JOURNAL OF COMMUTATIVE ALGEBRA
Volume 1, Number 3, Fall 2009

# GRASSMANNIANS AND REPRESENTATIONS

DAN EDIDIN AND CHRISTOPHER A. FRANCISCO

ABSTRACT. In this note we use Bott-Borel-Weil theory to compute cohomology of interesting vector bundles on sequences of Grassmannians.

**1. Introduction.** For any positive integer $n$ the higher cohomology of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^{m-1}$ vanishes. The dimension of the space of global sections of this bundle is easily calculated to be $\binom{n+m}{n}$ via the identification of $H^0(\mathbb{P}^{m-1}, \mathcal{O}(n))$ with the vector space of homogeneous forms of degree $n$ in $m$ variables.

If we view $\mathbb{P}^{m-1}$ as the space of lines in an $m$-dimensional vector space $V$, then the line bundle $\mathcal{O}(n)$ is the $n$-th tensor power of the dual of the tautological line subbundle $\mathcal{O}(-1)$. Generalizing to the Grassmannian of $k$-planes we are led to a number of questions about the cohomology of vector bundles on Grassmannians.

The most obvious question is the following: Let $V$ be a vector space of dimension $m$. Compute the dimension of the space of sections $H^0(\mathrm{Gr}(k, V), \mathcal{V}^{\otimes n})$ as a function of $k, m, n$, where $\mathcal{V}$ is the dual of the tautological rank $k$ subbundle on the Grassmannian of $k$-planes in $V$. Likewise, if we are interested in the dimension of linear systems we could ask for the dimension of the linear system $H^0(\mathrm{Gr}(k, V), (\det \mathcal{V})^{\otimes n})$.

In this note we use Bott-Borel-Weil theory to answer these questions. In particular we will compute for any (irreducible) representation $W$ of $\mathrm{GL}_k$ the dimension $H^0(\mathrm{Gr}(k, m), \mathcal{W})$ where $\mathcal{W}$ is the vector bundle on $\mathrm{Gr}(k, m)$ whose fiber at a point corresponding to a $k$-dimensional linear subspace $L$ is the dual vector space $W^*$ viewed as a $\mathrm{GL}(L)$-module. In addition we explain when the higher cohomology vanishes.

The results we obtain are variants of results that are well known in representation theory. See, for example, Section 4.1 of Weyman's book

2000 AMS *Mathematics subject classification.* Primary 14M15; Secondary 20G05.
The first author was partially supported by N.S.A. grant MDA904-03-1-0040.
Received by the editors on November 12, 2007.
DOI:10.1216/JCA-2009-1-3-381 

[**6**]. The main point of this article is to repackage the classical Bott-Borel-Weil theory in terms of asymptotic behavior of spaces of global sections of bundles on $\mathrm{Gr}(k,m)$ for fixed $k$ and varying $m$.

Our motivation for studying this question came from trying to understand cohomology of vector bundles on classifying spaces. It is also closely related to the study of representations of *ind*-groups ([**2**]). The classifying space, $B\mathbb{G}_m$, for the multiplicative group $\mathbb{G}_m$, is the infinite projective space $\mathbb{P}^\infty$. The sequence of bundles $\{\mathcal{O}_{\mathbb{P}^m}(n)\}_{m=1}^\infty$ defines a line bundle on $B\mathbb{G}_m$ whose global sections is the limit of the countable sequence of finite dimensional vector spaces $H^0(\mathbb{P}^m, \mathcal{O}(n))$.

In a similar way the classifying space $B\mathrm{GL}_k$ may be identified with the infinite Grassmannian $\mathrm{Gr}(k,\infty)$ which is the limit of the finite-dimensional Grassmannians $\mathrm{Gr}(k,m)$. For a given $\mathrm{GL}_k$-module $W$, the associated sequence of vector bundles on $\mathrm{Gr}(k,m)$ defines a vector bundle on the classifying space $B\mathrm{GL}_k$. As we show below, the global sections of these bundles on $Gr(k,m)$ have natural $\mathrm{GL}_m$-module structures and the sequence of global sections $\{H^0(Gr(k,m),\mathcal{W})\}_{m=k}^\infty$ defines a representation of the *ind*-group $\mathrm{GL}_\infty$. When $W$ is irreducible this is an irreducible representation of $\mathrm{GL}_\infty$ in the sense of [**2**].

**2. Definitions and statement of results.** We work over an arbitrary algebraically closed field $K$.

For any $k$ and $m \geq k$ we may construct the Grassmannian $\mathrm{Gr}(k,m)$ as follows: Let $U_{k,m} \subset \mathbb{A}^{mk}$ be the open set parametrizing $m \times k$ matrices of rank $k$. There is a free action of $\mathrm{GL}_k$ on $U_{k,m}$ given by right matrix multiplication and $\mathrm{Gr}(k,m) = U_{k,m}/\mathrm{GL}_k$. The principal bundle $U_{k,m} \to \mathrm{Gr}(k,m)$ is the *frame bundle*; i.e., the fiber over a point of $\mathrm{Gr}(k,m)$ corresponding to a linear subspace $L$ is the set of all possible bases for $L$ in the vector space $K^m$. When $k=1$, this gives the familiar construction of $\mathbb{P}^{m-1}$ as the quotient of $\mathbb{A}^m - \{0\}$ by $\mathbb{G}_m$.

If $W$ is a (left) $\mathrm{GL}_k$-module then we obtain a vector bundle on $\mathrm{Gr}(k,m)$ by taking the quotient $\mathcal{W} = U_{k,m} \times_{\mathrm{GL}_k} W$ where $\mathrm{GL}_k$ acts diagonally on the product $U_{k,m} \times W$. When $W$ is the defining representation of $\mathrm{GL}_k$, then $\mathcal{W}$ is the dual of the tautological rank $k$ subbundle of $\mathrm{Gr}(k,m)$. Likewise if $W$ is the determinant character, then $\mathcal{W}$ is the line bundle which gives the Plücker embedding.

Since $\mathrm{GL}_m$ acts on $U_{k,m}$ by left matrix multiplication the quotient bundle $\mathcal{W} = U_{k,m} \times_{\mathrm{GL}_k} W$ is $\mathrm{GL}_m$-equivariant, so the cohomology of $\mathcal{W}$ is a (left) $\mathrm{GL}_m$-module. The main result of this paper is the determination of these cohomology modules.

To state our theorem we recall some notation about $\mathrm{GL}_k$-modules. Let $T_k \subset \mathrm{GL}_k$ be the group of diagonal matrices and let $B_k$ the group of upper triangular matrices. Any nonincreasing sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ determines a $\mathrm{GL}_k$-module $V_\lambda^{(k)}$ which is defined as follows: The sequence of integers $\lambda$ determines a character of $T_k$ which we also call $\lambda$. This character extends to a representation of $B_k$. Let $L_\lambda$ be the line bundle $\mathrm{GL}_k \times_{B_k} \lambda$, where $B_k$ acts diagonally[1], and set $V_\lambda^{(k)} = H^0(\mathrm{GL}_k/B_k, L_\lambda)$. When the characteristic of $K$ is 0 then $V_\lambda^{(k)}$ is irreducible. In any characteristic, the groups $H^i(GL_k/B_k, L_\lambda)$ vanish for $i > 0$ [**1**].

*Remark* 2.1. With this notation $V_\lambda$ and $V_{\lambda'}$ restrict to the same representation of $\mathrm{SL_k}$ if and only if $\lambda_l - \lambda'_l$ is constant (i.e. independent of $l$). In this case $V_\lambda = V_{\lambda'} \otimes D^{\otimes r}$ where $r = \lambda_l - \lambda'_l$ and $D$ is the determinant character of $\mathrm{GL}_k$.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a sequence of non-increasing integers and let $V_\lambda^{(k)}$ be the associated representation of $\mathrm{GL}_k$. Let $\mathcal{V}_\lambda$ be the corresponding vector bundle on $\mathrm{Gr}(k, m)$.

**Theorem 2.2.** *(a) If $\lambda_k \geq 0$, then $H^0(\mathrm{Gr}(k,m), \mathcal{V}_\lambda) = V_\lambda^{(m)}$, and $H^i(\mathrm{Gr}(k,m), \mathcal{V}_\lambda) = 0$ for $i > 0$. Here $V_\lambda^{(m)}$ denotes the $\mathrm{GL}_m$-module with highest weight $(\lambda_1, \ldots, \ldots \lambda_k, 0, \ldots, 0)$.*

*(b) If $\lambda_k < 0$, then for sufficiently large $m$, $H^i(\mathrm{Gr}(k,m), \mathcal{V}_\lambda) = 0$ for all $i$.*

*Remark* 2.3. As noted in the introduction similar results to Theorem 2.2 appear in the literature (cf. [**6**, Corollary 4.19]). Again the main difference is in perspective. We view our result as inducing from a single representation of $\mathrm{GL}_k$ a family of representations of $\mathrm{GL}_m$ for all $k \geq m$.

[1] In [**3**, p.393] the line bundle whose global sections is the highest weight module $V_\lambda$ is denoted $L_{-\lambda}$.

**3. Bott-Borel-Weil theory for parabolics.** The material here is well known, but we do not know a reference with algebraic proofs.

Let $H$ be an affine algebraic group and let $\pi\colon X \to Y$ be an $H$-principal bundle[2]. If $\mathcal{F}$ is an $H$-equivariant $\mathcal{O}_X$-module, set $\mathcal{F}^{\mathcal{H}}$ to be subsheaf of invariant sections of $\pi_*\mathcal{F}$. Given an $\mathcal{O}_Y$-module $\mathcal{G}$, $\pi^*\mathcal{G} = \pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X$ has a natural $H$-action given by the action of $H$ on $\mathcal{O}_X$. Because $H$ acts freely, the functors $\mathcal{G} \mapsto \pi^*\mathcal{G}$ and $\mathcal{F} \mapsto \mathcal{F}^{\mathcal{H}}$ are inverse to each other.

Let $X' \stackrel{\pi'}{\to} Y'$ and $X \to Y$ be principal $H$-bundles. Let $q\colon X' \to X$ be a $H$-equivariant. Then there is an induced map $p\colon Y' \to Y$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "q"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "p"] & Y \end{tikzcd} \tag{1}$$

commutes.

**Lemma 3.1.** *Diagram* (1) *is cartesian.*

*Proof.* Since the diagram commutes there is an $H$-equivariant map $X' \to Y' \times_Y X$. Now $X' \to Y'$ and $Y' \times_Y X \to Y'$ are both principal bundles over $Y$. After base change by a flat surjective morphism $X' \to Y'$ both bundles are trivialized. Thus the map of $Y'$-schemes, $X' \to Y' \times_Y X$ is an isomorphism after flat surjective base change. Therefore it is an isomorphism. $\square$

**Lemma 3.2.** *Let* $\begin{tikzcd} X' \arrow[r, "q"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "p"] & Y \end{tikzcd}$ *be a commutative (hence cartesian) diagram of principal $H$-bundles as above. Then for any $H$-equivariant $\mathcal{O}_{X'}$-module $\mathcal{F}$ there are natural isomorphism of $\mathcal{O}_Y$-modules $(R^iq_*\mathcal{F})^H = R^ip_*(\mathcal{F}^H)$ for any $i \geq 0$.*

[2]This means [**5**, Definition 0.10] that $\pi$ is flat, $X \to Y$ is a geometric quotient and that $X \times_Y X$ is $H$-equivariantly isomorphic to $H \times X$. This definition implies that the action of $H$ on $X$ is free.

*Proof.* Because $\mathcal{F} = \pi'^*\mathcal{F}^{\mathcal{H}}$, and cohomology commutes with base change [**4**, Proposition 9.3], there are natural isomorphisms $R^iq_*\mathcal{F} = \pi^*R^ip_*\mathcal{F}^H$. Pushing forward by $\pi$ and taking $H$-invariants yields the desired isomorphism. □

Let $G$ be an algebraic group, $B \subset G$ a Borel subgroup and $P \supset B$ a parabolic subgroup. Let $p\colon G/B \to G/P$ be the projection.

**Proposition 3.3.** *If $\lambda$ is a $B$-module, then $G\times_P H^i(P/B, P\times_B\lambda) = R^ip_*(G \times_B \lambda)$ as $G$-equivariant bundles on $G/P$.*

*Proof.* The map $G \times P/B \to G/B$ given by $(g, pB) \mapsto gpB$ is a $P$-principal bundle, where $P$ acts freely on $G \times P/B$ by the formula $q \cdot (g, pB) = (gq, q^{-1}pB)$ where $q \in P$.

Thus we have a commutative diagram with the vertical arrows proper and the horizontal arrows quotient maps by the free action of $B$.

$$\begin{tikzcd}
G \times P/B \arrow[r, "q"] \arrow[d] & G \arrow[d] \\
G/B \arrow[r, "p"] & G/P
\end{tikzcd}$$

Consider the $P$-equivariant line bundle $L = G \times (P \times_B \lambda)$ on $G \times P/B$. Since $q$ is a projection, $R^iq_*L = G \times H^i(P/B, P \times_B \lambda)$. On the other hand $L^P = G \times_B \lambda$ so so by Lemma 3.2, $G \times_P H^i(P/B, P \times_B \lambda)$ is naturally isomorphic to $R^ip_*(G \times_B \lambda)$. Since this isomorphism is natural, it is equivariant for the $G$-actions on these bundles. □

**4. Proof of Theorem 2.2.** Let $G = \mathrm{GL}_m$, and let $B_m$ be the Borel subgroup of upper triangular matrices. Let $P$ be the parabolic subgroup of matrices of the form $\begin{pmatrix} A_k & U \\ 0 & A_{m-k} \end{pmatrix}$ with $A_k \in \mathrm{GL}_k$, $A_{m-k} \in \mathrm{GL}_{m-k}$ and $U$ an arbitrary $k \times (m-k)$ matrix. The subgroup $N$ of matrices of the form $\begin{pmatrix} I & U \\ 0 & A_{m-k} \end{pmatrix}$ is normal and $P/N = \mathrm{GL}_k$.

**Lemma 4.1.** *$U_{k,m}$ is isomorphic to the homogeneous space $G/N$.*

*Proof.* Any $m \times k$ matrix of maximal rank can be viewed as the first $k$ columns of a nonsingular $m \times m$ matrix. Since $\mathrm{GL}_m$ acts transitively on itself by left multiplication, it also acts transitively on the space $U_{k,m}$ of $m \times k$ matrices of maximal rank. The stabilizer of the $m \times k$ matrix $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$ is the subgroup $N \subset \mathrm{GL}_k$. □

Using Lemma 4.1 we obtain the well known identification of $\mathrm{Gr}(k, m) = U_{k,m}/\mathrm{GL}_k = G/P$. Likewise for any $\mathrm{GL}_k$-module, the vector bundle $U_{k,m} \times_{\mathrm{GL}_k} V$ is identified with $G \times_P V$ where $V$ is made into a $P$-module via the surjective map $P \to P/N = \mathrm{GL}_k$.

Since $P$ is a parabolic subgroup containing $B$ we have a proper map $G/B \xrightarrow{p} G/P$. Let $L_\lambda$ be the line bundle on $G/B$ corresponding to the weight $\lambda = (\lambda_1, \dots, \lambda_k, 0, \dots, 0)$. Then by Bott-Borel-Weil $H^0(G/B, L_\lambda) = V_\lambda^{(m)}$.

**Lemma 4.2.** *$p_*(L_\lambda) = \mathcal{V}_\lambda$ as $\mathrm{GL}_m$ equivariant bundles on $\mathrm{Gr}(k, m)$.*

*Proof.* By Proposition 3.3, $p_*(L_\lambda) = G \times_P H^0(P/B, P \times_B L_\lambda)$. Thus to prove the Lemma we must compute $H^0(P/B, P \times_B L_\lambda)$.

Since $P$ is a unipotent extension of $GL_k \times GL_{m-k}$, the homogeneous space $P/B$ is isomorphic to the product $GL_k/B_k \times GL_{m-k}/B_{m-k}$, where $B_k = B \cap (\mathrm{GL}_k \times I_{m-k})$ and $B_{m-k} = B \cap (I_k \times \mathrm{GL}_{m-k})$. Thus

$$H^0(P/B, P \times_B \lambda) = H^0(\mathrm{GL}_k/B_k, \mathrm{GL}_k \times_{B_k} (\lambda|B_k)) \otimes \\ H^0(\mathrm{GL}_{m-k}/B_{m-k}, \mathrm{GL}_{m-k} \times_{B_{m-k}} (\lambda|B_{m-k})).$$

Since $\lambda = (\lambda_1, \dots, \lambda_k, 0, \dots, 0)$, its restriction $B_k$ is the highest weight vector $(\lambda_1, \dots, \lambda_k)$, and its restriction to $B_{m-k}$ is trivial. □

Part (a) of the theorem now follows easily from Lemma 4.2. First observe that $H^0(\mathrm{Gr}(k, m), \mathcal{V}_\lambda) = H^0(G/P, p_* L_\lambda) = H^0(G/B, L_\lambda)$. If all of the $\lambda_k$'s are nonnegative then $H^0(G/B, L_\lambda) = V_\lambda^{(m)}$ and $H^i(G/B, L_\lambda) = 0$ for $i > 0$. It follows from the Leray spectral sequence that that $H^i(G/P, p_* L_\lambda) = 0$ as well.

We now prove (b). Suppose that $\lambda_k < 0$. It suffices to show that, for $m$ sufficiently large, that $H^i(G/B, G \times_B \lambda) = 0$ where is the character of the maximal torus with weight $\lambda = (\lambda_1, \dots, \lambda_k, 0, \dots, 0)$.

Let $\{0\} = V_0 \subset V_1 \cdots \subset V_m$ be the flag fixed by the Borel subgroup of upper triangular matrices $B_m \subset \mathrm{GL}_m$. For any $l < m$ let $P_l$ be the parabolic subgroup fixing the subflag $V_0 \subset V_1 \subset \cdots \subset V_l$. Then $P_l \subset P_{l-1}$ and the fibers of the map $p_l : G/P_l \to G/P_{l-1}$ are isomorphic to the projective space $\mathbb{P}^{m-l}$. In this way the flag variety $G/B = G/P_{m-1}$ is realized as being at the top of the tower of projective bundles

$$G/B \stackrel{p_{m-1}}{\to} G/P_{m-2} \to \cdots \to G/P_2 \stackrel{p_2}{\to} G/P_1 = \mathbb{P}^{m-1}.$$

The group $P_l$ is isomorphic to the product $B_l \times \mathrm{GL}_{m-l}$ where $B_l$ is the Borel group of upper triangular matrices in $\mathrm{GL}_l$. Thus, irreducible representations of $P_l$ are of the form $\chi \times W$ where $\chi$ is a character of the maximal torus in $B_l$ and $W$ is an irreducible representation of $GL_{m-l}$. If $\chi = (a_1, \dots, a_l)$ is a character of the maximal torus of $B_l$ then the same argument used in the proof Proposition 3.3 shows that $R^i p_{l*}(G \times_{P_l} \chi) = G \times_{P_{l-1}} H^i(P_{l-1}/P_l, P_{l-1} \times_{P_l} \chi)$. Under the identification $P_{l-1}/P_l = \mathbb{P}^{m-l+1}$ the line bundle $P_{l-1} \times_{P_l} \chi$ corresponds to $\mathcal{O}_{\mathbb{P}^{m-l+1}}(a_l)$.

If $\lambda$ is a character of $T_m$ with $\lambda_{k+1} = \cdots \lambda_m = 0$ then we may view $\lambda$ as $P_l$-module for $l \geq k$. The argument of the previous paragraph implies that $H^i(G/B, G \times_B \lambda) = H^i(G/P_k, G \times_{P_k} \lambda)$. If $\lambda_k < 0$ then, for all $j$, $H^j(\mathbb{P}^{m-k}, \mathcal{O}_{\mathbb{P}^{m-k}}(\lambda_k)) = 0$ whenever $m \geq -\lambda_k + k$. Thus $R^j p_{k*}(G/ \times_{P_k} \lambda) = 0$ for all $j$ as long as $m$ is sufficiently large. Therefore, by the Leray spectral sequence, it follows that $H^i(G/B, L_\lambda) = 0$ for all $i$.

**5. Some explicit dimension computations.** We conclude by using Theorem 2.2 to give formulas in terms of binomial coefficients for the dimensions of $H^0(\mathrm{Gr}(k,m), \mathcal{V}_\lambda)$ for some particular $\mathcal{V}_\lambda$ involving symmetric powers and the determinant. To do this, we combine Theorem 2.2 with [**3**, Theorem 6.3] to obtain a convenient formula for the dimensions. Again the main point is to package the dimension formulas in terms of the dimensions of a family of cohomology groups for fixed $k$ and varying $m$.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a sequence of nonincreasing integers, and let $V^{(k)}_\lambda$ be the associated representation of $\mathrm{GL}_k$ with $\mathcal{V}_\lambda$ the corresponding vector bundle on $\mathrm{Gr}(k, m)$.

**Lemma 5.1.** *If all $\lambda_i \geq 0$, then*

$$\dim H^0(\mathrm{Gr}(k,m), \mathcal{V}_\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

*where $j$ runs from 2 to $m$, and $\lambda_{k+1} = \cdots = \lambda_m = 0$.*

The first proposition gives a symmetry result about the dimension of a power of the determinant. Throughout the rest of this section, let $V$ denote the standard $k$-dimensional representation of $\mathrm{GL}_k$ and $\mathcal{V}$ the corresponding tautological rank $k$ vector bundle on $\mathrm{Gr}(k, m)$.

**Proposition 5.2.** *For all $0 < k \leq m$ and $l > 0$,*

$$\begin{aligned} \dim H^0(\mathrm{Gr}(k,m), (\det\mathcal{V})^l) &= \dim H^0(\mathrm{Gr}(k, k+l), (\det\mathcal{V})^{m-k}) \\ &= \prod_{j=k+1}^{m} \frac{\binom{l+j-1}{l}}{\binom{l+j-k-1}{l}}. \end{aligned}$$

*Proof.* The partition $\lambda$ associated to $(\det\mathcal{V})^l$ is $\lambda = (l, \ldots, l, 0, \ldots, 0)$. Hence by Lemma 5.1,

$$\begin{aligned} &\dim H^0(\mathrm{Gr}(k,m), (\det\mathcal{V})^l) \\ &\quad = \prod_{j=k+1}^{m} \prod_{i=1}^{k} \frac{l + j - i}{j - i} \\ &\quad = \prod_{j=k+1}^{m} \frac{l!(l+j-1)!(j-k-1)!}{l!(j-1)!(l+j-k-1)!} \\ &\quad = \prod_{j=k+1}^{m} \frac{\binom{l+j-1}{l}}{\binom{l+j-k-1}{l}} = \prod_{j=1}^{m-k} \frac{\binom{l+j+k-1}{l}}{\binom{l+j-1}{l}}, \end{aligned}$$

$$\begin{aligned}
&\dim H^0(\mathrm{Gr}(k,k+l),(\det\mathcal{V})^{m-k}) \\
&\quad= \prod_{j=k+1}^{k+l}\prod_{i=1}^{k}\frac{m-k+j-i}{j-i} \\
&\quad= \prod_{j=k+1}^{m}\frac{(m-k)!(m-k+j-1)!(j-k-1)!}{(m-k)!(m+j-1)!(j-1)!} \\
&\quad= \prod_{j=k+1}^{k+l}\frac{\binom{m-k+j-1}{m-k}}{\binom{m-k+j-k-1}{m-k}} = \prod_{j=1}^{l}\frac{\binom{m+j-1}{m-k}}{\binom{m+j-k-1}{m-k}}.
\end{aligned}$$

To show that the two dimensions are equal, we induct on $l$. When $l=1$, it is easy to compute that both dimensions are $\binom{m}{k}$. Suppose now that $s>1$, and we have proven that the dimensions are the same for $l=s-1$. The induction hypothesis gives us that

$$\prod_{j=1}^{s-1}\frac{\binom{m+j-1}{m-k}}{\binom{m+j-k-1}{m-k}} = \prod_{j=1}^{m-k}\frac{\binom{s+j+k-2}{s-1}}{\binom{s+j-2}{s-1}}.$$

Hence, letting $l=s$,

$$\begin{aligned}
\prod_{j=1}^{s}\frac{\binom{m+j-1}{m-k}}{\binom{m+j-k-1}{m-k}} &= \frac{\binom{m+s-1}{m-k}}{\binom{m+s-k-1}{m-k}}\prod_{j=1}^{m-k}\frac{\binom{s+j+k-2}{s-1}}{\binom{s+j-2}{s-1}} \\
&= \frac{\binom{m+s-1}{m-k}}{\binom{m+s-k-1}{m-k}}\prod_{j=1}^{m-k}\frac{s+j-1}{s+j+k-1}\prod_{j=1}^{m-k}\frac{\binom{s+j+k-1}{s}}{\binom{s+j-1}{s}}.
\end{aligned}$$

The rightmost product is what we want, so it suffices to show that the product of the first two factors is one. Note that

$$\prod_{j=1}^{m-k}\frac{s+j-1}{s+j+k-1} = \frac{\frac{(s+m-k-1)!}{(s-1)!}}{\frac{(s+m-1)!}{(s+k-1)!}} = \frac{\binom{m+s-k+1}{m-k}}{\binom{m+s-1}{m-k}},$$

which completes the proof. □

*Remark* 5.3. Assume that the characteristic of the ground field is 0. If we fix $k$ and $l$ and let $m$ go to infinity, then sequence of irreducible $\mathrm{GL}_m$

representations $\{V^{(m)}_{(l,\dots,l)}\}$ determines an irreducible representation of the *ind*-group $\mathrm{GL}_\infty$. Proposition 5.2 implies that the dimensions of the finite dimensional representations $V^{(m)}_{(l,l,\dots,l)}$ are the same as the dimensions of the irreducible $\mathrm{GL}_{k+l}$-modules $V^{(l+k)}_{(m-k,\dots,m-k)}$.

*Remark* 5.4. As noted in [**2**] the infinite Grassmannian $\mathrm{Gr}(k,\infty)$ is an *ind*-projective variety. For fixed $l$, the sequence of $\mathrm{GL}_m$-modules $\{\mathrm{Sym^l(\wedge^k V_m)/V^{(m)}_{(l,\dots,l)}}\}$ is the *ind*-representation of $\mathrm{GL}_\infty$ corresponding to the degree $l$ component of the ideal of Plücker relations for the embedding of $\mathrm{Gr}(k,\infty)$ into the infinite projective space given by the sequence of projective spaces $\{\mathbb{P}(\wedge^k V_m)\}$. (Here $V_m$ denotes the standard representation of $\mathrm{GL}_m$.)

Proposition 5.2 allows us to compute the sequence of dimensions of the Plücker relations in degree $l$. In particular the dimension of the $\mathrm{GL}_m$-module of Plücker relations in degree $l$ is

$$\binom{\binom{m}{k}+l}{l} - \prod_{j=k+1}^{m} \frac{\binom{l+j-1}{l}}{\binom{l+j-k-1}{l}}.$$

**Proposition 5.5.** *For all $r \geq 0$ and $0 < k \leq m$,*

$$\dim H^0(\mathrm{Gr}(k,m), \mathrm{Sym^r}\mathcal{V}) = \binom{\mathrm{r+m-1}}{\mathrm{r}}.$$

*Proof.* Use Lemma 5.1 and the partition $(r,0,\dots,0)$. □

**Proposition 5.6.** *For all $r \geq 0$, $l \geq 0$, and $0 < k \leq m$,*

$$\dim H^0(\mathrm{Gr}(k,m), \mathrm{Sym^r}\mathcal{V} \otimes (\det\mathcal{V})^{\mathrm{l}}) = \frac{\binom{r+l+m-1}{r}}{\binom{r+l+k-1}{l}} \binom{l+m-1}{l} \prod_{j=k+1}^{m} \frac{\binom{l+j-2}{l}}{\binom{l+j-k-1}{l}}.$$

As we would expect, when $l = 0$, we get $\dim H^0(\mathrm{Gr}(k,m), \mathrm{Sym}^{\mathrm{r}}\mathcal{V})$, and when $r = 0$, we have (after a bit of manipulation) the dimension of $H^0(\mathrm{Gr}(k,m), (\det\mathcal{V})^l)$. Also, the product over $j$ is exactly $\dim H^0(\mathrm{Gr}(k-1, m-1), (\det\mathcal{V})^l)$ (again easy to see after a little algebra).

*Proof.* We use the partition $\lambda = (r+l, l, \ldots, l, 0, \ldots, 0)$, where $\lambda_j = 0$ for all $k+1 \le j \le m$. By Lemma 5.1, the dimension is

$$\prod_{j=2}^{k} \frac{r+j-1}{j-1} \prod_{j=k+1}^{m} \frac{r+l+j-1}{j-1} \prod_{j=k+1}^{m} \prod_{i=2}^{k} \frac{l+j-i}{j-i}.$$

The first product is $\binom{r+k-1}{r}$. Consider the second product. We have

$$\begin{aligned}\prod_{j=k+1}^{m} \frac{r+l+j-1}{j-1} &= \frac{\frac{(r+l+m-1)!}{(r+l+k-1)!}}{\frac{(m-1)!}{(k-1)!}} = \frac{(r+l+m-1)!(k-1)!}{(r+l+k-1)!(m-1)!} \\ &= \frac{(r+l+m-1)!}{r!(l+m-1)!} \frac{r!(l+m-1)!(k-1)!}{(r+l+k-1)!(m-1)!} \frac{l!(r+k-1)!}{l!(r+k-1)!} \\ &= \binom{r+l+m-1}{r}\binom{r+l+k-1}{l}^{-1}\binom{l+m-1}{l}\binom{r+k-1}{r}^{-1}.\end{aligned}$$

The last factor is

$$\begin{aligned}\prod_{j=k+1}^{m} \prod_{i=2}^{k} \frac{l+j-i}{j-i} &= \prod_{j=k+1}^{m} \frac{\frac{(r+l+m-1)!}{(r+l+k-1)!}}{\frac{(m-1)!}{(k-1)!}} \\ &= \prod_{j=k+1}^{m} \frac{(l+j-2)!(j-k-1)!}{(l+j-k-1)!(j-2)!}.\end{aligned}$$

Multiplying the top and bottom by $l!$ yields

$$\prod_{j=k+1}^{m} \frac{\binom{l+j-2}{l}}{\binom{l+j-k-1}{l}}.$$

Finally, multiplying all three products together completes the proof. $\square$

*Remark* 5.7. Using Theorem 2.2 it is easy to show that if $d \leq k$ then $H^0(\mathrm{Gr}(k,m), \mathcal{V}^{\otimes d}) = V_m^{\otimes d}$ where $V_m$ is the standard representation of $\mathrm{GL}_m$. In particular $\dim H^0(\mathrm{Gr}(k,m), \mathcal{V}^{\otimes d}) = m^d$. An interesting question is to determine whether there are relatively simple formulas for the dimension of $H^0(\mathrm{Gr}(k,m), \mathcal{V}^{\otimes d} \otimes (\det \mathcal{V})^l)$. If $l \geq 0$ then Theorem 2.2 gives the summation formula

$$\dim H^0(\mathrm{Gr}(k,m), \mathcal{V}^{\otimes d} \otimes (\det \mathcal{V})^l) = \sum_{\lambda} m_\lambda \dim V_{\lambda+l}^{(m)}$$

where the sum is over all $k$-element partitions $\lambda$ of $d$, $m_\lambda$ is the product of the hook lengths in the tableau corresponding to the partition $(\lambda_1, \ldots \lambda_k)$, and $V_{\lambda+l}^{(m)}$ refers to the $\mathrm{GL}_m$-module with highest weight $(\lambda_1 + l, \ldots \lambda_k + l, 0, \ldots 0)$. Unfortunately, we have do not know how to simplify this sum.

Other summation formulas $\dim H^0(\mathrm{Gr}(k,m), \mathcal{V}^{\otimes d} \otimes (\det \mathcal{V})^l)$ can be obtained using the localization theorem in equivariant $K$-theory for the action of a maximal torus $T \subset \mathrm{GL}_m$. Unfortunately, such formulae involve sums over fixed points which in general we are not able to simplify.

## REFERENCES

**1.** M. Demazure, *Une démonstration algébrique d'un théoréme de Bott*, Inv. Math. **5** (1968), 349–356.

**2.** I. Dimitrov, I. Penkov, J.A. Wolf, *A Bott-Borel-Weil theory for direct limits of algebraic groups*, American J. Math. **124** (2002), 955–998.

**3.** W. Fulton, J. Harris, *Representation theory, a first course*, Springer-Verlag, New York, (1991).

**4.** R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, (1977).

**5.** D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Springer-Verlag, Berlin, (1994).

**6.** J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge University Press, (2003).

Department of Mathematics, University of Missouri, Columbia, MO 65211
**Email address: edidind@missouri.edu**

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078
**Email address: chris@math.okstate.edu**