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ARTINIANNESS OF LOCAL COHOMOLOGY

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ABSTRACT. Let $R = k[[x, y, u, v]]$ over a field k, $I = \langle u, v\rangle$ and $p = xu + yv$. Hartshorne has proved that $H^2_I(R/pR)$ is not artinian. We show that the same is true for *every* element p of $(x, y)R$. In fact, we show an even stronger statement. We use Matlis duals of local cohomology modules.

1. Introduction. It is an interesting question to determine if a given local cohomology module $H^i_I(M)$ is artinian, where I is an ideal of a local ring (R, m) and M is a finite R-module; this is one of Huneke's problems on local cohomology (see [**3**, third problem]). In this note, we prove that a large class of local cohomology modules is not artinian:

Theorem 1.1. *Let (R, m) be a local, complete ring, $n = \dim(R) \geq 4$. Let I be an ideal of R of height $n-2$ such that $H^{n-1}_I(R) = H^n_I(R) = 0$. Let $a, b \in R$ such that $(a, b)R$ is a prime ideal of height two and such that a, b defines a system of parameters for R/I. Then, for every $p \in (a, b)R$, $H^{n-2}_I(R/pR)$ is not artinian.*

We actually prove something stronger: the Matlis dual $D(H^{n-2}_I(R/pR))$ has infinitely many associated prime ideals and is therefore not noetherian.

Theorem 1.1 immediately specializes to the following result which was proved by Hartshorne ([**2**, Section 3]): $H^2_I(R)$ is not artinian, where $R = k[[x, y, u, v]]/(xu+yv)$, k is a field and $I \subseteq R$ is the ideal generated by the classes of u and v in R; in fact, according to Theorem 1.1, we can replace $xu+yv$ by any element of $(x, y)R$ and the statement is still true.

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In Theorem 2.3 of [**6**], Marley and Vassilev generalize Hartshorne's example in a different direction; due to different hypotheses, their and our generalization can be compared only in a special case, see Example 2.7 for details.

2. Results. Let (R, m) be a noetherian local ring; by E we denote a fixed R-injective hull of R/m and by D the Matlis dual functor for R-modules, i e. $D(M) := \mathrm{Hom}_R(M, E)$ for an R-module M.

Theorem 2.1. *Let (R, m) be a local, noetherian ring, $n := \dim(R) \geq 4$. Let I be an ideal of R of height $n-2$ such that $H_I^{n-1}(R) = H_I^n(R) = 0$. Let $a, b \in R$ such that $(a, b)R$ is a prime ideal of height two and such that a, b defines a system of parameters for R/I. Then, for every $p \in (a, b)R$, the set*

$$\{Q \in \mathrm{Ass}_R(D(H_I^{n-2}(R)))|p \in Q\}$$

is infinite. In particular, if R is complete, $H_I^{n-2}(R/pR)$ is not artinian.

Proof. We may assume that $p \notin bR$, because: If $p \in bR$, then we first replace p by b (this is possible because $V(bR) \subseteq V(pR)$); now we have $p \notin (b-a)R$ (because $p \in (b-a)R$ implies $b - a \mid p = b$ and hence $(a, b)R = (b-a, b)R = (b-a)R$, contradiction) and replace b by $b - a$.

Because of $p \notin bR$, there exists, by Krull's Intersection Theorem, $q \in \mathbb{N}^+$ such that

$$p \in (a^q, b)R \setminus (a^{q+1}, b)R .$$

In particular, there are $f, g \in R$ such that

$$p = fa^q + gb .$$

f is not an element of $(a, b)R$, because otherwise there would be $u, v \in R$ such that $f = ua + vb$ and hence $p = ua^{q+1} + (g + va^q)b \in (a^{q+1}, b)R$.

Now, choose $x \in I \setminus (a, b)R$ arbitrary (this is possible as the height of $I + (a, b)R$ is n which is > 2). For every $l \in \mathbb{N}^+$, we have

$$p = f(a^q + x^l g) + g(b - x^l f)$$

and hence $p \in I_l := (a^q + x^l g, b - x^l f)R$. The height of I_l is two (because $\sqrt{I_l + xR} = \sqrt{(a, b, x)R}$ and $3 = \mathrm{height}(a, b, x)R = \mathrm{height}(I_l + xR) \leq$

height$(I_l)+1)$. Clearly, $a^q+x^lg, b-x^lf$ defines a system of parameters for R/I; this implies that there exists a $p_l \in \mathrm{Ass}_R(D(H_I^{n-2}(R)))$ containing I_l (because

$$\begin{aligned}\mathrm{Hom}_R(R/I_l, D(H_I^{n-2}(R))) &= \mathrm{Hom}_R(R/I_l, \mathrm{Hom}_R(H_I^{n-2}(R), E))\\ &= D((R/I_l)\otimes_R H_I^{n-2}(R))\\ &= D(H_I^{n-2}(R/I_l))\\ &= D(H_m^{n-2}(R/I_l))\\ &\neq 0\end{aligned}$$

by the right-exactness of H_I^{n-2}; note that the dimension of R/I_l is precisely $n-2$, because: It is at least $n-2$ because I_l is generated by two elements and it is at most $n-2$ because the height of I_l is two). The height of p_l is two by Lemma 2.3. For $l, l' \in \mathbb{N}^+$, $l \neq l'$, we have

$$\sqrt{I_l+I_{l'}} = \sqrt{(x,a,b)R} \cap \sqrt{(f,g,a,b)R}\ ;$$

in particular, the height of $I_l+I_{l'}$ is three (both f and x are not in $(a,b)R$ by construction); therefore, $p_l \neq p_{l'}$.

$H_I^{n-2}(R/pR)$ is not artinian because its Matlis dual is

$$\begin{aligned}D(H_I^{n-2}(R/pR)) &= \mathrm{Hom}_R(H_I^{n-2}(R/pR), E)\\ &= \mathrm{Hom}_R((R/pR)\otimes_R H_I^{n-2}(R), E)\\ &= \mathrm{Hom}_R(R/pR, \mathrm{Hom}_R(H_I^{n-2}(R), E))\\ &= \mathrm{Hom}_R(R/pR, D(H_I^{n-2}(R)))\ ;\end{aligned}$$

but the latter module has infinitely many associated prime ideals and thus is not noetherian. □

Remark 2.2. Note that, in the situation of Theorem 2.1, it can easily happen that $H_I^{n-2}(R/pR)$ consists only of m-torsion: This is for example the case if R is complete and p is a prime element of R such that $\dim(R/pR) = n-1$ and all minimal prime divisors of $I+pR$ have height $n-2$ (because then, for every prime ideal $q \neq m$ of R/pR one has $H_I^{n-2}(R/pR)_q = H_{I+pR}^{n-2}((R/pR)_q) = 0$ by Hartshorne-Lichtenbaum vanishing). In those cases the socle dimension of $H_I^{n-2}(R/pR)$ is infinite because of Theorem 2.1 and the general fact that an R-module is artinian iff it is only m-torsion and its socle has finite (vector space) dimension.

Lemma 2.3. *Let (R, m) be a noetherian local ring, $n := \dim(R)$. Let $I \subseteq R$ be an ideal of height $n-2$ such that $H_I^{n-1}(R) = H_I^n(R) = 0$. Set $D := D(H_I^{n-2}(R))$. Then* height $q \leq 2$ *for every $q \in \operatorname{Ass}_R(D)$.*

Proof. For $q \in \operatorname{Ass}_R(D)$, one has

$$\begin{aligned} 0 &\neq \operatorname{Hom}_R(R/q, D) \\ &= \operatorname{Hom}_R(R/q, \operatorname{Hom}_R(H_I^{n-2}(R), E)) \\ &= \operatorname{Hom}_R((R/q) \otimes_R H_I^{n-2}(R), E) \\ &= \operatorname{Hom}_R(H_I^{n-2}(R/q), E) \end{aligned}$$

and it follows that $\dim(R/q) \geq 2$ and hence height $q \leq 2$. □

Corollary 2.4. *Let (R, m) be a noetherian, local, complete, regular ring containing a separably closed coefficient field, $\dim(R) := n$. Let $I \subseteq R$ be an height $n-2$ ideal such that R/I is Cohen-Macaulay. Let $a, b \in R$ such that $(a, b)R$ is a prime ideal of R and such that a, b define a system of parameters for R/I. Then, for every $p \in (a, b)R$, the set*

$$\{Q \in \operatorname{Ass}_R(D(H_I^{n-2}(R))) | p \in Q\}$$

is infinite. In particular, $H_I^{n-2}(R/pR)$ is not artinian.

Proof. All hypotheses of Theorem 2.1 are fulfilled: The height of $(a, b)R$ is necessarily two because $\sqrt{(a, b)R + I} = m$ and R is regular (the height of the sum of two ideals is at most the sum of their heights). $\operatorname{Spec}(R/I)$ is two-dimensional and is connected in codimension one because R/I is Cohen-Macaulay (this is well-known; it follows e. g. from a Mayer-Vietoris-sequence argument); hence, by a well-known vanishing theorem (e. g. [**4**, Theorem 2.9]), one has $H_I^{n-1}(R) = H_I^n(R) = 0$. □

Remark 2.5. Note that the statement of Corollary 2.4 remains true for an arbitrary field k if we assume instead that $H_I^{n-1}(R) = 0$;

this holds for example if I is generated by a regular sequence or if $\operatorname{Spec}(R/I)\setminus\{m/I\}$ is formally geometrically connected, see [**4** Theorem 2.9].

In particular, we may set $R := k[[x,y,u,v]]$ where k is a field with variables x,y,u,v, $I := (u,v)R$, $p := xu+yv$ and we get Hartshorne's example: $H_I^{n-2}(R/pR)$ is not artinian.

Marley and Vassilev have shown

Theorem. ([6, Theorem 2.3]) *Let (T,m) be a noetherian local ring of dimension at least two. Let $R = T[x_1,\dots,x_n]$ be a polynomial ring in n variables over T, $I = (x_1,\dots,x_n)$, and $f \in R$ a homogenous polynomial whose coefficients form a system of parameters for T. Then the *socle of $H_I^n(R/fR)$ is infinite dimensional. In particular, $H_I^n(R/fR)$ is not artinian.*

Theorem 2.1 and Theorem 2.3 of [**6**] are both generalizations of Hartshorne's example and can be compared in the following special case:

Example 2.6. Let k be a field, $n \geq 4$,

$$R_0 = k[[x_{n-1},x_n]][x_1,\dots,x_{n-2}] \quad , \qquad R = k[[x_1,\dots,x_n]] \quad ,$$

$I = (x_1,\dots,x_{n-2})R$, p a homogenous element of R_0. Then [**6**, Theorem 2.3] says (implicitly) that

$$H_I^{n-2}(R/pR)$$

is not artinian, if the coefficients of $p \in R_0$ in $k[[x_{n-1},x_n]]$ form a system of parameters for $k[[x_{n-1},x_n]]$, while Theorem 2.1 says that $H_I^{n-2}(R/pR)$ is not artinian if p is contained in $(x_{n-1},x_n)R$, i. e. if none of the coefficients of p is a unit in $k[[x_{n-1},x_n]]$.

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