\sigma)$ is the number of descents of the permutation σ (see, e.g., [**11**], page 22). Writing

$$A_n(t) = A_{n,1} t + \cdots + A_{n,n} t^n,$$

the coefficients $A_{n,k}$ are called *Eulerian numbers*. Like binomial coefficients, they satisfy a recurrence relation:

$$A_{n,i+1} = (n-i)A_{n-1,i} + (i+1)A_{n-1,i+1}.$$

For trees, the above concepts are related:

Proposition 4.4. *Let T be a tree with $n \geq 1$ edges. Then the Hilbert series of its cut variety X_T is*

$$H_{X_T}(t) = \frac{A_{n,1} + A_{n,2}t + \cdots + A_{n,n}t^{n-1}}{(1-t)^{n+1}}.$$

Moreover, the Castelnuovo-Mumford regularity is $\operatorname{reg} X_T = n$.

Proof. By Theorem 4.1, we know that the Hilbert series of X is

$$H_X(t) = \sum_{i \geq 0} (i+1)^n t^i.$$

It is known (see, for example, [**11**], page 209) that the Eulerian polynomials satisfy

$$\sum_{i \geq 0} i^n t^i = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Dividing by t provides the desired formula for the Hilbert series of X.

Finally, since X is arithmetically Cohen-Macaulay, the regularity of its homogeneous coordinate ring is the degree of the numerator polynomial in the Hilbert series. Hence $A_{n,n} = 1$ provides $\operatorname{reg} X = n$. □

As X_T is arithmetically Gorenstein, its h-vector is symmetric, that is $A_{n,k} = A_{n,n+1-k}$. This also follows directly form the interpretation of the Eulerian number $A_{n,k}$ as the number of permutations in $\mathcal{S}_n$ with $k-1$ excedances (see [**11**], Proposition 1.3.12).

5. Disjoint unions. We want to show that the cut ideal of a disjoint union of two graphs can be studied by means of their zero-sum.
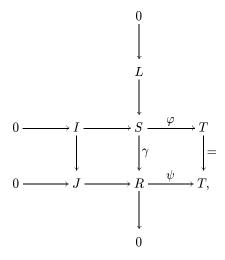
We need a general fact:

Lemma 5.1. *Let $R = K[x_1, \ldots, x_n]$ and $S := K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be polynomial rings in n and $2n$ variables, respectively. Let $\psi : R \to T$ be any K-algebra homomorphism and consider the homomorphism $\varphi : S \to T$ that is defined by $\varphi(y_i) = \varphi(x_i) := \psi(x_i),\ i = 1, \ldots, n.$ Set $I := \ker\varphi$ and $J := \ker\psi.$ Then*

$$I = J \cdot S + (x_1 - y_1, \ldots, x_n - y_n)$$

and $S/I \cong R/J.$

Moreover, if, for some monomial order on R, $\mathbf{F}$ is a Gröbner basis of J, then $\mathbf{F} \cup \{x_1 - y_1, \ldots, x_n - y_n\}$ is a Gröbner basis of I with respect to some monomial order on S.

Proof. This is probably well-known to specialists. For the convenience of the reader we provide a short proof. The K-algebra homomorphism $\gamma : S \to R$ that maps x_i and y_i onto x_i induces the following commutative diagram with exact rows and column

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & L & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & I & \longrightarrow & S & \overset{\varphi}{\longrightarrow} & T & & \\
 & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \parallel{\scriptstyle =} & & \\
0 & \longrightarrow & J & \longrightarrow & R & \overset{\psi}{\longrightarrow} & T, & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

where L is the ideal $L := (x_1 - y_1, \ldots, x_n - y_n).$ Thus, we get an exact sequence

$$0 \to L \to I \to J \to 0.$$

Using also the natural embedding of R as a subring of S, we conclude that $I = J \cdot S + L$, as desired.

The claim about the Gröbner bases follows by using an elimination order on S that extends the term order on R used for computing the Gröbner basis $\mathbf{F}$ with the property that each variable $y_1 > y_2 > \cdots > y_n$ is greater than any monomial in R. □

Let G_1 and G_2 be (non-empty) graphs on v_1 and v_2 vertices. Consider the zero-sum $G_0 := G_1 \# G_2$ obtained by joining the two graphs at any vertex. Its cut ideal lies in a polynomial ring R with $2^{v_1+v_2-2}$ variables. The disjoint union of the two graphs $G_\sqcup := G_1 \sqcup G_2$ defines a cut ideal in a polynomial ring S in $2^{v_1+v_2-1}$ variables. Using this notation, the main result of this section is:

Proposition 5.2. *There is an injective, graded K-algebra homomorphism $\alpha : R \to S$ mapping the variables of R onto variables of S such that*
$$I_{G_1 \sqcup G_2} = \alpha(I_{G_0}) + L,$$
where $L \subset S$ is an ideal that is minimally generated by $2^{v_1+v_2-2}$ linear forms.

Furthermore, the cut variety $X_{G_1 \sqcup G_2} \subset \mathbb{P}^{2^{v_1+v_2-1}-1}$ is isomorphic to the Segre embedding of $X_{G_1} \times X_{G_2}$ into $\mathbb{P}^{2^{v_1+v_2-2}-1}$.

Proof. To simplify notation, set $G_\sqcup := G_1 \sqcup G_2$. Moreover, denote the polynomial rings that are used to define cut ideals of $G_\sqcup$ and the zero sum G_0 by S, S', R, R', that is, $I_{G_\sqcup}$ is the kernel of $\varphi_{G_\sqcup} : S \to S'$ and I_{G_0} is the kernel of $\varphi_{G_0} : R \to R'$.

Let $x \in V(G_1)$ and $y \in V(G_2)$ be the vertices of G_1 and G_2 that are identified in the 0-sum G_0. It will be convenient to denote the resulting vertex in G_0 by z.

There is a natural bijection $\widetilde{\beta} : E(G_0) \to E(G_\sqcup)$, defined by
$$\{i,j\} \mapsto \widetilde{\beta}(\{i,j\}) := \begin{cases} \{i,j\} & \text{if} \quad z \notin \{i,j\} \\ \{i,x\} & \text{if} \quad j = z, i \in G_1 \\ \{i,y\} & \text{if} \quad j = z, i \in G_2 \end{cases}$$
It induces an isomorphism $\beta : R' \to R$.

Now consider any unordered partition $A|B$ of the vertex set of G_0. We may assume that $z \in A$. Then we define a partition $A'|B$ of the vertex set of $G_\sqcup$ by setting $A' := (A \setminus \{z\}) \cup \{x, y\}$. This induces an injective K-algebra homomorphism $\alpha : R \to S$ that maps the variable $q_{A|B} \in R$ onto the corresponding variable $q_{A'|B} \in S$.

Observing that $\widetilde{\beta}$ maps $Cut(A|B)$ onto $Cut(A'|B)$, we get a commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\varphi_{G_0}} & R' \\
\downarrow{\alpha} & & \downarrow{\beta} \\
S & \xrightarrow{\varphi_{G_\sqcup}} & S'.
\end{array}
$$

Since α is injective, it follows immediately that $J := \alpha(I_{G_0}) \subset I_{G_\sqcup}$.

We now consider the set $\mathcal{P}$ of partitions of the vertex set of $G_\sqcup$. We decompose it as
$$\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2,$$
where
$$\mathcal{P}_1 := \{A|B \in \mathcal{P} : x, y \in A\}$$
and
$$\mathcal{P}_2 := \{A|B \in \mathcal{P} : x \in A, \ y \in B\}$$

Given a partition $A|B \in \mathcal{P}_1$, define sets C, D:

$$
\begin{aligned}
C &:= (A \cap V(G_1)) \cup (B \cap V(G_2)) \\
D &:= (B \cap V(G_1)) \cup (A \cap V(G_2)).
\end{aligned}
$$

Then $C|D$ is a partition in $\mathcal{P}_2$, thus we get a map $\varepsilon : \mathcal{P}_1 \to \mathcal{P}_2$. Note that this map is bijective, thus $|\mathcal{P}_1| = \frac{1}{2}|\mathcal{P}| = 2^{v_1 + v_2 - 1}$. Moreover, the cut sets of $A|B \in \mathcal{P}_1$ and $C|D = \varepsilon(A|B)$ are the same. Using also that α maps the variables in R onto variables in S indexed by partitions in $\mathcal{P}_1$, Lemma 5.1 provides

$$(1) \qquad\qquad I_{G_\sqcup} = J + (q_{A|B} - q_{\varepsilon(A|B)} : A|B \in \mathcal{P}_1)$$

and an isomorphisms between the homogeneous coordinate rings of the cut varieties defined by $G_\sqcup$ and G_0. Since taking 0-sums corresponds to forming Segre products, it follows that $X_{G_\sqcup} \cong X_{G_1} \times X_{G_2}$, and the proof is complete. $\square$

The above proof also implies:

Corollary 5.3. *If the cut ideals of G_1 and G_2 admit a squarefree Gröbner basis, then so does the cut ideal of their disjoint union.*

Proof. The assumption implies that the cut ideal of the zero sum of G_1 and G_2 admits a squarefree Gröbner basis. Hence, using the second assertion of Lemma 5.1, Equation (1) provides the claim. □

Now we address the transfer of the Cohen-Macaulay property under forming disjoint unions.

Corollary 5.4. *Let G_1 and G_2 be two graphs such that that their cut varieties are arithmetically Cohen-Macaulay. Then the cut variety associated to the disjoint union of G_1 and G_2 is arithmetically Cohen-Macaulay if and only if the two varieties defined by G_1 and G_2 are Hilbertian.*

Proof. Denote by T, T_1, and T_2 the homogeneous coordinate rings of the cut varieties associated to $G_1 \sqcup G_2, G_1$, and G_2, respectively. Proposition 5.1 provides that T is isomorphic to the Segre product $T_1 \boxtimes T_2$. If, say, G_1 does not have any edge, then T_1 is isomorphic to the a polynomial ring in one variable. Thus it is Hilbertian and T is isomorphic to T_2.

If both graphs G_1 and G_2 have at least one edge, the Krull dimension of T_1 and T_2 is at least two. Then the claim is a consequence of Lemma 2.2. □

Recall that a *forest* is a disjoint union of trees. Thus we get:

Corollary 5.5. *The cut variety defined by any forest is arithmetically Gorenstein.*

Proof. This follows immediately by combining Theorem 4.1 and Proposition 5.2. □

6. Cut ideals of ring graphs. We are ready to analyze cut ideals of more complicated graphs using our results for cut ideals of trees and cycles. In order to describe the graphs, we need additional vocabulary.

A vertex v of a graph G is called a *cutvertex* if the number of connected components of the vertex-induced subgraph on $V(G) \setminus \{v\}$ is larger than that of G. Similarly, an edge e is called a *bridge* if the number of connected components of $G \setminus \{e\}$ is larger than that of G. A *block* of G is a maximal connected subgraph of G without cut vertices. It follows that each block of a graph is either an isolated vertex, a bridge, or a maximal 2-connected subgraph.

Definition 6.1. A *ring graph* is a graph G with the property that each block of G which is not a bridge or a vertex can be constructed from a cycle by successively adding cycles of length at least 3 using the edge-sum (1-sum) construction.

Gitler, Reyes and Villarreal study the family of graphs whose number of primitive cycles equals its cycle rank. They show that this family is precisely the family of ring graphs. Ring graphs were first introduced in [**6**] and [**7**]. They are a subclass of series-parallel graphs. For further background on series-paraller graphs, the reader should refer to [**4**].

Examples of ring graphs include trees and cycles. More precisely, ring graphs are those graphs that can be obtained from trees and cycles by performing clique sums over vertices or edges. They form a large subclass of series parallel graphs, since they are free of K_4-minors ([**6**]).

Combining our results from the previous sections, we obtain the following more precise version of Theorem 1.2:

Theorem 6.2. *If G is a ring graph, then the cut variety X_G is generated by quadrics. In addition, there exists a term order for which its defining ideal I_G has a squarefree quadratic Gröbner basis.*

Therefore, such varieties X_G are Hilbertian and arithmetically Cohen-Macaulay, but not arithmetically Gorenstein in general.

Proof. The theorem follows by applying Proposition 3.2, Theorem 4.1, Corollary 5.3, and Theorem 2.1 repeatedly. □

Recall that if an ideal I admits a quadratic Gröbner basis, then the coordinate ring S/I is Koszul (see [2]). Thus, we get

Corollary 6.3. *The coordinate ring of the cut variety associated to an arbitrary ring graph is Koszul.*

Remark 6.4. Theorem 1.3 of [15] characterizes those graphs whose cut ideals have squarefree reverse-lexicographic initial ideals. Arbitrary ring graphs *do not* fall into that category; however, our result shows that they *do* have squarefree initial ideals with respect to another term order.

We conclude this note with an estimate on the degrees of the higher syzygies of cut ideals. In general, the Castelnuovo-Mumford regularity of a homogeneous ideal can grow doubly exponentially in the number of variables. However, the following result shows that cut ideals of ring graphs have much better properties also in this respect. Recall that the cut ideal of a graph with n vertices lives in a polynomial ring in 2^{n-1} variables.

Corollary 6.5. *If G is a ring graph with e edges, then the Castelnuovo-Mumford regularity of the cut variety X_G satisfies*

$$\operatorname{reg} X_G \leq e + 1.$$

Proof. The least integer r such that the Hilbert function and the Hilbert polynomial of a graded algebra in each degree $j \geq r$ is often called the index of regularity. By Theorem 6.2 we know that the coordinate ring A_G of any cut variety associated to a ring graph G is Hilbertian, i.e., its index of regularity is at most zero. However, in case of Cohen-Macaulay rings the index of regularity and the Castelnuovo-Mumford regularity are related (see, e.g.,[9, Lemma 8]). It follows that $\operatorname{reg} X_G - 1 = \operatorname{reg} A_G < \dim A_G$. Since the dimension of the ring A_G is one plus the number of edges of G, the argument is complete. □

Remark 6.6. (i) Note that by Proposition 4.4 the above bound is almost sharp if the graph is a tree.

(ii) The above arguments use only the fact that cut ideals of ring graphs admit a squarefree Gröbner basis. Thus, they provide the following observation:

If the cut ideal of a graph G admits a squarefree Gröbner basis, then the Castelnuovo-Mumford regularity of the cut variety X_G is bounded above by one plus the number of edges of G.

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