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RINGS OF FINITE RANK AND FINITELY GENERATED IDEALS

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ABSTRACT. Here we provide examples of rings of minimum rank n for every positive integer n. We also introduce a tool that is used to count irreducible elements in finitely generated prime ideals in atomic domains.

1. Introduction. The concept of finite generation is prominent in commutative algebra. Indeed, Noetherian rings, where every ideal is finitely generated, form one of the richest and most fruitful classes of commutative rings with identity. In this paper we consider two classes of rings: the class of rings of finite rank, which is a restrictive subclass of Noetherian rings, introduced by Cohen [**1**]; the class of rings that satisfy the n-generator property, which is a restrictive subclass of Prüfer domains, introduced by Gilmer [**2**]. Both of these classes are defined below.

Let I be an ideal of R. We say that I is n-generated if it can be generated by a set of n elements. Using notation from [**3**], we denote by $\mu(I)$ the minimal number of generators of I. Additionally, we say that the ring R is of finite rank n if every ideal of R is n-generated. For convenience, we will say that a ring that is an element of the class of rings that are off finite rank n is M_n.

The concept of mininum rank will also prove useful. We say that the ring R has minimum rank n if it is M_n and contains an ideal I such that $\mu(I)$ is n. Following the notation from [**4**], we use $\mu_*(R)$ to denote the minimum rank of R. For an example, note that if R is a Dedekind domain, then $\mu_*(R)$ is 1 if $|\mathrm{Cl}(R)|$ is 1 and is 2 otherwise.

It is also natural to generalize these concepts to the realm of non-Noetherian domains. One could consider a ring to have the n-generator

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property if, for a fixed n, every finitely generated ideal is n-generated. For convenience, we will say a ring that is an element of the class of rings satisfying the n-generator property is q-M_n. For a number of years, this was a question of import in the study of Prüfer domains. Indeed, since Dedekind domains are M_2, it is natural to ask if Prüfer domains are q-M_2. In [**3**] and [**5**], Heitmann and Swan answered these problems more or less completely.

The classes of rings of finite rank and the n-generator property have been studied fairly extensively in the literature. After the seminal work of Cohen [**1**], rings of finite rank were considered by Gilmer [**2**], Heitmann [**3**], and most recently Pettersson [**4**], to name a few. One missing facet in the study of finite generation is the existence for each n of rings whose minimum rank is n.

2. Results Regarding Prime Ideals. We begin by connecting finitely generated prime ideals in atomic domains to the irreducible elements of the domain. Afterwards, we will pursue our main result of the existence of rings of minimum rank n for every n. We first recall a definition from factorization theory.

A nonzero nonunit r of a domain R is irreducible if whenever r can be written as ab, then either a or b must be a unit of R. R is an atomic domain if every nonzero nonunit can be written as a finite product of irreducible elements. On the other hand, R is an antimatter domain if R contains no irreducible elements.

Theorem 2.1. *Let R be an atomic domain. Any finitely generated nonzero proper prime ideal P of R can be minimally generated by irreducible elements.*

Proof. Let P be a nonzero finitely generated prime ideal of an atomic domain R. Since P is finitely generated, there exists some minimal set $\{x_1, \dots, x_n\}$ that generates P. As P is nonzero, we can take the generators to be nonzero. Since R is atomic, there exists a factorization of each x_i as a finite product of irreducible elements of R: $\tau_{i1} \cdots \tau_{ik_i}$. Since P is a prime ideal, τ_{ij} must be an element of P for some j. Let τ_i denote the first irreducible τ_{ij} such that τ_{ij} is in P. It is straightforward to see that P can be generated by $\{\tau_i\}_{i=1}^n$. This allows any finitely generated prime ideal P of the atomic domain R to be minimally generated by irreducible elements of R. $\square$

The hypothesis that R is atomic is essential. In [**6**], J. Coykendall and T. Dumitrescu construct a finitely generated maximal ideal in an antimatter domain. Since the ideal is maximal, it must be prime. Since the ring is an antimatter domain, it must contain no irreducible elements. Accordingly the finitely generated prime ideal cannot be generated by irreducible elements.

This next corollary establishes boundaries for the number of irreducible elements in a given prime ideal of an atomic domain.

Corollary 2.2. *Let P be a nonzero proper prime ideal in an atomic domain R. If $\mu(P)$ is n then P contains at least n non-associate irreducible elements. If P cannot be finitely generated, then P contains infinitely many non-associate irreducible elements of R.*

Proof. Let P be a nonzero prime ideal in an atomic domain such that $\mu(P)$ is n. By Theorem 2.1, any finitely generated prime ideal in an atomic domain can be generated minimally by a set of irreducible elements. This forces there to be a generating set $\{\tau_i\}_{i=1}^n$ consisting solely of irreducible elements. Since this generating set is a minimal generating set, the irreducible elements must be pairwise non-associate. Therefore, P contains at least n pairwise non-associate irreducible elements.

On the other hand, let P be a prime ideal in an atomic domain that cannot be finitely generated. Let Γ denote the set of all pairwise non-associate irreducible elements of R contained in P. Assume for the moment that Γ is a finite set. Since P is not finitely generated, Γ does not generate all of P; and, there exists some element x in P which is not in the ideal generated by Γ. Since Γ contains all pairwise non-associate irreducible elements of P, the ideal generated by Γ must contain all irreducible elements of P. Since R is atomic and x is a nonzero nonunit of R, x can be written as a finite product of irreducibles: $\tau_1 \cdots \tau_n$. The fact that P is prime forces τ_j to be an element of P for some j. By construction of Γ, τ_j must be an element of the ideal generated by Γ, contradicting the fact that x is not in the ideal generated by Γ.

Therefore, Γ cannot be finite and hence P contains infinitely many non-associate irreducible elements. $\square$

In this next example, we consider a UFD of dimension 2 where every prime ideal is generated by 2 elements or fewer. This example illustrates that it does not suffice to check prime ideals in order to determine whether or not a ring is of finite rank. In [**7**], Bresinsky shows existence of a field K for every degree $d \geq 4$ such that $K[x_1, \ldots, x_d]$ contains a prime ideal which is not d-generated. It is not known to the author whether or not this example has previously been done in literature.

Example 2.3. The ring $\mathbb{F}[x, y]$ is a UFD with the property that every prime ideal is two generated, however, there exist ideals which require an arbitrarily large number of generators.

Proof. Notice that for any given n, the ideal $(x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n)$ of $\mathbb{F}[x, y]$ requires $n+1$ generators. Accordingly, $\mathbb{F}[x, y]$ is not M_n for any n. To establish the rest of the claim, we consider prime ideals of $\mathbb{F}[x, y]$. Let P be such a prime ideal. If P is the zero ideal, then we are done.

If P is a height one prime ideal, which forces there to be no prime ideals strictly between 0 and P, then we will show that P must be principal. Since $\mathbb{F}[x, y]$ is a UFD, P must contain a nonzero principal prime element, p. As p is a nonzero prime element of $\mathbb{F}[x, y]$, the ideal (p) is a prime ideal strictly containing zero. Since p is an element of P, we arrive at the chain: $0 \subset (p) \subseteq P$. Since P is a height one prime ideal, P must be the same as the ideal (p) and hence is principal.

If P is not a height one prime, then the fact that $\mathbb{F}[x, y]$ is a Noetherian UFD yields the existence of infinitely many principal prime ideals (p_i) such that (p_i) strictly contains 0 and is strictly contained in P. Let p_1 and p_2 be two such prime elements of $\mathbb{F}[x, y]$ such that (p_1) is distinct from (p_2). Since p_1 and p_2 are distinct prime elements of $\mathbb{F}[x, y]$, p_1 and p_2 are relatively prime in $\mathbb{F}(x)[y]$. As relatively prime elements of a principal ideal domain, there exist k_1 and k_2 in $\mathbb{F}(x)[y]$ such that $k_1p_1 + k_2p_2$ is 1. Now k_i is an element of $\mathbb{F}(x)[y]$ and hence can be written as $\dfrac{\hat{k_i}}{f_i}$ where $\hat{k_i}$ is an element of $\mathbb{F}[x, y]$ and f_i is an element of $\mathbb{F}[x]$. Clearing denominators, we attain that $f_2\hat{k_1}p_1 + f_1\hat{k_2}p_2$ must be f_1f_2 where f_1f_2 is an element of $\mathbb{F}[x]$. Notice also that $f_2\hat{k_1}p_1 + f_1\hat{k_2}p_2$, and hence f_1f_2, is an element of P. Since $\mathbb{F}[x]$ is a UFD, f_1f_2 can be written as a product of prime elements each of which must be an element of $\mathbb{F}[x]$. Since P is a prime ideal, one of those prime elements must be an element of P. Let $p(x)$ be such a prime element.

Consider the ring $\dfrac{\mathbb{F}[x,y]}{p(x)\mathbb{F}[x,y]}$. This ring is isomorphic to $\dfrac{\mathbb{F}[x]}{p(x)}[y]$. Notice that as $p(x)$ is a nonzero prime element in the one dimensional domain $\mathbb{F}[x]$, $(p(x))$ is a maximal ideal in $\mathbb{F}[x]$. Accordingly, $\dfrac{\mathbb{F}[x]}{p(x)}$ is a field and hence $\dfrac{\mathbb{F}[x]}{p(x)}[y]$ is a principal ideal domain. Look at the image of P in $\dfrac{\mathbb{F}[x,y]}{p(x)\mathbb{F}[x,y]}$. This ideal, $\hat{P}$, must be principally generated by some element $w(x,y) + p(x)\mathbb{F}[x,y]$ where $w(x,y)$ is an element of P.

We will show that the prime ideal P can be generated by w and p. Certainly, $\{w,p\}$ is a subset of P. Let $r(x,y)$ be an arbitrary element of P. As $r(x,y)$ is an element of P, $r(x,y) + p(x)\mathbb{F}[x,y]$ is an element of $(w(x,y) + p(x)\mathbb{F}[x,y])$. Accordingly, there exists $f(x,y) + p(x)\mathbb{F}[x,y]$ in $\dfrac{\mathbb{F}[x,y]}{p(x)\mathbb{F}[x,y]}$ such that $r(x,y) + p(x)\mathbb{F}[x,y]$ can be written as $(f(x,y) + p(x)\mathbb{F}[x,y])(w(x,y) + p(x)\mathbb{F}[x,y])$ or, more simply, $f(x,y)w(x,y) + p(x)\mathbb{F}[x,y]$. This makes $r(x,y)$ an $\mathbb{F}[x,y]$-linear combination of $w(x,y)$ and $p(x)$; and hence, P is two generated. Consequently, every prime ideal of $\mathbb{F}[x,y]$ can be generated by two elements or fewer. $\square$

3. Existence of Rings of Minimum Rank n as Subrings of Dedekind Domains. We now focus our attention on rings that are of finite rank. Since any M_n domain must be one dimensional and Noetherian, its integral closure must be Dedekind. It is natural to study subrings of Dedekind domains in the pursuit of rings of finite rank. This next theorem connects rings of algebraic integers to finite generation and builds the foundations for the existence of rings of minimum rank n for each n.

Theorem 3.1. *Let D be a Dedekind domain with quotient field $\mathbb{F}$ such that there exists a principal ideal domain R with quotient field K such that $\overline{R_{\mathbb{F}}}$ is D and the degree of $\mathbb{F}$ over K is some integer n. Any ring A which is a subring of D, contains R and has quotient field $\mathbb{F}$ must be M_n.*

Proof . Before we begin, we gather the hypotheses in the following diagram.

$$\begin{tikzcd}
A \arrow[r] & D \arrow[r] & \mathbb{F} \\
 & R \arrow[ul] \arrow[u] \arrow[r] & K \arrow[u, "n"']
\end{tikzcd}$$

Since the degree of $\mathbb{F}$ over K is finite, D must be a free module over the PID, R, of finite rank. Accordingly, A must be a free R-module of rank less than or equal to n and every submodule of A must be generated by n elements or fewer. Since every ideal of A is also a submodule of A, every ideal of A must be generated by n elements or fewer as an R-module. Since every generator is originally an element of A and R is contained in A, this allows the same generating set to generate I as an ideal of A. Thus A must be M_n. $\square$

Before we continue, we will look at an example that illustrates the necessity of some of the hypotheses. Notice that the ring $\mathcal{Q}[x]$ is an M_1 domain whose quotient field is not a finite field extension over $\mathcal{Q}$. The subring $\mathbf{Z}[x]$ shares a quotient field with $\mathcal{Q}[x]$, however, is not M_k for any k since for every n, the ideal $(2^n, 2^{n-1}x, \ldots, 2x^{n-1}, x^n)$ requires $n+1$ generators.

Here we introduce the main result.

Theorem 3.2. *For every natural number n, there exist rings of minimum rank n.*

Proof. We have seen examples of principal ideal rings and Dedekind domains which have minimum rank 1 and 2 respectively. We introduce here an example of a ring which has minimum rank n.

Let R and T denote the rings $\mathbf{Z}[2 \cdot 2^{\frac{1}{n}}, \ldots, 2 \cdot 2^{\frac{n-1}{n}}]$ and $\mathbf{Z}[2^{\frac{1}{n}}]$

respectively. Notice that $2^{\frac{1}{n}}$ is a prime element of T. Let $I = (2, 2{\cdot}2^{\frac{1}{n}}, \dots, 2{\cdot}2^{\frac{n-1}{n}})$. We will show that this ideal requires n generators and that n is an upper bound for the required number of generators for ideals of R. For a contradiction to the first claim, assume that I can be generated by the set $\{\xi_i\}_{i=1}^{k}$ where k is less than n. Note that each ξ_i can be written as $(2^{\frac{1}{n}})^{\beta_i} t_i$ where t_i is an element of T that $2^{\frac{1}{n}}$ does not divide and β_i is a nonnegative integer. Reindex the generating set such that β_i is no greater than β_{i+1} for each i. Notice that since 2 generates I in T, 2 must divide each ξ_i in T. This forces β_i to be at least n for each index i.

To begin, we will show that β_1 must be n. Assume that β_1 is strictly greater than n. As 2 is an element of the ideal generated by $\{\xi_i\}_{i=1}^{k}$, 2 can be written as an R-linear combination $\sum_{i=1}^{k} r_i \xi_i$ where each r_i is an element of R. We now lift to T and compute:

$$
\begin{aligned}
2 &= \sum_{i=1}^{k} r_i \xi_i \\
2 &= \sum_{i=1}^{k} r_i (2^{\frac{1}{n}})^{\beta_i} t_i \\
1 &= \sum_{i=1}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n} t_i \\
1 &= 2^{\frac{1}{n}} \left(\sum_{i=1}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n - 1} t_i \right).
\end{aligned}
$$

This forces $2^{\frac{1}{n}}$ to be a unit of T which is a contradiction. Therefore β_1 must be n. Assume that there exists some index j such that β_j is not $n + j - 1$. Take m to be the smallest such index. Look now at $2 \cdot 2^{\frac{m}{n}}$. As an element of I, it can be written as $\sum_{i=1}^{k} r_i \xi_i$ where each r_i is an element of R. We will show that for each i less than m, 2 divides r_i in T and use this to attain a contradiction. We again lift to T and compute:

$$\begin{aligned}
2 \cdot 2^{\frac{m}{n}} &= \sum_{i=1}^{k} r_i \xi_i \\
2 \cdot 2^{\frac{m}{n}} &= \sum_{i=1}^{k} r_i (2^{\frac{1}{n}})^{\beta_i} t_i \\
2^{\frac{m}{n}} &= r_1 t_1 + \sum_{i=2}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n} t_i \\
r_1 t_1 &= (2^{\frac{1}{n}})^m - \sum_{i=2}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n} t_i \\
r_1 t_1 &= 2^{\frac{1}{n}} \left((2^{\frac{1}{n}})^{m-1} - \sum_{i=2}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n - 1} t_i \right).
\end{aligned}$$

Recall here that $2^{\frac{1}{n}}$ is prime in T and hence must divide either r_1 or t_1. By construction, $2^{\frac{1}{n}}$ does not divide t_1. Accordingly, $2^{\frac{1}{n}}$ divides r_1 in T. All elements of R that, in T, have a factor of $2^{\frac{1}{n}}$ must have at least n factors of $2^{\frac{1}{n}}$ in order to remain elements of R. Therefore, 2 divides r_1 in T. Write r_1 as $2\hat{t_1}$ where $\hat{t_1}$ is an element of T. We now continue our computation:

$$\begin{aligned}
2\hat{t_1} t_1 &= (2^{\frac{1}{n}})^m - \sum_{i=2}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n} t_i \\
2^{\frac{1}{n}} r_2 t_2 &= (2^{\frac{1}{n}})^m - 2\hat{t_1} t_1 - \sum_{i=3}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n} t_i \\
r_2 t_2 &= (2^{\frac{1}{n}})^{m-1} - (2^{\frac{1}{n}})^{n-1} \hat{t_1} t_1 - \sum_{i=3}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n - 1} t_i \\
r_2 t_2 &= 2^{\frac{1}{n}} \left((2^{\frac{1}{n}})^{m-2} - (2^{\frac{1}{n}})^{n-2} \hat{t_1} t_1 - \sum_{i=3}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n - 2} t_i \right).
\end{aligned}$$

We attain again that $2^{\frac{1}{n}}$ divides r_2 or t_2 in T. By construction, $2^{\frac{1}{n}}$ does not divide t_2 so must divide r_2. This forces 2 to divide r_2 in T.

In general, for l less than m, we have:

$$r_l t_l = 2^{\frac{1}{n}} \left((2^{\frac{1}{n}})^{m-l} - \sum_{j=1}^{l-1} (2^{\frac{1}{n}})^{n-l-1+j} \hat{t_j} t_j - \sum_{i=l+1}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n - l} t_i \right).$$

This forces 2 to divide r_l for the same reason that 2 divided r_1 and r_2 previously. Writing r_i as $2\hat{t_i}$, whenever i is less than or equal to m, we are now prepared to return to the original equation:

$$\begin{aligned} 2 \cdot 2^{\frac{m}{n}} &= \sum_{i=1}^{k} r_i (2^{\frac{1}{n}})^{\beta_i} t_i \\ 2 \cdot 2^{\frac{m}{n}} &= \sum_{j=1}^{m-1} 2\hat{t_j} t_j (2^{\frac{1}{n}})^{n+j-1} + \sum_{i=m}^{k} r_i (2^{\frac{1}{n}})^{\beta_i} t_i \\ 1 &= \sum_{j=1}^{m-1} (2^{\frac{1}{n}})^{n-m+j-1} \hat{t_j} t_j + \sum_{i=m}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n - m} t_i \\ 1 &= 2^{\frac{1}{n}} \left(\sum_{j=1}^{m-1} (2^{\frac{1}{n}})^{n-m+j-2} \hat{t_j} t_j + \sum_{i=m}^{k} r_i (2^{\frac{1}{n}})^{\beta_i - n - m - 1} t_i \right). \end{aligned}$$

Notice that $n - m + j - 2$ is a nonnegative number since m is strictly less than n and j is at least 1. This forces $2^{\frac{1}{n}}$ to be a unit in T which is a contradiction; therefore, each β_i is exactly $n + i - 1$. To finish the proof, look at the element $2 \cdot 2^{\frac{n-1}{n}}$:

$$2 \cdot 2^{\frac{n-1}{n}} = \sum_{i=1}^{k} r_i 2^{\frac{n+i-1}{n}} t_i .$$

The same argument from before will yield that 2 divides each r_i since i is less than or equal to k and k is less than n. This yields:

$$1 = \sum_{i=1}^{k} \hat{t_i} 2^{\frac{i}{n}} t_i .$$

Again forcing $2^{\frac{1}{n}}$ to be a unit in T since i ranges from 1 through k. Therefore k cannot be less than n and I requires n generators. This forces the ring $\mathbf{Z}[2 \cdot 2^{\frac{1}{n}}, \ldots, 2 \cdot 2^{\frac{n-1}{n}}]$ to have minimum rank at least n. Since $\mathbf{Z}[2 \cdot 2^{\frac{1}{n}}, \ldots, 2 \cdot 2^{\frac{n-1}{n}}]$ is a subring a Dedekind domain, contains $\mathbf{Z}$, and has the same quotient field as the Dedekind domain whose field extension over $\mathcal{Q}$ is n, Theorem 3.1 forces the ring to be strictly M_n. $\square$

Here we have established existence of rings of finite rank n, where n is the minimum rank of the ring, for every n.

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