

CELLULAR RESOLUTIONS OF COHEN-MACAULAY MONOMIAL IDEALS

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ABSTRACT. We investigate monomial labelings on cell complexes, giving a minimal cellular resolution of the ideal generated by these monomials, and such that the associated quotient ring is Cohen-Macaulay. We introduce a notion of such a labeling being maximal. There is only a finite number of maximal such labelings for each cell complex, and we classify these for trees, subdivisions of polygons, and some classes of selfdual polytopes.

1. Introduction. In this paper we study cellular resolutions of monomial ideals which have a Cohen-Macaulay quotient ring. Cellular resolutions of monomial ideals, introduced in [2] and [3], is a very natural technique for constructing resolutions of monomial ideals, and appealing in its blending of topological constructions, combinatorics and algebraic ideas. Much activity has centered around it in the last decade, and good introductions and surveys may be found in [8] and [11]. Usually one starts with a monomial ideal and finds a suitable labeled cell complex giving a (preferably minimal) resolution of the monomial ideal. It was hoped that a minimal resolution of a monomial ideal was always cellular, but this was shown recently not to be so, [10].

Here we turn this around and start with the cell complex, and ask what monomial labellings are such that this cell complex gives a minimal cellular resolution of the ideal formed by the monomials in the labeling. To limit the task we assume that the monomial labeling is such that the monomial quotient ring is Cohen-Macaulay, and the cell complex gives a minimal cellular resolution of it. Such a labeling will be called a Cohen-Macaulay (CM) monomial labelling.

For a given cell complex, we define a notion of maximal CM monomial labeling. These are essentially labellings by monomials $\mathbf{x}^{\mathbf{a}_i}$ in a

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polynomial ring $\mathbb{k}[x_1, \dots, x_n]$ such that any CM monomial labeling of the cell complex by monomials $\mathbf{y}^{\mathbf{b}_i}$ in $\mathbb{k}[y_1, \dots, y_m]$ may be obtained by a multigraded homomorphism $\mathbb{k}[x_1, \dots, x_n] \rightarrow \mathbb{k}[y_1, \dots, y_m]$ sending the monomial $\mathbf{x}^{\mathbf{a}_i}$ to $\mathbf{y}^{\mathbf{b}_i}$. For any cell complex there turns out to be a finite number of maximal CM monomial labellings, and we are in particular concerned with classifying these labellings.

First we consider the case where the cell complex is one-dimensional, it must then be a tree. We show that any CM monomial quotient ring of codimension two has a cellular resolution given by a tree. Then we show that for a given tree there is a unique maximal CM monomial labeling up to isomorphism.

Then we consider the case where the cell complex is two-dimensional. First we look at the case of a polygon. If it is an n -gon with n even, there are no CM monomial labellings, and if n is odd there is a unique CM monomial labeling consisting of monomials of degree $(n - 1)/2$ in n variables. (This is known but we do not know of a specific reference.) We then proceed to consider subdivisions of polygons. By the techniques we use this is quite hard and we only do this in the case of a polygon with a single chord. We show that there are then two maximal CM monomial labellings. The description of them splits into the cases of whether we have an n -gon, where n is even or odd. In all these cases the monomials are in $n + 1$ variables. This makes it reasonable to conjecture that in a subdivision of an n -gon with r chords, any maximal CM monomial labeling consists of monomials in $n + r$ variables.

An interesting example is the subdivision of the hexagon.

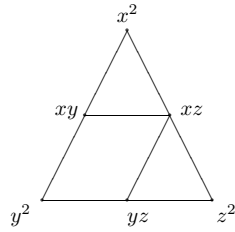


Figure 0.1

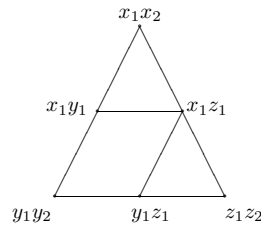


Figure 0.2

A CM monomial labelling is given by Figure 0.1. One may polarise this and get a CM monomial labelling in six variables, Figure 0.2. However this is not maximal. A maximal monomial labelling is given by Figure 3.4 in Subsection 3.2, and consists of monomials in eight variables.

In the end we consider CM monomial labellings of polytopes of dimension three and larger. We classify the maximal CM monomial labelling on pyramids over self-dual polytopes X , provided we know the maximal CM labellings of X . We also consider the elongated pyramid over X which is a union of $X \times [0, 1]$ and the pyramid over X glued together at $X \times \{1\}$. Given a maximal CM labelling of X , we construct such a labelling over the elongated pyramid. We also give several examples of CM labellings of three-dimensional self-dual polytopes, which give cellular resolutions of Gorenstein Stanley-Reisner rings of codimension four.

The organisation of the paper is as follows. In Section 1 we define the notion of a maximal CM monomial labelling. We show that there is a finite number of such for any cell complex, and we give a topological characterisation of such labellings. In Section 2, 3, and 4 we consider maximal CM monomial labellings of cell complexes of dimension 1, 2, and 3 and higher, respectively. In Section 2 we consider the case of trees, and show that there is a unique maximal CM monomial labelling, up to isomorphism. In Section 3 we consider the case of subdivisions of polygons, and in Section 4 we give maximal CM monomial labellings of self-dual polytopes, as well as examples of monomial labellings of three-dimensional self-dual polytopes giving cellular resolutions of Gorenstein Stanley-Reisner rings of codimension four.

1. Maximal Cohen-Macaulay monomial labellings. Let $\mathbb{k}[x_1, \dots, x_r]$ be a polynomial ring, which we may identify with the semi-group ring $\mathbb{k}[\mathbb{N}^r]$. Given an integer n . We shall consider ordered sets of monomials $(\mathbf{x}^{\mathbf{a}_1}, \mathbf{x}^{\mathbf{a}_2}, \dots, \mathbf{x}^{\mathbf{a}_n})$ where none divide any other, i.e. they form a set of minimal generators for an ideal.

A semi-group homomorphism $\mathbb{N}^r \xrightarrow{\phi} \mathbb{N}^s$ maps this ordered set of monomials to another ordered set $(\mathbf{y}^{\mathbf{b}_1}, \mathbf{y}^{\mathbf{b}_2}, \dots, \mathbf{y}^{\mathbf{b}_n})$ given by $\phi(\mathbf{a}_i) = \mathbf{b}_i$. In this way we get a category $\text{Mon}(n)$ whose objects are pairs $(\mathbb{N}^r, \mathbf{a})$ where \mathbf{a} is an n -tuple of elements of \mathbb{N}^r and morphisms are given by semi-group homomorphisms as above, mapping the n -tuples to each other.

Now consider the full subcategory $\text{CM}(n, c)$ of $\text{Mon}(n)$ consisting of those ordered sets of monomials generating an ideal I such that the quotient ring $\mathbb{k}[x_1, \dots, x_r]/I$ is a Cohen-Macaulay ring of codimension c .

Example 1.1 Monomial ideals which are not square free may be polarised. For instance (a^2, ab, b^2) is in $\text{CM}(3, 2)$ and polarises to (a_1a_2, a_1b_1, b_1b_2) also in $\text{CM}(3, 2)$ (with $a_1 - a_2, b_1 - b_2$ as a regular sequence in the quotient ring). This leads us to think of non square free monomial ideals as somewhat compressed monomial ideals. Alternatively polarisation is a “loosening up” of the non square free monomial ideal. However, also square free monomial ideals can be “loosened up”. For instance (ca, ab, bc) in $\text{CM}(3, 2)$ may be “loosened up” to (c_1a, ab, bc_2) isomorphic to the polarisation above (here $c_1 - c_2$ is a regular element in the quotient ring). A central theme of this paper is to investigate the most “free” or “loosened up” monomial ideals. We term these maximal monomial ideals. Here is the formal definition.

Definition 1.2. An object $(\mathbb{N}^r, \mathbf{a})$ in $\text{CM}(n, c)$ is *maximal* if whenever there is a morphism $\phi : (\mathbb{N}^s, \mathbf{b}) \rightarrow (\mathbb{N}^r, \mathbf{a})$, the map $\mathbb{N}^s \rightarrow \mathbb{N}^r$ is a surjection and there is a splitting $\psi : \mathbb{N}^r \rightarrow \mathbb{N}^s$, i.e. $\phi \circ \psi$ is the identity on \mathbb{N}^r .

Example 1.3. The pair $(\mathbb{N}^n, (x_1, \dots, x_n))$ is maximal in $\text{CM}(n, n)$ and is, up to isomorphism, i.e. permutation of variables, the only such object. For, instance, the pair $(\mathbb{N}^3, (x_1, x_2x_3))$ is not maximal in $\text{CM}(2, 2)$ because there is a morphism ϕ to it from $(\mathbb{N}^2, (x_1, x_2))$ such that $e_1 \mapsto e_1, e_2 \mapsto e_2 + e_3$. Note that there is also a morphism from $(\mathbb{N}^3, (x_1, x_2x_3))$ to $(\mathbb{N}^2, (x_1, x_2))$ sending $e_1 \mapsto e_1, e_2 \mapsto e_2$, and $e_3 \mapsto 0$. The morphism ϕ is a splitting of it, consistent with $(\mathbb{N}^2, (x_1, x_2))$ being maximal.

Remark 1.4. Another paper that considers maps of monomial generators is [7]. There one studies the LCM lattice of the monomials and considers a map ϕ between two such lattices which induces an isomorphism on the atoms i.e. the monomial generators. Note that this is a somewhat different situation from ours since in our case, the map on

monomials is induced from a map of semigroup rings. If the map ϕ they consider preserves joins, they show that if F is a free resolution of the ideal generated by the first set of monomials, there is a construction of a complex $\phi(F)$ which is a free resolution of the ideal generated by the second set of monomials. If ϕ is an isomorphism of lattices, then F is a minimal resolution iff $\phi(F)$ is a minimal resolution.

Lemma 1.5. *Let $(\mathbb{N}^r, \mathbf{a})$ and $(\mathbb{N}^s, \mathbf{b})$ be two maximal elements in $CM(n, c)$. Then they are either isomorphic or there are no morphisms between them.*

Proof. If $(\mathbb{N}^s, \mathbf{b}) \rightarrow (\mathbb{N}^r, \mathbf{a})$ is a morphism, then due to the maximality of $(\mathbb{N}^r, \mathbf{a})$, there is a splitting $(\mathbb{N}^r, \mathbf{a}) \rightarrow (\mathbb{N}^s, \mathbf{b})$, so $s \geq r$. But similarly since $(\mathbb{N}^s, \mathbf{b})$ is maximal, we must have $s \leq r$. So $s = r$ and the semigroup homomorphism $\mathbb{N}^r \rightarrow \mathbb{N}^s$ is an isomorphism. \square

Proposition 1.6. *In $CM(n, c)$ there is a finite set of maximal objects, each of which consists of square free monomials. To any object $(\mathbb{N}^r, \mathbf{a})$ in $CM(n, c)$ there is a morphism from some maximal object to this object, i.e. there is a maximal object $(\mathbb{N}^s, \mathbf{b})$ and a semi-group homomorphism $\mathbb{N}^s \rightarrow \mathbb{N}^r$ taking \mathbf{b} to \mathbf{a} .*

Proof. If the monomials are not square free, we can polarise the monomials. So we get a morphism $(\mathbb{N}^s, \mathbf{b}) \rightarrow (\mathbb{N}^r, \mathbf{a})$, where the \mathbf{b} 's are 0, 1-vectors. Clearly there cannot be any splitting ϕ in the reverse direction if \mathbf{a}_i is not square free, since then $\phi(\mathbf{a}_i)$ would not be either. Thus all maximal objects must be square free.

Now given an ordered set of square-free monomials (m_1, \dots, m_n) in $\mathbb{k}[x_1, \dots, x_r]$. To each variable x_p we associate the subset V_p of $[n]$ consisting of those positions i such that x_p divides m_i . This gives us a multiset of subsets of $[n]$, and this multiset determines the isomorphism class in $CM(n, c)$ of the ordered set of monomials. If $V_p = V_q$ for some $p < q$, we get a morphism from some $(\mathbb{N}^{r-1}, \mathbf{b})$ to $(\mathbb{N}^r, \mathbf{a})$ by sending e_i to e_i for $i \neq p, q$ and sending e_p to $e_p + e_q$. But then $(\mathbb{N}^r, \mathbf{a})$ cannot be maximal (there cannot be a splitting due to the ranks of semigroups). Iterating this process we can in the end assume that we to our monomial labelling have associated a family of distinct subsets of $[n]$. A maximal

object must be of this kind. Since there is only a finite number of families of subsets of $[n]$, there is only a finite number of maximal objects. \square

If m_1, \dots, m_n are square free monomials, we may to each variable x_p associate the set V_p of all i in $[n]$ such that x_p divides m_i . If the monomials give a maximal object, we know from the proof above that the V_p are all distinct, thus forming a family of subsets of $[n]$. We let $\text{CM}_*(n, c)$ denote the full subcategory of $\text{CM}(n, c)$ consisting of $(\mathbb{N}^r, \mathbf{a})$ such that the monomials $\mathbf{x}^{\mathbf{a}_i}$ are square free and the subsets $V_p \subseteq [n]$ associated to the variables are all distinct. Note that this family of subsets determines the isomorphism class of the object $(\mathbb{N}^r, \mathbf{a})$. Also if $(\mathbb{N}^r, \mathbf{a})$ and $(\mathbb{N}^s, \mathbf{b})$ are objects in $\text{CM}_*(n, c)$ with associated families \mathcal{F} and \mathcal{G} of subsets of $[n]$, then there is a morphism from the first to the latter iff every element of \mathcal{G} is a disjoint union of elements of \mathcal{F} . This lead us to on the families of subsets of $[n]$ to consider the refinement partial order given by $\mathcal{F} \succ \mathcal{G}$ iff \mathcal{F} consists of refinements of elements of \mathcal{G} together with additional subsets of $[n]$. (A refinement of a set S are subsets of it such that S is a disjoint union of them.)

If \mathcal{F} is a family of subsets of $[n]$ we let its *reduction* \mathcal{F}^{red} be the subfamily of \mathcal{F} consisting of those elements (which are subsets of $[n]$) which are not disjoint unions of other elements of \mathcal{F} .

Proposition 1.7. *a. If \mathcal{F} corresponds to an object in $\text{CM}_*(n, c)$, then \mathcal{F}^{red} corresponds to an object in this category.*

b. An object in $\text{CM}_(n, c)$ is maximal iff the associated family \mathcal{F} is reduced and is maximal among reduced associated families for the refinement order.*

Proof. a. Let \mathcal{F} correspond to $(\mathbb{N}^r, \mathbf{a})$. The elements of \mathcal{F} are indexed by basis elements e_i of \mathbb{N}^r . Let $\mathcal{F} \setminus \mathcal{F}^{\text{red}}$ consist of the sets S_{t+1}, \dots, S_r corresponding to e_{t+1}, \dots, e_r , so $\mathbb{N}^r = \mathbb{N}^t \oplus \bigoplus_{i=t+1}^r \mathbb{N}e_i$. Then \mathcal{F}^{red} corresponds to the monomials \mathbf{b}_i we get as the images of \mathbf{a}_i by the projection $\mathbb{N}^r \rightarrow \mathbb{N}^t$. Alternatively the ring $\mathbb{k}[x_1, \dots, x_t]/(\mathbf{x}^{\mathbf{b}_1}, \dots, \mathbf{x}^{\mathbf{b}_r})$ is obtained from $\mathbb{k}[x_1, \dots, x_r]/(\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r})$ by dividing out by $x_i - 1$ for $i = t + 1, \dots, r$. Now the codimension of the latter ring is the minimal number of sets in \mathcal{F} covering $[n]$. Similarly the codimension of

the former ring is the minimal number of sets in \mathcal{F}^{red} covering $[n]$. But the codimension of the former ring is greater or equal to that of the latter ring since $\mathcal{F}^{\text{red}} \subseteq \mathcal{F}$. Since the latter ring is Cohen-Macaulay, their codimensions must in fact be equal, and by [5, Prop. 18.13], the first is also Cohen-Macaulay. Thus \mathcal{F}^{red} corresponds to an object in $\text{CM}_*(n, c)$.

b. Note that we have a morphism $(\mathbb{N}^t, \mathbf{b}) \rightarrow (\mathbb{N}^r, \mathbf{a})$ since each element of \mathcal{F} is a disjoint union of elements of \mathcal{F}^{red} . Thus if \mathcal{F} is maximal it must be equal to \mathcal{F}^{red} . Clearly then $(\mathbb{N}^r, \mathbf{a})$ is maximal iff the associated family \mathcal{F} is maximal among reduced associated families for the refinement order. \square

Now we shall consider some subcategories of $\text{CM}(n, c)$. First let X be a regular cell complex (see [4] for definition) of dimension $d = c - 1$, where the vertices are labeled by elements of $[n] = \{1, 2, \dots, n\}$, i.e. they are ordered. Let $\text{CM}(X)$ be the subcategory of $\text{CM}(n, c)$ consisting of all objects such that when the vertices of X are labeled with the monomials in this object, the cellular complex associated to this monomial labeling gives a minimal free resolution of the ideal generated by these monomials. Such a labeling will be called a Cohen-Macaulay (CM) labeling of X .

Proposition 1.8. *In $\text{CM}(X)$ there is a finite set of maximal objects. These objects lie in the subcategory $\text{CM}_*(X)$, which is the intersection of $\text{CM}(X)$ and $\text{CM}_*(n, c)$.*

Proof. This goes completely as the proof of Proposition 1.6 \square

Remark 1.9. Another variant is to fix an object $A = (\mathbb{N}^r, \mathbf{a})$ in $\text{CM}(n, c)$ and define $\text{CM}(A)$ to be all objects B in $\text{CM}(n, c)$ which has a map $B \rightarrow A$. In [6] we consider the case when A consists of all square free monomials of degree d in m variables. This is an object of $\text{CM}(\binom{m}{d}, m - d + 1)$. Conjecture 1, in Section 4 in [6] may be formulated as saying that every maximal object over A consist of monomials in $dm - 2\binom{d}{2}$ variables or less. We showed that this number of variables may be attained. The ideas implicit in this conjecture was a motivating factor for this paper. Conjeture 3.16 in the present paper has a similar flavor.

To an isomorphism class of objects in $\text{CM}_*(X)$ there is associated a family \mathcal{F} of subsets of $[n]$ which determines this isomorphism class. In order for a family of subsets of $[n]$ to correspond to an object of $\text{CM}_*(X)$ some conditions must be fulfilled.

Proposition 1.10. *Let $\dim X = d$. A family of subsets \mathcal{F} of $[n]$ corresponds to an object in $\text{CM}_*(X)$ iff the following conditions hold.*

1. *No d of the subsets in \mathcal{F} cover $[n]$.*
2. *Let W be a union of subsets of \mathcal{F} . Then the restriction of X to the complement of W is acyclic.*
3. *For every pair $F \subsetneq G$ of (vertices of) faces of X , there is an S in \mathcal{F} such that $S \cap F$ is empty, but $S \cap G$ is nonempty.*

Remark 1.11. Letting F be the empty set and G consist of a single vertex v in condition 3, we see that the elements of \mathcal{F} cover $[n]$.

Remark 1.12. In brief condition 2. shows that X gives a cellular resolution of the ideal, condition 3. shows the minimality of this resolution, and condition 1. (together with the fact that the elements of \mathcal{F} cover $[n]$) shows the ideal has codimension $\geq \dim X + 1$. Thus condition 1. and 2. gives that the monomial quotient ring is Cohen-Macaulay.

Proof. We first show that condition 2. holds if and only if X gives a cellular resolution of the ideal associated to the monomial labelling. The latter is equivalent to the subcomplex $X_{\leq \mathbf{b}}$, induced on the vertices corresponding to monomials $\mathbf{x}^{\mathbf{a}}$ with $\mathbf{a} \leq \mathbf{b}$, being acyclic for every \mathbf{b} .

Suppose now condition 2. holds. Then $X_{\leq \mathbf{b}}$ is X restricted to the set U of vertices i such that m_i divides $\mathbf{x}^{\mathbf{b}}$. If there is a zero in position p in \mathbf{b} , then clearly V_p is disjoint from U . So all such V_p are subsets of the complement $W = \overline{U}$. But the union of these must be all of W , since if q is in W then m_q does not divide $\mathbf{x}^{\mathbf{b}}$ and so there must be some variable x_p in m_q not in $\mathbf{x}^{\mathbf{b}}$, and so q is in V_p . Thus $X_{\leq \mathbf{b}}$ is X restricted to the complement of a union of V_p 's, and so is acyclic.

Now suppose $X_{\leq \mathbf{b}}$ is always acyclic. If W is a union of V_p 's, let \mathbf{b} be the 0,1-vector with 0 in positions p . Then X restricted to the complement of W is $X_{\leq \mathbf{b}}$, and so acyclic.

Now consider condition 1. That the monomials in the labelling generate an ideal I of codimension $\geq \dim X + 1$ is equivalent to there being no $\dim X$ variables whose associated vertex sets cover the vertices of X .

Condition 3. gives the condition of minimality of the cellular resolution. In fact, minimality is equivalent to the fact that for each pair $F \subsetneq G$ the monomial label $\mathbf{x}^{\mathbf{b}}$ associated to G is strictly larger than the monomial labelling $\mathbf{x}^{\mathbf{a}}$ associated to F . Since we are considering square free monomials, some variable x_p must occur in $\mathbf{x}^{\mathbf{b}}$ and not in $\mathbf{x}^{\mathbf{a}}$. Hence $V_p \cap G$ is nonempty while $V_p \cap F$ is empty. \square

An extra condition that must be fulfilled if the family \mathcal{F} corresponds to a maximal object is the following.

Lemma 1.13. *If a family of subsets \mathcal{F} of $[n]$ corresponds to a maximal object in $CM_*(X)$, then for every S in \mathcal{F} , the restriction of X to S is connected.*

Proof. Suppose X restricted to S is not connected, and let S be $S_1 \cup S_2$ such that $X|_S$ is the disjoint union of X_{S_1} and X_{S_2} . We want to show that $\mathcal{F}' = \mathcal{F} \cup \{S_1, S_2\}$ fulfils the criteria of Proposition 1.10. But then $\mathcal{F}'^{\text{red}}$ would give us a larger family of subsets for the refinement order, contradicting the fact that \mathcal{F} corresponds to a maximal object.

The criteria 1. and 3. hold for \mathcal{F}' given that they hold for \mathcal{F} . We must show that 2. holds. Let \mathcal{G} be the set of complements of sets in \mathcal{F} , and let T be the intersection of elements in a subfamily of \mathcal{G} . Let T_1 and T_2 be the complements $\overline{S_1}$ and $\overline{S_2}$ respectively. We know that X restricted to T and to $T \cap T_1 \cap T_2$ are acyclic. We must show that X restricted to $T \cap T_1$ and to $T \cap T_2$ is acyclic. This follows from the following.

Claim 1. Suppose Y_1 and Y_2 are open subsets of Y such that $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2$ are acyclic. Then Y_1 and Y_2 are acyclic.

This claim follows from the Mayer-Vietoris sequence. \square

We also have the following property of a maximal family.

Proposition 1.14. *Let \mathcal{F} be a maximal family in $CM_*(X)$. Let $t \in T \in \mathcal{F}$. Then there exists $S_1, \dots, S_{\dim X}$ in \mathcal{F} such that*

1. $T \cup \bigcup_{i=1}^{\dim X} S_i$ covers X ,
2. t is not in any S_i .

Proof. By Proposition 1.10.3 there are S_1, \dots, S_r in \mathcal{F} whose union contains all the neighbour vertices of t , but not t itself.

Note that since the complement of $S_1 \cup \dots \cup S_r$ must be connected, this complement is simply $\{t\}$ and so S_1, \dots, S_r cover $X \setminus \{t\}$.

Let \mathcal{G} be the family consisting of the S_i and T . Then X restricted to every complement of a union of elements of \mathcal{G} is acyclic. Hence the associated monomial labeling of X gives a cellular resolution of the associated ideal. By the Auslander-Buchsbaum theorem, the corresponding quotient ring then has codimension $\leq \dim X + 1$. Therefore one must be able to cover X with $\dim X + 1$ subsets in the family \mathcal{G} , and this cover must contain T since only T contains $t \in V$. \square

We let $CM_{\dagger}(X)$ be the subcategory of $CM_*(X)$ such that the associated family \mathcal{F} also fulfils the condition of Lemma 1.13. Then all maximal monomial labelings of X lie in this subcategory.

2. Cellular resolutions of projective dimension 2. Let an ordered set of monomials generate an ideal I such that S/I is Cohen-Macaulay of codimension two. A minimal cellular resolution of S/I must then be an acyclic graph, a tree. We first show that such a cellular resolution exists, describing in principle all such graphs.

2.1 Existence of cellular resolution. Let m_1, \dots, m_n be the monomials, and let K be the complete graph whose vertices are $[n]$. Label vertex i with m_i and the edge $\{i, j\}$ with $\text{lcm}(m_i, m_j)$. For $d \in \mathbb{N}$, let $K_{\leq d}$ be the subgraph of K , consisting of all vertices and edges labeled with a monomial of total degree $\leq d$. Let $\{F_i\}$ be a sequence of subgraphs of K such that the following holds.

- i. $F_i \subseteq F_j$ for $i \leq j$,
- ii. F_i is a spanning forest for $K_{\leq i}$.

As soon as $K_{\leq d}$ contains all vertices and is connected, F_i will be F_d for $i \geq d$ and F_d is a spanning tree T for K .

Proposition 2.1. *Let m_1, \dots, m_n be generators of I such that S/I is Cohen-Macaulay of codimension two.*

a. *The labelled tree T constructed above gives a minimal cellular resolution for I .*

b. *If a tree T labelled by the monomial m_1, \dots, m_n gives a minimal cellular resolution of S/I , then T may be obtained by the construction above.*

Proof. a. Let

$$\bigoplus_{i=1}^n S e_i \xrightarrow{d} S \rightarrow S/I$$

be the start of the minimal resolution and let \mathcal{K} be the first syzygy module, the kernel of d .

Since the Taylor complex, the cellular complex associated to the $n-1$ -simplex labelled by m_1, \dots, m_n , gives a resolution of S/I , the first syzygy module will be generated by

$$\sigma_{i,j} = \frac{\text{lcm}(m_i, m_j)}{m_j} e_i - \frac{\text{lcm}(m_i, m_j)}{m_i} e_j.$$

Each such syzygy corresponds to an edge in the complete graph K .

Let e be the least integer for which $K_{\leq e}$ contains an edge. The edges in F_e give an injective map

$$\bigoplus_{\{i,j\} \in F_e} S e_{i,j} \rightarrow \mathcal{K}$$

It is injective because F_e does not have homology in (homological) degree 1. Also the syzygies corresponding to the edges in F_e generate $\mathcal{K}_{\leq e}$. To see this let $\sigma_{i,j}$ be a syzygy, associated to an edge $\{i, j\}$ not in F_e , then $F_e \cup \{i, j\}$ will contain a cycle $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_r \rightarrow i_1$. Considering $S(-e)^r \rightarrow \mathcal{K}$ defined by these edges and letting M be the least common multiple of the m_{i_j} , we get a minimal syzygy

$$\sum_j M/\text{lcm}(m_{i_j}, m_{i_{j+1}}) \sigma_{i_j, i_{j+1}} = 0$$

where the coefficient of $\sigma_{i,j}$ is nonzero. Since I has projective dimension one, some coefficient must be constant here, and since all $\sigma_{i_j, i_{j+1}}$ have degree e , all coefficients must be constant. Hence $\sigma_{i,j}$ is a linear combination of syzygies corresponding to edges in F_e .

Now we get further a map $\bigoplus_{\{i,j\} \in F_{e+1}} S e_{i,j} \rightarrow \mathcal{K}$. Again this map is injective. We may also argue as above that it is surjective on $\mathcal{K}_{\leq e+1}$: If $\{i, j\}$ is an edge in $K_{\leq e+1}$ not in F_{e+1} or $K_{\leq e}$, then adjoining it to F_{e+1} we get again a cycle and a syzygy. This must have degree $e + 1$ and since $\sigma_{i,j}$ has degree $e + 1$ the coefficient of $\sigma_{i,j}$ must be a constant and so it is a linear combination of the other syzygies. In this way we may continue and get that $\bigoplus_{\{i,j\} \in T} S e_{i,j} \rightarrow \mathcal{K}$ is injective and surjective and so an isomorphism.

b. Let T be a tree labeled by m_1, \dots, m_n giving a minimal cellular resolution of S/I . We will show that $T_{\leq i}$ is a spanning forest for $K_{\leq i}$.

All edges of $T_{\leq i}$ are contained in $K_{\leq i}$. The uniqueness of graded Betti numbers in a minimal free resolution, implies that the cardinality of $T_{\leq i}$ equals the cardinality of $F_{\leq i}$ in the resolution constructed in a. Hence $T_{\leq i}$ must be a spanning forest for $K_{\leq i}$. \square

Example 2.2. The ideal $(x^n, x^{n-1}y, \dots, y^n)$ is of codimension two with Cohen-Macaulay quotient ring. There is a unique tree giving a cellular resolution of this ideal, namely the linear graph on $n+1$ vertices.

On the opposite side of the spectrum one has the following.

Example 2.3. Let x_1, \dots, x_n be the variables. For $i = 1, \dots, n$ let m_i be the monomial $\prod_{p \neq i} x_p$. These generate an ideal of codimension two whose quotient ring is Cohen-Macaulay. By the construction in the theorem, any tree T with vertices $[n]$, gives a cellular resolution of the ideal I .

2.2. Maximal CM monomial labelings. Now given a tree T with vertex set $[n]$, we shall show that, up to isomorphism, there is a unique maximal monomial labelling in $\text{CM}(T)$. Let us describe this.

Maximal monomial labelling of T . Orientate T , i.e. give each edge an orientation. Each edge $s \xrightarrow{e} t$ disconnects the tree into two parts. For each edge e associate a variable x_e to all nodes in the connected component of s and y_e to the connected component of t . To each node v in T there will now be a map $\{\text{edges in } T\} \xrightarrow{v} \{x, y\}$. Here $v(e) = x$ if x_e is associated to v and correspondingly for y . We label the vertex v with the product of all the variables associated to v , i.e. with

$$(1) \quad M_v = \prod_{v(e)=x} x_e \times \prod_{v(e)=y} y_e$$

Theorem 2.4. *Let T be a tree on the vertices $[n]$. In $CM(T)$ there is, up to isomorphism, a unique maximal object M given by the monomial labeling (1). Moreover, for every monomial labeling L in $CM(T)$ there is a unique morphism $M \rightarrow L$ to this object from the maximal object.*

Proof. Given an element in $CM(T)$ where the monomials m_i are in the polynomial ring $\mathbb{k}[z_1, \dots, z_r]$. If $s \xrightarrow{e} t$ is an edge, let the reduced expression for m_t/m_s be $\mathbf{z}^{a_e}/\mathbf{z}^{b_e}$. The sought for morphism of semigroup rings from $\mathbb{k}[\{x_e\}_{e \in E}, \{y_e\}_{e \in E}]$ to $\mathbb{k}[z_1, \dots, z_r]$ must send $M_t/M_s = y_e/x_e$ to m_t/m_s . Since the images of y_e and x_e must be relatively prime (otherwise all the monomials would have a common factor), the only possibility is sending y_e to \mathbf{z}^{a_e} and x_e to \mathbf{z}^{b_e} . Hence the uniqueness of the morphism is clear. We shall now show that this morphism actually does send the monomial M_v in (1) to m_v .

Consider a variable, say z_1 . To each vertex v we associate the exponent of z_1 in m_v . This set of z_1 -exponents must be convex in the sense that given a path (in the non-oriented graph) $s_1 \rightarrow s_2 \rightarrow \dots \rightarrow s_{n-1} \rightarrow s_n$, the z_1 -exponents on this path from s_1 to s_n must first be nonincreasing and then nondecreasing. To see this, suppose to the contrary that there were $i < j < k$ where the exponents fulfilled $n_i < n_j > n_k$. Then letting M be $\text{lcm}(m_i, m_k)$, the subgraph $T_{\leq M}$ would not be connected, contradicting the fact that T gives a cellular resolution.

Now orientate the graph so that all arrows point towards v . There will be a vertex u such that z_1 does not occur in m_u (otherwise the ideal would not have codimension two). So let

$$u = u_0 \xrightarrow{e_1} u_1 \xrightarrow{e_2} \dots \xrightarrow{e_r} u_r = v$$

be the path from u to v . Its z_1 exponents must be non-decreasing. Now $M_v = \prod_{i=1}^r y_{e_i} \times \prod_{e \neq e_i} y_e$ and the z_1 exponent of the image of the first factor is precisely the exponent of z_1 in m_v . We must then show that the z_1 exponent of the image of the second factor is zero, i.e. z_1 does not occur in \mathbf{z}^{a_e} for any $e \neq e_i$.

If $s \xrightarrow{e} t$ is any other edge in the tree, then deleting e , note that t, v and u will be in the same connected component. So the path from s to u must pass through t . But then the z_1 exponents along this path must be non-increasing, and then \mathbf{z}^{a_e} does not contain z_1 .

In view of the uniqueness of the morphism, it is immediate that the labelling (1) is maximal. \square

Remark 2.5 In [9], J.Phan studies the LCM lattice of a monomial ideal, or rather he starts from an atomic lattice and studies monomial ideals with this as their LCM lattice. He shows that for every such lattice there is a distinguished square free monomial ideal, which he calls a minimal monomial ideal, which has this lattice as the LCM lattice.

If one considers the maximal monomial labeling we have on a tree, this is in fact a minimal monomial ideal on the atomic lattice it generates. Note however that for a given tree T , the objects in $\text{CM}(T)$ will give many different LCM lattices. In particular Example 2.3 shows that on any tree with n vertices there is a monomial labeling whose atomic lattice is the lattice \hat{L} where $\hat{L} \setminus \{0, 1\}$ is the antichain of n elements. In fact this monomial labeling is also the minimal monomial ideal for this lattice.

An interesting question is : Given two ideals I and J in $\text{CM}(n, 2)$ which have the same LCM lattice, do they have the same set of trees which give a cellular resolution? If so it would induce a nice correspondence between families of trees and families of atomic lattices.

3. Subdivisions of polygons. Consider a polygon whose vertices are labeled by $0, 1, \dots, n-1$ and let X be a subdivision without introducing new vertices. I.e. we get the subdivisions by introducing chords in the polygon. We shall use the notation $[i, j]$, called a *string* to denote the vertices obtained by starting at i and increasing by one each step modulo n until we reach j . The length of a string is the number of vertices in it. We denote by V the set of vertices $\{0, 1, \dots, n-1\}$.

We want to describe the maximal monomial labelings in $\text{CM}(X)$. By Lemma 1.13 we may assume that these are in $\text{CM}_\dagger(X)$. To a monomial labeling on X , there is associated a family \mathcal{F} of subsets of the vertices V which determines the monomial labeling up to isomorphism (i.e. permutation of variables).

Lemma 3.1. *Let \mathcal{F} be the family of subsets of V associated to an object in $\text{CM}_\dagger(X)$. Then each element in \mathcal{F} is a string $[i, j]$.*

Proof. Let s be an element of \mathcal{F} and suppose s is not a string. Then s is a disjoint union of two or more strings, none of which are adjacent. If there is a chord between any of these strings, then X restricted to the complement of s is not connected, contradicting Proposition 1.10. Hence there are no chords between strings in s , and so X restricted to s is not connected, contradicting Lemma 1.13.

Lemma 3.2. *For every vertex i there is a string in \mathcal{F} ending at i and a string in \mathcal{F} starting at i .*

Proof. Since the monomials constitute a minimal generating set for the ideal, there must be some variable in m_i which is not in m_{i-1} . But then to this variable the associated vertex subset must be a string starting at i . Similarly we get a string ending at i by considering m_{i+1} . \square

Lemma 3.3 *Let i be a vertex which is not the end of a chord. Then in \mathcal{F} there is a string ending at $i-1$ and a string starting at $i+1$ whose union cover $V \setminus \{i\}$.*

Proof. By Lemma 3.2 there are strings ending at $i-1$ and starting at $i+1$. Letting W be their union, we know that X restricted to the complement of W is acyclic. But i is an isolated vertex here, and so this must be the only vertex in the complement of W . \square

Lemma 3.4. *Given a string s in \mathcal{F} starting at i , which is not the endpoint of a chord. Then there is a string in \mathcal{F} starting at $i+1$ of length greater or equal to that of s . Similarly if s ends at j , not the endpoint of a chord, there is a string ending at $j-1$ of greater or equal length than that of s .*

Proof. There is a string t starting at $i + 1$ which together with a string ending at $i - 1$ covers $V \setminus \{i\}$. If t was contained in s , then s and the string ending in $i - 1$ would cover V . Impossible. Hence t is not contained in s , and its length must be at least that of s . \square

Now we are ready to do the case when X is a polygon, i.e. it contains no chords.

Theorem 3.5. *Let X be a polygon and \mathcal{F} the family of strings associated to an object in $CM_{\dagger}(X)$.*

a. *If the number of vertices is odd, $n = 2r + 1$, then \mathcal{F} consists of all strings of length r . Hence, up to isomorphism, there is only one object in $CM_{\dagger}(X)$ and this is the only maximal monomial labeling of X (up to isomorphism).*

b. *If the number of vertices is even, there are no monomial labelings of X giving ideals of codimension three with Cohen-Macaulay quotient ring, i.e. $CM(X)$ is empty.*

Remark 3.6. This result is known but we don't know of a specific reference. It is of course closely related to the Buchsbaum-Eisenbud structure theorem in [1, Thm. 2.1].

Proof. Let L be the length of the longest string in \mathcal{F} . By Lemma 3.4 all strings of this length must be in \mathcal{F} . Since two strings cannot cover the vertices, we must have $2L < n$. If n is even, equal to $2r$, then $L < r$. But then there will be two disjoint nonadjacent strings of length L , and so X restricted to the complement of the union of these is disconnected. Hence n cannot be even. If n is odd, equal to $2r + 1$, a similar argument gives that L cannot be less or equal to $r - 1$. Hence we must have $L = r$. Again an argument as above gives that there can not be in addition any string of length less or equal to $r - 1$. And so all strings have length r . It is easy to check that the family of all strings of length r fulfils the criteria of Proposition 1.10, and so it corresponds to an object in $CM_{\dagger}(X)$. Being the only object of $CM_{\dagger}(X)$ (up to isomorphism), it must be a maximal monomial labeling of X . \square

Now in the rest of this section, we shall consider the case of a polygon with one chord. In this case we shall show that there are exactly two maximal objects in $\text{CM}(X)$. They must be in $\text{CM}_\dagger(X)$ and so the families \mathcal{F}_1 and \mathcal{F}_2 of subsets of vertices will consist of strings. Let us describe these families of strings explicitly.

Maximal monomial labelings of X . Let the chord be between 0 and a where $2a \leq n$. Suppose the number of vertices n is odd, equal to $2r + 1$. The first family \mathcal{F}_1 is the family in Theorem 3.5 a. extended by adding one string. It consists of :

- a. All strings of length r .
- b. The string $[1, a - 1]$.

The family \mathcal{F}_2 is given by :

- a. All strings of length $r + 1$ containing $[0, a]$.
- b. All strings of length r containing i) 0 but not $a - 1$, or ii) a but not 1 .
- c. All strings of length $r - 1$ disjoint from $[0, a]$.
- d. The string $[1, a - 1]$.

In both cases the families consist of $n + 1$ subsets of vertices, and hence the maximal labellings of X are monomials in $n + 1$ variables.

Suppose the number of vertices n is even, equal to $2r$. The family \mathcal{F}_1 is given by :

- a. All strings of length r containing 0 .
- b. All strings of length $r - 1$ not containing 0 and 1 .
- c. The string $[1, a - 1]$.

The family \mathcal{F}_2 is the mirror image of the family \mathcal{F}_1 :

- a. All strings of length r containing a .
- b. All strings of length $r - 1$ not containing a and $a - 1$.
- c. The string $[1, a - 1]$.

Again in both cases the families consist of $n + 1$ subsets of vertices, and hence the maximal monomial labellings of X are monomials in $n + 1$ variables.

Theorem 3.7. *Let X be a polygon with a chord. Then there are two maximal objects in $CM(X)$, and their associated families of subsets of vertices are given by \mathcal{F}_1 and \mathcal{F}_2 above.*

We shall proceed to prove this theorem through a series of lemmata. In the lemmata below we let \mathcal{F} be the family of subsets of vertices arising from an object of $CM_{\dagger}(X)$.

Lemma 3.8. *There is no string in \mathcal{F} containing the complement of $(0, a)$.*

Proof. Let i be in the complement of $[0, a]$. By Lemma 3.3 there is a string ending in $i - 1$ and a string starting in $i + 1$, together covering $V \setminus \{i\}$. At least one of them has length $\geq (n - 1)/2$, say the one ending in $i - 1$. We can then by Lemma 3.4 push it backwards till it ends in a . Then it must start in 1 or earlier. If there were a string containing the complement of $(0, a)$, these two strings would cover V . Impossible.

□

3.1. *The case of n even.* We now assume that the polygon has an even number of vertices $2r$.

Lemma 3.9. *Let i be in $(0, a)$. Let s_1 be a string ending in $i - 1$ and s_2 a string starting in $i + 1$. Then s_1 has length r and s_2 length $r - 1$, or conversely.*

Proof. Suppose s_1 has length $\geq r + 1$. By Lemma 3.4 we can then step by step push it back to a string ending at 0 which also has length $\geq r + 1$. It does not begin in $[0, a]$ by Lemma 3.8, so we can also push it forward to a string starting at 0 and having length $\geq r + 1$. Together with the one ending in 0, these two would cover the vertices of X . Impossible.

Thus s_1 has length $\leq r$, and similarly s_2 has length $\leq r$. If s_2 has length $\leq r - 2$, the complement of $s_1 \cup s_2$ consists of $\{i\}$ together with another disjoint component. Impossible. Thus the strings have length r or $r - 1$. If both had length $r - 1$, the complement of $s_1 \cup s_2$ will

again consist of at least two components. If both had length r , we could push the end of s_2 one step back to get s'_2 such that $s_1 \cup s'_2$ covers V . Impossible. Thus the lengths are r and $r - 1$. \square

Thus either there is a string of length r containing 0 but not a , or there is a string of length r containing a but not 0. These cases turn out to be mutually exclusive and separates the treatment into two cases. These are symmetric and we shall consider the first case.

Lemma 3.10. *Suppose a string in \mathcal{F} containing 0 but not a , has length r .*

a. *Then all strings in \mathcal{F} containing 0 have length r , and all strings of length r containing 0 are in \mathcal{F} .*

b. *All strings in \mathcal{F} which do not contain 0 or 1 have length $r - 1$, and all strings of length $r - 1$ not containing 0 and 1 are in \mathcal{F} .*

Proof. Let s be the string of length r in \mathcal{F} containing 0 but not a . We can push it successively forward until we get a string starting in 0. We can also start with s and push it successively backwards to a string ending in 0. If the length jumps up at some stage, these two strings will cover V . Impossible. Thus all these strings have length r .

Now let s be a string starting in 2. By Lemma 3.9, s has length $r - 1$. We can successively push it forward, one step a time, until it starts in a . If it now had length $\geq r$, then since there is a string of length r ending in $a - 1$, these would cover X . Impossible. Thus all of these strings have length $r - 1$.

Let now s be a string in \mathcal{F} ending in -1 . We can successively push it backwards, getting s' ending in $-r$. Since there is a string $[-r + 1, 0]$, s' must start at 2 or later, in order to avoid two strings covering V . Thus all such strings have length $\leq r - 1$. Now there is a string t starting at 2 of length $r - 1$. If s (ending in -1) had length $\leq r - 2$, the complement of $s \cup t$ would be disconnected. Thus s has length $r - 1$, and we get part b. \square

Lemma 3.11. *In the situation of Lemma 3.10, the family \mathcal{F} has one string starting in 1 and this is the string $[1, a - 1]$.*

Proof. Let s begin in 1. Then s has length $\leq r - 1$ since else we could push it forward one step and get a string of length r starting in 2. Impossible. If s ends in a point $\geq a$, let t end in -1 , note that t has length $r - 1$. Then $s \cup t$ has a complement which is disconnected. Hence s ends in a point in $(0, a)$, According to Lemma 3.9 its endpoint must be $a - 1$. \square

Proof of Theorem 3.7 [Even number of vertices] The Lemmata above show that a maximal family in $\text{CM}_\dagger(X)$ must be either \mathcal{F}_1 or \mathcal{F}_2 . It is an easy matter to verify that they also satisfy the criteria of Proposition 1.10. \square

3.2. *The case of n odd.* We now assume that the number of vertices of the polygon is odd equal to $2r + 1$. We let \mathcal{F} be a maximal CM family of subsets of vertices.

Lemma 3.12. a. *Let i be in $(0, a)$. Let s_1 end in $i - 1$ and s_2 begin in $i + 1$. Then both s_1 and s_2 have length r .*

b. *All strings in \mathcal{F} have length $\leq r + 1$.*

Proof. Part a. goes as Lemma 3.9. To show part b., let s be a string of length $\geq r + 2$. It does not both start and end in $[0, a]$ by Lemma 3.8. Suppose it ends outside of $[0, a]$. Then we can push it backward to a string containing 0, and then forward to a string starting in 0, of length $\geq r + 2$. But by part a. there is a string ending in 0 of length r and these two would cover V . Impossible. \square

Lemma 3.13. *All strings in \mathcal{F} containing $[0, a]$ have the same length, and this length is either r or $r + 1$. Either i) all strings of length $r + 1$ containing $[0, a]$ are in \mathcal{F} , or ii) all strings of length r containing $[0, a]$ are in \mathcal{F} , in which case $a < r$, or iii) there are no strings containing $[0, a]$, in which case $a = r$.*

Proof. There is a string s ending in 0, it has length r by Lemma 3.12. By Lemma 3.8 it does not start in $[0, a]$. We may then push it forward to a string t starting in 0, having length $\leq r + 1$. Suppose the

length is $r + 1$. We can freely push it back and forth over $[0, a]$ and its length will not decrease by Lemma 3.4, and not increase by Lemma 3.12, so all such strings are in \mathcal{F} . Suppose there in this case also were a string s in \mathcal{F} of length r containing $[0, a]$. Suppose the distance from its start point $i + 1$ to 0 is less or equal to the distance from its end point to a . There is a string u ending in $i - 1$ which together with s covers $V \setminus \{i\}$. The length of u cannot be $r + 1$ since we then could push u one step forward and this string together with s would cover V . Thus u has length r . But then the complement of u would be a string u' containing $[0, a]$ of length $r + 1$ and thus be in \mathcal{F} . Thus s and u' together would cover V . Impossible.

Suppose now there is a string t of length $\leq r - 1$ containing $[0, a]$, and let t end in $i - 1$. There is a string s starting in $i + 1$. Since $s \cup t$ must be $V \setminus \{i\}$, s must have length $\geq r + 1$. We can then successively push s forward till it starts in $r + 1$. Its length must by Lemma 3.12.b be $r + 1$, and so it ends in 0. But this is impossible by part a. of the same lemma. Thus the length of t is r or $r + 1$.

Suppose now the string t in the beginning of the proof had length r . (Then $a < r$.) We can push it back and forth over $[0, a]$. All these strings are in \mathcal{F} , and the length does not increase since by the first part of the proof, strings covering $[0, a]$ cannot both have length r and $r + 1$. \square

The argument now splits into two cases corresponding to whether \mathcal{F} contains strings of length $r + 1$ or not. These will give the two maximal families \mathcal{F}_1 and \mathcal{F}_2 .

Lemma 3.14. a. *Suppose \mathcal{F} has strings of length $r + 1$. Then all strings in \mathcal{F} disjoint from $[0, a]$ have length $r - 1$ and all such strings of length $r - 1$ are in \mathcal{F} .*

b. *Suppose \mathcal{F} has no strings of length $r + 1$. Then all strings in \mathcal{F} disjoint from $[0, a]$ have length r and all such strings of length r are in \mathcal{F} .*

Proof. a. Suppose s is disjoint from $[0, a]$. If its length is r , its complement has length $r + 1$ and covers $[0, a]$. Thus it is also in \mathcal{F} . Impossible since the vertices of X cannot be covered by two sets in \mathcal{F} . The length of s cannot be $\geq r + 1$ by a similar argument. Suppose s has length $\leq r - 2$. So suppose it ends in $i - 1$ distinct from -1 . If t begins in $i + 1$ it must together with s cover $V \setminus \{i\}$ and thus have length $\geq r + 2$. Impossible. Similarly it cannot start in a vertex distinct from $a + 1$. So it would have to start in $a + 1$ and end in -1 . But that is not possible if the length is $\leq r - 2$. Thus s is of length $r - 1$.

Now there is a string starting in $a + 1$. Since it cannot cover V together with a string containing $[0, a]$, its endpoint must be before 0 . Thus it has length $r - 1$. The same holds also for strings ending in -1 .

We now push the string starting in $a + 1$ successively forward until we get a string t starting in $-r$. The endpoint of t is not ≥ 0 since it would then cover V with the string $[0, r]$. Thus t is in the complement of $[0, a]$ and has length $r - 1$, and ends in -2 .

The proof of part b. is analogous. \square

Lemma 3.15. a. *Suppose \mathcal{F} has strings of length $r + 1$. Then $[1, a - 1]$ is in \mathcal{F} and is the only string starting in 1 or ending in $a - 1$.*

b. *Suppose \mathcal{F} has no strings of length $r + 1$. Then $[1, r]$ is in \mathcal{F} , and the only other possible string in \mathcal{F} starting in 1 is $[1, a - 1]$. Correspondingly $[a - r, a - 1]$ is in \mathcal{F} and the only other possible string ending in $a - 1$ is $[1, a - 1]$.*

Proof. a. Let s start in 1. Then it is of length $\leq r$ since otherwise we could push it forward to a string starting in 2 of length $\geq r + 1$. Impossible by Lemma 3.12. If it ends in a point $\geq a$, let t be a string of length $r - 1$ ending in -1 . Then the complement of $s \cup t$ would be disconnected. Impossible. Thus s ends in a point in $(0, a)$. According to Lemma 3.12 it must then end in $a - 1$. The argument concerning the string ending in $a - 1$ is analogous.

b. Let s start in 1. As above its length must be $\leq r$. Suppose the length is $< r$. If it ends in $(0, a)$ its endpoint must be $a - 1$ according to Lemma 3.12. That s ends in a point $\geq a$ is as above impossible.

The string starting in 2 has length r . Pushing it backward one step, its length cannot increase. Thus we get a string of length r starting in 1. \square

Proof of Theorem 3.7 [Odd number of vertices] The lemmata above show that if \mathcal{F} has a string of length $r + 1$, it must be the family \mathcal{F}_2 . Also if \mathcal{F} does not contain strings of length $r + 1$, it must contain all strings of length r and the only possible extra string being $[1, a - 1]$. But adding this string gives a family fulfilling the criteria of Proposition 1.10. Hence if \mathcal{F} is maximal it must be \mathcal{F}_1 . \square

We formulate the following conjecture concerning maximal CM monomial labelings of subdivisions of polygons.

Conjecture 3.16. *Given an n -gon with a subdivision of k additional edges (but no additional vertices). Then any maximal CM monomial labeling has $n + k$ variables.*

Example 3.17. A cellular resolution of the monomials of degree two in three variables is given by removing one of the interior edges in Figure 3.1.

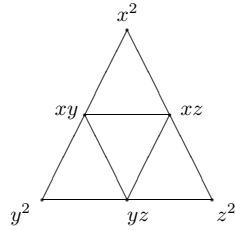


Figure 3.1

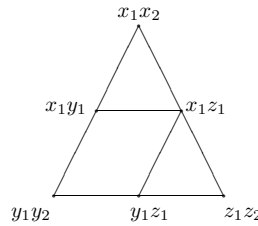


Figure 3.2

We may polarise and get Figure 3.2. However this is still not a maximal monomial labeling. Removing the edge between xy and yz in the first diagram, there is another way of “polarising” this, Figure 3.3. (The point here is that the indices are always 1 and 2.)

Combining these two last diagrams we get a maximal monomial labeling with eight variables in Figure 3.4. Note that there are two possible maps from this monomial labeling to the one in Figure 3.1,

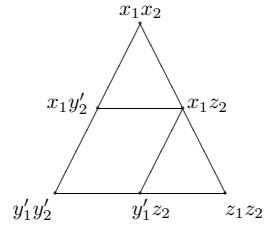


Figure 3.3

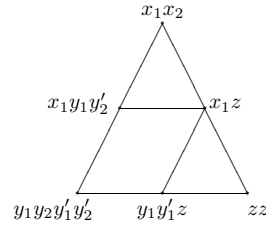


Figure 3.4

factoring through the second and third figure respectively.

4. Cohen-Macaulay monomial labelings of higher dimensional polytopes. This section investigates CM cellular resolutions supported on polytopes. We give constructions of CM monomial labelings on some classes of selfdual polytopes where the class contains polytopes of arbitrary dimensions. These labelings are shown to be maximal. We also conjecture that any polytope supporting a CM cellular resolution must be selfdual.

In the end we consider an example of a three dimensional selfdual polytope and construct a maximal CM monomial labeling of it. Along the way we give several examples of how the labeled polytopes give cellular resolutions of Stanley-Reisner rings of simplicial polytopes.

In this section our cell complex X will be a convex polytope.

4.1 Necessary conditions on CM monomial labelings.

Lemma 4.1. *Let P be a polytope supporting a CM cellular resolution. Then its f -vector is symmetric, i.e. if f_i is the number of i -dimensional cells, then $f_i = f_{\dim P - 1 - i}$.*

Proof. By polarising we may assume the monomial labeling of P is square free. Since P is a polytope, the corresponding cellular resolution has type 1. But a simplicial complex which is Cohen-Macaulay of type 1 is Gorenstein. Hence the resolution is self-dual and so $f_i = f_{\dim P - 1 - i}$.

□

Conjecture 4.2. *If P is a polytope supporting a CM cellular resolution, it is a selfdual polytope.*

Example 4.3. A three-dimensional polytope which is a bipyramid cannot support a CM cellular resolution. If the base is an n -gon, where $n \geq 3$, it has $n + 2$ vertices and $2n$ faces. But then $2n \neq n + 2$.

4.2. *CM monomial labellings of pyramids.* Given a polytope X , let PX be the pyramid over X , i.e. the convex hull of X and a point t outside the linear space where X lives. The polytope X is selfdual iff PX is.

Theorem 4.4. *Given a CM monomial labeling $(\mathbf{a}_1, \dots, \mathbf{a}_v)$ of X . Let y be a variable.*

- a. *Then $(\mathbf{a}_1, \dots, \mathbf{a}_v, y)$ is a CM monomial labeling of PX .*
- b. *The labeling of X is maximal iff the associated labeling of PX is maximal.*
- c. *Every maximal CM monomial labeling of PX is of this form.*

Proof. a. If C is a cellular resolution over S of the quotient ring $S/(a_1, \dots, a_v)$, then the cone of $C \otimes_k k[y](-1) \xrightarrow{-y} C \otimes_k k[y]$ is a resolution of the quotient ring of $S \otimes_k k[y]$ given by the labeling of PX .

b. Let the monomial labeling $(\mathbf{a}_1, \dots, \mathbf{a}_v)$ correspond to a family \mathcal{F} of subsets of V . We need to show that \mathcal{F} is maximal iff $\mathcal{F} \cup \{\{t\}\}$ is maximal.

If \mathcal{F} is maximal, then if we could add a subset \tilde{T} of $V \cup \{t\}$ to $\mathcal{F} \cup \{\{t\}\}$ and still have the conditions of Proposition 1.10 holding, then we could add $\tilde{T} \cap V$ to \mathcal{F} , Proposition 1.10 still holding. Since \mathcal{F} is maximal, $\tilde{T} \cap V$ would have to be a disjoint union of sets in \mathcal{F} . But then the same would hold for \tilde{T} . Therefore $\mathcal{F} \cup \{\{t\}\}$ is maximal.

Conversely, if $\mathcal{F} \cup \{\{t\}\}$ is maximal, then if we add a subset T to \mathcal{F} and still have the criteria of Proposition 1.10 holding, then we could add this to $\mathcal{F} \cup \{\{t\}\}$ with the criteria still valid. Hence T is a disjoint union of sets in $\mathcal{F} \cup \{\{t\}\}$ and so it must be a disjoint union of sets in \mathcal{F} , and so \mathcal{F} is maximal.

c. Suppose the maximal family \mathcal{F} on PX contains $S \cup \{t\}$ where S is nonempty. Let $W = V \setminus S$ and $Y = X|_W$. Let $\mathcal{F}' = \{T \cap W \mid T \in \mathcal{F}\}$. Then the family \mathcal{F}' has the property that the complement of any union of elements in \mathcal{F}' gives an acyclic restriction. Hence the family \mathcal{F}' determines a cellular resolution of the ideal generated by $\{\mathbf{a}_i\}_{i \in W}$. Since $\dim Y < \dim X$, the quotient ring by this ideal will have codimension $\leq \dim Y + 1 \leq \dim X$. Thus Y can be covered by $\leq \dim X$ elements in \mathcal{F}' . But then PX can be covered by $S \cup \{t\}$ together with these, a total of $\leq \dim X + 1$ elements of \mathcal{F} . But this is impossible for a CM labeling. \square

Example 4.5. A three-dimensional bipyramid is a simplicial polytope. In order for its Stanley-Reisner ideal to have a cellular resolution supported on a three dimensional polytope, the base of the bipyramid must be a pentagon (the number of vertices must be four more than the dimension). In this case a minimal cellular resolution is given by the pyramid over the pentagon labeled as follows. (A label ij is short for $x_i x_j$.)

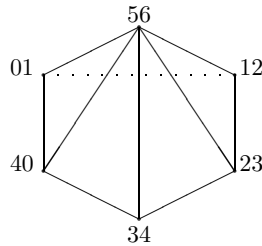


Figure 4.1

4.3. *CM monomial labelings of elongated pyramids.* The elongated pyramid EPX over X is the union of $X \times [0, 1]$ and the pyramid over X , with the base of the pyramid identified with $X \times 1$. The polytope EPX is selfdual iff X is. If V is the vertex set of X let EPV be the vertex set of EPX . It consists of $V \times \{0, 1\}$ and the vertex t of the pyramid. Given a family \mathcal{F} of subsets of V , we get a family $EP\mathcal{F}$ of subsets of EPV consisting of (letting S vary over \mathcal{F}) i) $S \times \{1\} \cup \{t\}$, ii) $S \times \{0, 1\}$, iii) $V \times \{0\}$.

Theorem 4.6. *Let \mathcal{F} be in $CM_*(X)$.*

- a. *The family $EP\mathcal{F}$ is in $CM_*(EPX)$.*
- b. *If \mathcal{F} is maximal, then $EP\mathcal{F}$ is maximal.*

Proof. a. The family \mathcal{F} fulfils the conditions of Proposition 1.10. We must show that $EP\mathcal{F}$ fulfils the same conditions.

Condition 3. Let $F \subseteq G$ be two distinct faces of EPX . Suppose G is contained in the pyramid over $X \times \{1\}$ with vertex t . If G is contained in the base, condition 3. is clear. If G is the pyramid over a face of X , then if F is strictly contained in this base face, condition 3. is clear. If F is the base face then there is a set S in \mathcal{F} disjoint from F (by considering the inclusion of faces $F \subseteq X$), and then condition 3. holds by considering $S \cup \{t\}$. If G is contained in $X \times \{0\}$, condition 3. clearly also holds. Suppose now that G is $A \times [0, 1]$ where A is the face of X . If F is contained in $B \times [0, 1]$ where B is a face strictly in A , condition 3. is clear. Otherwise F is either $A \times \{0\}$ or $A \times \{1\}$ in which case condition 3. is also clear.

Condition 1. Let $\tilde{S}_1 \cup \dots \cup \tilde{S}_m$ be a covering of EPV by sets of $EP\mathcal{F}$. If none of these are $V \times \{0\}$, then since they cover $V \times \{0\}$, a selection of, say r of these must be of the form $S \times \{0, 1\}$ where these S 's cover V . But then $r \geq \dim X + 1$. Since something also must contain t , we get $m \geq \dim X + 2$.

If one of the above is $V \times \{0\}$, the rest will cover $V \times \{1\}$. We need at least $\dim X + 1$ such and so $m \geq \dim X + 2$.

Condition 2. Let $\tilde{S}_1 \cup \dots \cup \tilde{S}_m$ be a union of subsets of $EP\mathcal{F}$. We will show that X restricted to its complement is acyclic. Suppose first that $V \times \{0\}$ is one of the sets in the union. If none of the \tilde{S}_i contains t , the complement is contractible. If some \tilde{S}_i contains t , the acyclicity of the complement follows by the fact that this is true for X .

Suppose then that $V \times \{0\}$ is not one of the sets in the union. Then the complement will consist of the union of $A \times \{1\}$ and $B \times \{0\}$, where $A \subseteq B$, and possibly t , where X restricted to A or B are both acyclic. Then EPX restricted to this complement is also acyclic.

b. We now want to show that if \mathcal{F} is a maximal family, then so is $EP\mathcal{F}$. Let $T \subseteq EPV$ be a set of vertices. Suppose $EP\mathcal{F} \cup \{T\}$ fulfils the criteria of Proposition 1.10.

Let $T^0 \times \{0\} = T \cap (V \times \{0\})$. Suppose T^0 is not in \mathcal{F} . The reason is either that i) it is empty or ii) X restricted to the complement C of the union of T^0 and various other elements of \mathcal{F} is not acyclic, or iii) T^0 covers V together with $\leq \dim X - 1$ elements in \mathcal{F} , or iv) it is a disjoint union of sets in \mathcal{F} .

In case ii) $C \times \{0\}$ will also be a complement of a union of elements of $EP\mathcal{F}$. Impossible. In case iii), let $T^0 \cup S_1 \cup \dots \cup S_r$ cover V where $r \leq \dim X - 1$. Now T , the subsets $S_i \times \{0, 1\}$, and $S_1 \times \{1\} \cup \{t\}$ comprise a total of $\dim X + 1$ sets. That is one short of possibly covering EPV . If $V \setminus T^0$ is not empty, the complement of $\cup_{i=1}^r (S_i \times \{1\} \cup \{t\}) \cup T$ will be disconnected: Its restriction to level 0 and level 1 are non-empty and disjoint, since T^0 and the S_i cover V . Impossible. So we can conclude that either T^0 is a disjoint union of sets in \mathcal{F} or it is empty, or it is V .

Now let $T^1 \times \{1\} = T \cap (V \times \{1\})$. Suppose T^1 is not in \mathcal{F} . Then the reason is either that i) it is empty or ii) X restricted to the complement C' of the union of T^1 and various other elements of \mathcal{F} is not acyclic, or iii) T^1 covers V together with $\leq \dim X - 1$ elements in \mathcal{F} , or iv) it is a disjoint union of sets in \mathcal{F} .

In case ii) $C' \times \{1\}$ will also be the complement of the union of elements in $EP\mathcal{F}$, since $V \times \{0\}$ is in $EP\mathcal{F}$. Impossible. In case iii) EPV will be covered by $\dim X + 1$ elements since $V \times \{0\}$ is in $EP\mathcal{F}$. Impossible. So we can conclude that T^1 is a disjoint union of sets in \mathcal{F} or it is empty.

Now suppose T^0 is V . If T^1 is empty, then since T is not $V \times \{0\}$ (since T is not in $EP\mathcal{F}$), T must be $V \times \{0\} \cup \{t\}$. But then EPX restricted to the complement of T is a sphere and so not acyclic. If T^1 is a disjoint union of sets in \mathcal{F} , EPV can be covered by $\dim X + 1$ elements according to Proposition 1.14. Impossible.

Now suppose T^0 is empty. If T^1 is empty then T must be $\{t\}$. Impossible since the complement of $T \cup \{V \times \{0\}\}$ will be a sphere and so have homology. If T^1 is a disjoint union of two or more sets in

\mathcal{F} , then by the following Lemma 4.7 there are sets $S_1, S_2, \dots, S_{\dim X - 1}$ in \mathcal{F} such that T^1 and these sets cover V . But then EPV will be covered by $\dim X + 1$ subsets of $EP\mathcal{F}$ if $\dim X \geq 2$. Impossible. If $\dim X = 1$, then either T contains t and EPV is covered by two subsets of $EP\mathcal{F}$, or T does not contain t and EPX restricted to the complement of T is disconnected, also impossible.

Hence T^0 must be a disjoint union of sets in \mathcal{F} . Proposition 1.13 guarantees that T^0 together with $d = \dim X$ elements S_1, \dots, S_d in \mathcal{F} covers X . If T^1 is empty, the complement of $\cup_1^d (S_i \times \{1\} \cup \{t\}) \cup T$ will be disconnected. Impossible. Therefore both T^0 and T^1 are disjoint unions of sets in \mathcal{F} .

If $T^0 = T^1$ and T is $T^0 \times \{0, 1\} \cup \{t\}$, then according to Proposition 1.14, we can cover EPV with $\dim X + 1$ elements. Impossible.

Suppose $T^0 \setminus T^1$ is nonempty. Let v be in the difference set. By Proposition 1.14 there exists a covering of V consisting of T^0 and S_1, \dots, S_r where T^0 is the only set containing v . Then the complement of $T \cup \cup_{i=1}^r (S_i \times \{1\} \cup \{t\})$ is disconnected. Impossible.

Hence $T^0 \subseteq T^1$. Now by Proposition 1.14 there exists $d = \dim X$ elements S_1, \dots, S_d of \mathcal{F} which together with T^0 cover V . If T contains t then T and the $S_i \times \{0, 1\}$, a total of $\dim X + 1$ sets, would cover EPX . Impossible. Hence t is not in T . If $T^1 \setminus T^0$ is nonempty. Let v be in the difference set. Again by Proposition 1.14 there exists a covering of V consisting of T^1 and S_1, \dots, S_r where v is only in T^1 . Then the complement of $T \cup \cup_{i=1}^r (S_i \times \{0, 1\})$ is disconnected. Impossible.

In conclusion $T^0 = T^1$ is a disjoint union of sets in \mathcal{F} , and T is $T^0 \times \{0, 1\}$. Thus T is already a disjoint union of sets in $EP\mathcal{F}$, and so this family is maximal. \square

Lemma 4.7. *Let \mathcal{F} be a maximal family of $CM_*(X)$ where X is a polytope. Let T_1 and T_2 be two disjoint sets in \mathcal{F} . Then there are $S_1, S_2, \dots, S_{\dim X - 1}$ in \mathcal{F} such that the union of the two T 's and the S 's cover the vertices of X .*

Proof. The restriction of X to the complement of $T_1 \cup T_2$ is acyclic. By Alexander duality, [12, Thm. 3.44], the restriction of X to $T_1 \cup T_2$ is also acyclic and hence connected. So there is an edge e connecting

points $t_1 \in T_1$ and $t_2 \in T_2$. For each face containing e there is a set S in \mathcal{F} disjoint from $\{t_1, t_2\}$ which includes a vertex of the face. The restriction of X to the complement W of the union of the S 's will be acyclic. In $X|_W$ the edge e is a maximal face. Hence $X|_W \setminus \{e\}$ is disconnected, since otherwise $X|_W$ would have nonvanishing \tilde{H}^1 -cohomology. Let U_i be the vertices in the connected component of t_i in $X|_W \setminus \{e\}$. Then $T_1 \supseteq U_1$ since otherwise X restricted to $W \setminus T_1$ would be disconnected and this cannot be so since $W \setminus T_1$ is the complement of a union of sets in \mathcal{F} . Similarly $T_2 \supseteq U_2$, and so the two T 's and the S 's cover the vertices of X . As in Proposition 1.14 we may conclude that there are $\dim X - 1$ of the S 's that together with the two T 's cover the vertices. \square

Example 4.8. The Stanley-Reisner ring of an octahedron with a stellar subdivision of one face has cellular resolution given by the elongated pyramid over a triangle.

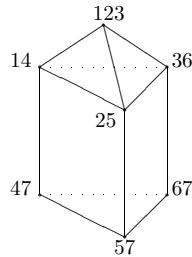


Figure 4.2

4.4. *CM labellings of a three-dimensional polytope.* Another family of selfdual polytopes of dimension three has plane diagrams given as follows. Given a $2n$ -gon labeled modulo $2n$ by vertices $0, 1, \dots, 2n - 1$. Add a vertex c at the centre and edges from the centre to each oddly labeled vertex. Also add edges from $2i$ to $2i + 2$ on the outside. This is the planar graph corresponding to a selfdual polytope with $2n + 1$ vertices. When $n = 4$ this may be displayed as follows.

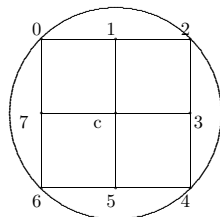


Figure 4.3

This has a maximal Cohen-Macaulay labeling given as follows.

Proposition 4.9. *Given the 3-polytope P above with $n = 4$. The family of subsets $\{0, 1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}, \{6, 7, 0\}, \{c, 1, 3\}, \{c, 3, 5\}, \{c, 5, 7\}, \{c, 7, 1\},$ and $\{1, 2, 3\}, \{3, 4, 5\}$ is a maximal family in $CM_*(X)$.*

Proof. It is a tedious but straightforward task to show that this family fulfils the criteria of Proposition 1.10. To show that it is maximal we must also show that this family cannot be extended, i.e. we cannot add another subset S of vertices, or refine a subset, and still have all the criteria 1, 2, and 3 of Proposition 1.10. This is laborious but straightforward. \square

Now this 3-polytope P gives a cellular resolution of the Stanley-Reisner ideal of various simplicial polytopes where the number of vertices is four more than the dimension. We give examples of such simplicial polytopes of dimension two and three.

Example 4.10. The hexagon has cellular resolution given by P when labeled as follows. (A label ij denotes $x_i x_j$.)

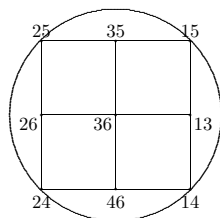


Figure 4.4

Example 4.11. Consider the bipyramid over the triangle.

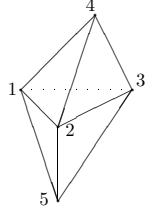


Figure 4.5

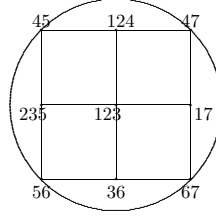


Figure 4.6

We can take stellar subdivisions of various pairs of faces. The cellular resolution of its Stanley-Reisner ring is then given by various labellings of P . With stellar subdivision of faces 124 and 235 we have Figure 4.6. With stellar subdivision of faces 124 and 234 we have Figure 4.7, and with stellar subdivision of faces 124 and 125 we have Figure 4.8.

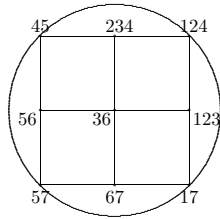


Figure 4.7

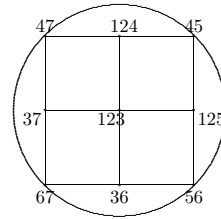


Figure 4.8

REFERENCES

1. D. Buchsbaum, D. Eisenbud, *Algebra structure for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math **99** (1977), 447–485.
2. D. Bayer, I. Peeva, and B. Sturmfels, *Monomial resolutions*, Math.Res.Lett. **5** (1998) no.1-2, pp. 31–46.
3. D. Bayer and B. Sturmfels, *Cellular resolutions of monomial ideals*, J. Reine Angew. Math. **102** (1998), pp. 123–140.
4. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics 39, Cambridge University Press (1993).

5. D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, GTM 150, Springer-Verlag, (1995).

6. G. Fløystad, J. E. Vatne, *(Bi)-Cohen-Macaulay simplicial complexes and their associated coherent sheaves*, Communications in algebra, **33** (2005) no.9, pp.3121–3136.

7. V. Gasharov, I. Peeva, and V. Welker, *The LCM-lattice in monomial resolutions*, Mathematical Research Letters, **6**, (1999), pp.521–532.

8. E. Miller, B. Sturmfels, *Combinatorial commutative algebra*, GTM 2005, Springer-Verlag, (2005).

9. J. Phan, *Minimal monomial ideals and linear resolutions*, preprint, arXiv:math/0511032v2 [math.AC].

10. M. Velasco, *Cellular resolutions and nearly Scarf ideals*, preprint (2005).

11. V. Welker, *Discrete Morse theory and free resolutions*, in Algebraic Combinatorics, Lectures at a summer school at Nordjordeid, Norway, Universitext, Springer Verlag, (2007).

12. A. Hatcher, *Algebraic topology*, Cambridge University Press (2002).

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