J. Math. Kyoto Univ. (JMKYAZ)
49-4 (2009), 771–783

On localizing subcategories of derived categories

By

Ryo Takahashi

Abstract

Let A be a commutative noetherian ring. In this paper, we interpret localizing subcategories of the derived category of A by using subsets of $\operatorname{Spec} A$ and subcategories of the category of A-modules. We unify theorems of Gabriel, Neeman and Krause.

1. Introduction

Let A be a commutative noetherian ring. In this paper, we investigate the relationship among subcategories of the derived category $\mathcal{D}(A)$ of A, subcategories of the category $\operatorname{Mod} A$ of A-modules, and subsets of the prime spectrum $\operatorname{Spec} A$ of A (i.e. $\operatorname{Spec} A$ is the set of prime ideals of A).

In the early 1960s, Gabriel [4] showed the following.

Theorem 1.1 (Gabriel). *There is an inclusion-preserving bijection between the set of localizing subcategories of* $\operatorname{Mod} A$ *and the set of subsets of* $\operatorname{Spec} A$ *closed under specialization.*

Thirty years later, Neeman [10] proved the following result, which generalizes a theorem of Hopkins [7].

Theorem 1.2 (Neeman). *The assignment* $\mathcal{X} \mapsto \operatorname{supp} \mathcal{X}$ *makes an inclusion-preserving bijection from the set of localizing subcategories of* $\mathcal{D}(A)$ *to the set of subsets of* $\operatorname{Spec} A$, *which induces an inclusion-preserving bijection from the set of smashing subcategories of* $\mathcal{D}(A)$ *to the set of subsets of* $\operatorname{Spec} A$ *closed under specialization. The inverse map sends a subset* Φ *of* $\operatorname{Spec} A$ *to the localizing subcategory of* $\mathcal{D}(A)$ *generated by* $\{k(\mathfrak{p})\}_{\mathfrak{p} \in \Phi}$.

Here, $\operatorname{supp} \mathcal{X}$ denotes the set of prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \in \operatorname{supp} X$ for some $X \in \mathcal{X}$, where $\operatorname{supp} X$ denotes the set of prime ideals $\mathfrak{p}$ such that $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X \neq 0$ in $\mathcal{D}(A)$ ($k(\mathfrak{p})$ denotes the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$).

Recently, Krause [9] generalized the above Gabriel's result, and corrected a theorem of Hovey [8].

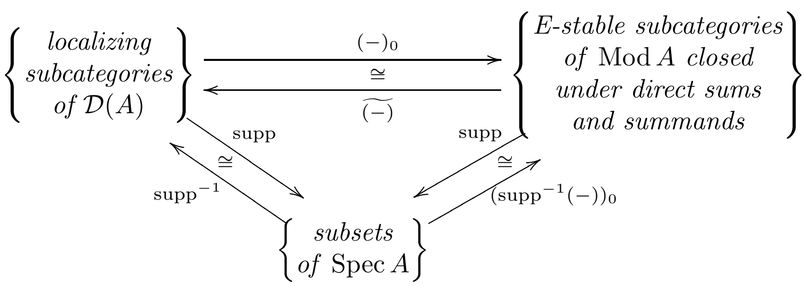
2000 *Mathematics Subject Classification*(*s*). 13C05, 16D90, 18E30.
Received February 16, 2009
Revised June 15, 2009

Theorem 1.3 (Krause). *The assignment* $\mathcal{M} \mapsto \operatorname{supp} \mathcal{M}$ *makes an inclusion-preserving bijection from the set of thick subcategories of* $\operatorname{Mod} A$ *closed under direct sums and the set of coherent subsets of* $\operatorname{Spec} A$. *The inverse map is given by* $\Phi \mapsto (\operatorname{supp}^{-1} \Phi)_0$.

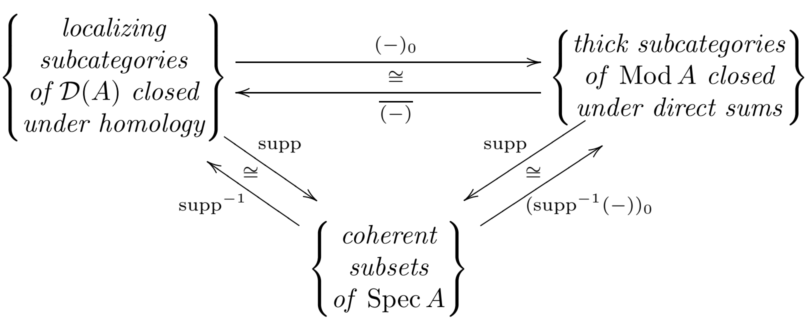
Here, $\operatorname{supp}^{-1} \Phi$ denotes the full subcategory of $\mathcal{D}(A)$ consisting of all complexes X such that $\operatorname{supp} X$ is contained in Φ, and for a subcategory $\mathcal{X}$ of $\mathcal{D}(A)$, $\mathcal{X}_0$ denotes the full subcategory of $\operatorname{Mod} A$ consisting of all modules whose corresponding complexes are in $\mathcal{X}$.

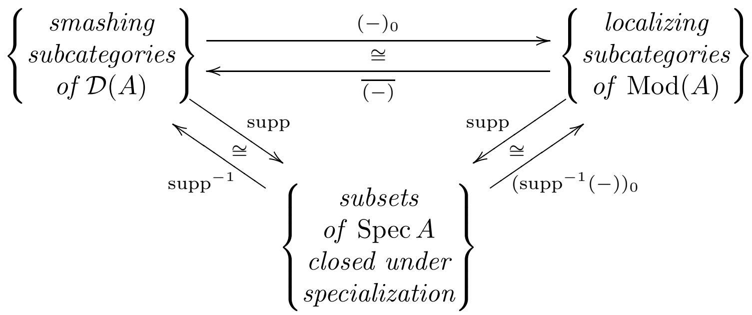
Let $E(M) = (0 \to E^0(M) \to E^1(M) \to E^2(M) \to \cdots)$ denote the minimal injective resolution of an A-module M. We say that a full subcategory $\mathcal{M}$ of $\operatorname{Mod} A$ is E-stable provided that a module M is in $\mathcal{M}$ if and only if so is $E^i(M)$ for all $i \geq 0$. We denote by $\widetilde{\mathcal{M}}$ the localizing subcategory of $\mathcal{D}(A)$ generated by $\mathcal{M}$, and by $\overline{\mathcal{M}}$ the localizing subcategory of $\mathcal{D}(A)$ consisting of all complexes each of whose homology modules is in $\mathcal{M}$. A subcategory $\mathcal{X}$ of $\mathcal{D}(A)$ is said to be closed under homology if (the corresponding complex of) any homology module of any complex in $\mathcal{X}$ is also in $\mathcal{X}$. Our main result is the following, which contains all of the above three theorems.

Main Theorem. *One has the following commutative diagram of inclusion-preserving bijections.*



Moreover, restricting this diagram, one has the following two commutative diagrams of inclusion-preserving bijections.





Thus, one obtains the following commutative diagram.

$$\begin{array}{ccccc}
\left\{\begin{array}{c}\textit{localizing}\\ \textit{subcategories}\\ \textit{of } \mathcal{D}(A)\end{array}\right\} & \cong & \left\{\begin{array}{c}\textit{subsets}\\ \textit{of } \operatorname{Spec} A\end{array}\right\} & \cong & \left\{\begin{array}{c}\textit{E-stable subcategories}\\ \textit{of } \operatorname{Mod} A \textit{ closed}\\ \textit{under direct sums}\\ \textit{and summands}\end{array}\right\} \\
\cup & & \cup & & \cup \\
\left\{\begin{array}{c}\textit{localizing}\\ \textit{subcategories}\\ \textit{of } \mathcal{D}(A) \textit{ closed}\\ \textit{under homology}\end{array}\right\} & \cong & \left\{\begin{array}{c}\textit{coherent}\\ \textit{subsets}\\ \textit{of } \operatorname{Spec} A\end{array}\right\} & \cong & \left\{\begin{array}{c}\textit{thick subcategories}\\ \textit{of } \operatorname{Mod} A \textit{ closed}\\ \textit{under direct sums}\end{array}\right\} \\
\cup & & \cup & & \cup \\
\left\{\begin{array}{c}\textit{smashing}\\ \textit{subcategories}\\ \textit{of } \mathcal{D}(A)\end{array}\right\} & \cong & \left\{\begin{array}{c}\textit{subsets}\\ \textit{of } \operatorname{Spec} A\\ \textit{closed under}\\ \textit{specialization}\end{array}\right\} & \cong & \left\{\begin{array}{c}\textit{localizing}\\ \textit{subcategories}\\ \textit{of } \operatorname{Mod}(A)\end{array}\right\}
\end{array}$$

There are some related works other than ones cited above; see [1, 3, 5, 6, 12, 13] for example. In the next section, we will prove this Main Theorem after stating precise definitions and showing preliminary results.

2. Proof of Main Theorem

Throughout this section, let A be a commutative noetherian ring. By a *subcategory*, we always mean a full subcategory which is closed under isomorphisms. We denote the category of A-modules by $\operatorname{Mod} A$ and the derived category of $\operatorname{Mod} A$ by $\mathcal{D}(A)$. For an A-module M, let

$$C_M = (\cdots \to 0 \to M \to 0 \to \cdots)$$

be the complex with M in degree zero. We will often identify M with C_M.

First of all, we recall the definitions of a triangulated subcategory and a localizing subcategory of $\mathcal{D}(A)$. For a (cochain) A-complex X and an integer n, we denote by $X[n]$ the complex X shifted by n degrees; its module in degree i is X^{n+i} for each integer i.

Definition 2.1. Let $\mathcal{X}$ be a subcategory of $\mathcal{D}(A)$.

(1) We say that $\mathcal{X}$ is *triangulated* provided that for every exact triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}(A)$, if two of X, Y and Z are in $\mathcal{X}$, then so is the third.

(2) We say that $\mathcal{X}$ is *localizing* if $\mathcal{X}$ is triangulated and closed under (arbitrary) direct sums.

Remark.

(1) Triangulated subcategories of $\mathcal{D}(A)$ are *closed under shifts*: if a complex X is in a triangulated subcategory $\mathcal{X}$ of $\mathcal{D}(A)$, then $X[n]$ is also in $\mathcal{X}$ for every integer n.

In fact, it follows from the triangle $X \xrightarrow{=} X \to 0 \to X[1]$ that 0 is in $\mathcal{X}$, and it follows from the triangles $X \to 0 \to X[1] \xrightarrow{=} X[1]$ and $X[-1] \to 0 \to X \xrightarrow{=} X$ that $X[1], X[-1]$ are in $\mathcal{X}$. An inductive argument shows that $X[n]$ is in $\mathcal{X}$ for every $n \in \mathbb{Z}$.

(2) Localizing subcategories of $\mathcal{D}(A)$ are closed under direct summands; see [11, Proposition 1.6.8].

The support of a complex is defined as follows.

Definition 2.2. The (small) *support* $\operatorname{supp} X$ of an A-complex X is defined as the set of prime ideals $\mathfrak{p}$ of A satisfying $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X \neq 0$ in $\mathcal{D}(A)$, where $k(\mathfrak{p})$ denotes the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ of the local ring $A_\mathfrak{p}$.

Here we state basic properties of support.

Lemma 2.1.

(1) *Let $X \to Y \to Z \to X[1]$ be an exact triangle in $\mathcal{D}(A)$. Then one has the following inclusion relations:*

$$\operatorname{supp} X \subseteq \operatorname{supp} Y \cup \operatorname{supp} Z,$$
$$\operatorname{supp} Y \subseteq \operatorname{supp} Z \cup \operatorname{supp} X,$$
$$\operatorname{supp} Z \subseteq \operatorname{supp} X \cup \operatorname{supp} Y.$$

(2) *The equality*

$$\operatorname{supp}\left(\bigoplus_{\lambda\in\Lambda} X_\lambda\right) = \bigcup_{\lambda\in\Lambda} \operatorname{supp} X_\lambda$$

holds for any family $\{X_\lambda\}_{\lambda\in\Lambda}$ of A-complexes.

(3) *Let s be an integer, and let $X = (\cdots \to X^{s-2} \to X^{s-1} \to X^s \to 0)$ be an A-complex. Then*

$$\operatorname{supp} X \subseteq \bigcup_{i\leq s} \operatorname{supp} X^i.$$

Proof. (1) Let $\mathfrak{p}$ be a prime ideal in $\operatorname{supp} X$. Then $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X$ is nonzero. There is an exact triangle

$$k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X \to k(\mathfrak{p}) \otimes_A^{\mathbf{L}} Y \to k(\mathfrak{p}) \otimes_A^{\mathbf{L}} Z \to k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X[1],$$

which says that either $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} Y$ or $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} Z$ is nonzero. Thus $\mathfrak{p}$ is in the union of $\operatorname{supp} Y$ and $\operatorname{supp} Z$. The other inclusion relations are similarly obtained.

(2) One has $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} (\bigoplus_{\lambda \in \Lambda} X_\lambda) \cong \bigoplus_{\lambda \in \Lambda} (k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X_\lambda)$ for $\mathfrak{p} \in \operatorname{Spec} A$. Hence $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} (\bigoplus_{\lambda \in \Lambda} X_\lambda)$ is nonzero if and only if $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X_\lambda$ is nonzero for some $\lambda \in \Lambda$.

(3) Assume that a prime ideal $\mathfrak{p}$ of A satisfies $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X^i = 0$ for every $i \leq s$. Let $F = (\cdots \to F^{-2} \to F^{-1} \to F^0 \to 0)$ be a free resolution of the A-module $k(\mathfrak{p})$. Then the complex $F \otimes_A X^i = (\cdots \to F^{-2} \otimes_A X^i \to F^{-1} \otimes_A X^i \to F^0 \otimes_A X^i \to 0)$ is exact for every $i \leq s$. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(\cdots & \longrightarrow & F^{-2} \otimes_A X^{s-2} & \longrightarrow & F^{-2} \otimes_A X^{s-1} & \longrightarrow & F^{-2} \otimes_A X^{s} & \longrightarrow & 0) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(\cdots & \longrightarrow & F^{-1} \otimes_A X^{s-2} & \longrightarrow & F^{-1} \otimes_A X^{s-1} & \longrightarrow & F^{-1} \otimes_A X^{s} & \longrightarrow & 0) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(\cdots & \longrightarrow & F^{0} \otimes_A X^{s-2} & \longrightarrow & F^{0} \otimes_A X^{s-1} & \longrightarrow & F^{0} \otimes_A X^{s} & \longrightarrow & 0) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact columns. Considering the spectral sequence of the double complex $F \otimes_A X$, we see that the total complex of $F \otimes_A X$ is exact. This means that $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} X = 0$. □

We say that a subcategory $\mathcal{X}$ of $\mathcal{D}(A)$ is *closed under left complexes* provided that for any A-complex $X = (\cdots \to X^{s-2} \to X^{s-1} \to X^s \to 0)$ bounded above, if each X^i is in $\mathcal{X}$, then X is also in $\mathcal{X}$. For a subset Φ of $\operatorname{Spec} A$, we denote by $\operatorname{supp}^{-1} \Phi$ the subcategory of $\mathcal{D}(A)$ consisting of all A- complexes X with $\operatorname{supp} X \subseteq \Phi$. The following proposition immediately follows from Lemma 2.1.

Proposition 2.1. *Let* Φ *be a subset of* $\operatorname{Spec} A$*. Then* $\operatorname{supp}^{-1} \Phi$ *is a localizing subcategory of* $\mathcal{D}(A)$ *closed under left complexes.*

We denote the set of associated primes of an A-module M by $\operatorname{Ass} M$, and the injective hull of M by $E(M)$.

Lemma 2.2.

(1) *For an A-module M we have a direct sum decomposition*

$$E(M) \cong \bigoplus_{\mathfrak{p} \in \operatorname{Ass} M} E(A/\mathfrak{p})^{\oplus \Lambda_{\mathfrak{p}}},$$

where $\Lambda_{\mathfrak{p}}$ is a nonempty set.

(2) *The equality*

$$\operatorname{supp} I = \operatorname{Ass} I$$

holds for every injective A-module I.

Proof. (1) This assertion can be shown by using [2, Theorem 3.2.8].

(2) Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of A. Then we easily see that there are isomorphisms

$$k(\mathfrak{p}) \otimes_A^{\mathbf{L}} (E(A/\mathfrak{q})_{\mathfrak{p}}) \cong k(\mathfrak{p}) \otimes_A^{\mathbf{L}} E(A/\mathfrak{q}) \cong (k(\mathfrak{p})_{\mathfrak{q}}) \otimes_A^{\mathbf{L}} E(A/\mathfrak{q}).$$

Therefore the complex $k(\mathfrak{p}) \otimes_A^{\mathbf{L}} E(A/\mathfrak{q})$ is nonzero if and only if $\mathfrak{p} = \mathfrak{q}$. The assertion follows from this fact and (1). □

We now recall the definition of a smashing subcategory.

Definition 2.3. Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{D}(A)$.

(1) An object $C \in \mathcal{D}(A)$ is called $\mathcal{X}$*-local* if $\operatorname{Hom}_{\mathcal{D}(A)}(X, C) = 0$ for any $X \in \mathcal{X}$.

(2) A morphism $f : C \to L$ is called a *localization* of C by $\mathcal{X}$ if L is $\mathcal{X}$-local, and $\operatorname{Hom}_{\mathcal{D}(A)}(f, L') : \operatorname{Hom}_{\mathcal{D}(A)}(L, L') \to \operatorname{Hom}_{\mathcal{D}(A)}(C, L')$ is an isomorphism for any $\mathcal{X}$-local object $L' \in \mathcal{D}(A)$.

(3) $\mathcal{X}$ is called *smashing* if localization by $\mathcal{X}$ commutes with direct sums.

For a subcategory $\mathcal{X}$ of $\mathcal{D}(A)$, we denote by $\operatorname{supp} \mathcal{X}$ the set of prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \in \operatorname{supp} X$ for some $X \in \mathcal{X}$. We describe a theorem of Neeman [10] in the following form.

Theorem 2.1.

(1) *One has maps*

$$\left\{ \begin{matrix} \textit{localizing} \\ \textit{subcategories} \\ \textit{of } \mathcal{D}(A) \end{matrix} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{matrix} \textit{subsets} \\ \textit{of } \operatorname{Spec} A \end{matrix} \right\}$$

defined by $f(\mathcal{X}) = \operatorname{supp} \mathcal{X}$ and $g(\Phi) = \operatorname{supp}^{-1} \Phi$. The map f is an inclusion-preserving bijection and g is its inverse map.

(2) *One has maps*

$$\left\{\begin{array}{c}\textit{smashing}\\ \textit{subcategories}\\ \textit{of } \mathcal{D}(A)\end{array}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{array}{c}\textit{subsets}\\ \textit{of } \operatorname{Spec} A\\ \textit{closed under}\\ \textit{specialization}\end{array}\right\}$$

defined by $f(\mathcal{X}) = \operatorname{supp}\mathcal{X}$ *and* $g(\Phi) = \operatorname{supp}^{-1}\Phi$. *The map* f *is an inclusion-preserving bijection and* g *is its inverse map.*

Proof. (1) Proposition 2.1 guarantees that g is well-defined. Let Φ be a subset of $\operatorname{Spec} A$. Then the inclusion $\operatorname{supp}(\operatorname{supp}^{-1}\Phi) \subseteq \Phi$ clearly holds. It follows from Lemma 2.2(2) that $\operatorname{supp} E(A/\mathfrak{p}) = \operatorname{Ass} E(A/\mathfrak{p}) = \{\mathfrak{p}\} \subseteq \Phi$ for every $\mathfrak{p} \in \Phi$, which yields the opposite inclusion $\operatorname{supp}(\operatorname{supp}^{-1}\Phi) \supseteq \Phi$. Therefore we have the equality $\operatorname{supp}(\operatorname{supp}^{-1}\Phi) = \Phi$, which shows that fg is the identity map. Since f is a bijective map by virtue of [10, Theorem 2.8], g is the inverse map of f. It is easy to check that f is inclusion-preserving.

(2) This follows from [10, Theorem 3.3] and (1). □

Combining Theorem 2.1(1) with [10, Theorem 2.8] and Proposition 2.1, we obtain the following result.

Corollary 2.1.

(1) *For every subset* Φ *of* $\operatorname{Spec} A$, $\operatorname{supp}^{-1}\Phi$ *is the localizing subcategory of* $\mathcal{D}(A)$ *generated by* $\{k(\mathfrak{p})\}_{\mathfrak{p}\in\Phi}$.

(2) *Any localizing subcategory of* $\mathcal{D}(A)$ *is closed under left complexes.*

Krause [9] introduces the notion of a coherent subset of $\operatorname{Spec} A$:

Definition 2.4. A subset Φ of $\operatorname{Spec} A$ is called *coherent* if every homomorphism $f : I^0 \to I^1$ of injective A-modules with $\operatorname{Ass} I^i \subseteq \Phi$ for $i = 1, 2$ can be completed to an exact sequence $I^0 \xrightarrow{f} I^1 \to I^2$ of injective A-modules with $\operatorname{Ass} I^2 \subseteq \Phi$.

To relate coherent subsets of $\operatorname{Spec} A$ to localizing subcategories of $\mathcal{D}(A)$, we make the following definition.

Definition 2.5. Let $\mathcal{X}$ be a subcategory of $\mathcal{D}(A)$.

(1) We say that $\mathcal{X}$ is *closed under homology* if $H^i(X)$ is in $\mathcal{X}$ for all $X \in \mathcal{X}$ and $i \in \mathbb{Z}$.

(2) We say that $\mathcal{X}$ is *H-stable* provided that a complex X is in $\mathcal{X}$ if and only if so is $H^i(X)$ for every $i \in \mathbb{Z}$.

We have the following one-to-one correspondence.

Theorem 2.2. *One has maps*

$$\left\{\begin{matrix}\textit{localizing}\\ \textit{subcategories}\\ \textit{of } \mathcal{D}(A) \textit{ closed}\\ \textit{under homology}\end{matrix}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\textit{coherent}\\ \textit{subsets}\\ \textit{of } \operatorname{Spec} A\end{matrix}\right\}$$

defined by $f(\mathcal{X}) = \operatorname{supp}\mathcal{X}$ *and* $g(\Phi) = \operatorname{supp}^{-1}\Phi$. *The map* f *is an inclusion-preserving bijection and* g *is its inverse map.*

Proof. Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{D}(A)$ closed under homology. Then we have $\mathcal{X} = \operatorname{supp}^{-1}(\operatorname{supp}\mathcal{X})$ by Theorem 2.1(1). It is seen from [9, Theorem 5.2] that $\operatorname{supp}\mathcal{X}$ is coherent. Hence f is well-defined. From Proposition 2.1 and [9, Theorem 5.2] we see that g is well-defined. Theorem 2.1(1) shows that f is an inclusion-preserving bijective map and g is the inverse map of f. □

By definition, an H-stable subcategory of $\mathcal{D}(A)$ is closed under homology. The converse of this statement also holds:

Corollary 2.2.

(1) *Any localizing subcategory of* $\mathcal{D}(A)$ *closed under homology is H-stable.*

(2) *Any smashing subcategory of* $\mathcal{D}(A)$ *is H-stable.*

Proof. (1) Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{D}(A)$ closed under homology. Then by Theorem 2.2 we have $\mathcal{X} = \operatorname{supp}^{-1}\Phi$ for some coherent subset Φ of $\operatorname{Spec} A$. It follows from [9, Theorem 5.2] that $\mathcal{X}$ is H-stable.

(2) Let $\mathcal{X}$ be a smashing subcategory of $\mathcal{D}(A)$. Then $\Phi := \operatorname{supp}\mathcal{X}$ is closed under specialization by Theorem 2.1(2). It follows from [9, Proposition 4.1(2)] that Φ is coherent. Theorem 2.1(1) yields $\mathcal{X} = \operatorname{supp}^{-1}\Phi$, which is H-stable by Theorem 2.2 and (1). □

Following [9], we define a thick subcategory of modules as follows.

Definition 2.6. A subcategory $\mathcal{M}$ of $\operatorname{Mod} A$ is called *thick* provided that for any exact sequence

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

of A-modules, if M_i is in $\mathcal{M}$ for $i = 1, 2, 4, 5$, then so is M_3.

Remark.

(1) A subcategory of $\operatorname{Mod} A$ is thick if and only if it is closed under kernels, cokernels and extensions.

(2) If a subcategory of $\operatorname{Mod} A$ is closed under kernels or cokernels, then it is closed under direct summands. In particular, every thick subcategory of $\operatorname{Mod} A$ is closed under direct summands, and contains the zero module 0.

Indeed, assume that the direct sum $M = N \oplus L$ of two A-modules N, L is in a subcategory $\mathcal{M}$ of $\operatorname{Mod} A$. Then the exact sequence

$$0 \to N \to M \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} M \to N \to 0$$

of A-modules shows that N is in $\mathcal{M}$ if $\mathcal{M}$ is closed under kernels or cokernels.

For an A-module M, let

$$E(M) = (0 \to E^0(M) \to E^1(M) \to E^2(M) \to \cdots)$$

denote the minimal injective resolution of M. (Recall that a minimal injective resolution of a given A-module is uniquely determined up to isomorphism; see [2, Page 99].)

Definition 2.7. We say that a subcategory $\mathcal{M}$ of $\operatorname{Mod} A$ is *E-stable* provided that a module M is in $\mathcal{M}$ if and only if so is $E^i(M)$ for every $i \geq 0$.

Proposition 2.2. *Every thick subcategory of* $\operatorname{Mod} A$ *closed under direct sums is E-stable.*

Proof. Let $\mathcal{M}$ be a thick subcategory of $\operatorname{Mod} A$ closed under direct sums. Then $\mathcal{M}$ is closed under cokernels and injective hulls by [9, Lemma 3.5]. Hence $E^i(M)$ is in $\mathcal{M}$ for every $M \in \mathcal{M}$ and $i \geq 0$. Conversely, let M be an A-module with $E^i(M) \in \mathcal{M}$ for any $i \geq 0$. There is an exact sequence

$$0 \to M \to E^0(M) \to E^1(M)$$

of A-modules, and M is in $\mathcal{M}$ by the closedness of $\mathcal{M}$ under kernels. Consequently, $\mathcal{M}$ is E-stable. □

For a subcategory $\mathcal{M}$ of $\operatorname{Mod} A$, we denote by $\operatorname{supp} \mathcal{M}$ the set of prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \in \operatorname{supp} M$ for some $M \in \mathcal{M}$. For a subcategory $\mathcal{X}$ of $\mathcal{D}(A)$, we denote by $\mathcal{X}_0$ the subcategory of $\operatorname{Mod} A$ consisting of all A-modules M with $C_M \in \mathcal{X}$. Now we can construct the following one-to-one correspondence.

Theorem 2.3. *One has maps*

$$\left\{\begin{array}{c}\textit{E-stable subcategories}\\ \textit{of } \operatorname{Mod} A \textit{ closed}\\ \textit{under direct sums}\\ \textit{and summands}\end{array}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{array}{c}\textit{subsets}\\ \textit{of } \operatorname{Spec} A\end{array}\right\}$$

defined by $f(\mathcal{M}) = \operatorname{supp} \mathcal{M}$ *and* $g(\Phi) = (\operatorname{supp}^{-1} \Phi)_0$. *The map* f *is an inclusion-preserving bijection and* g *is its inverse map.*

Proof. Let Φ be a subset of $\operatorname{Spec} A$, and put $\mathcal{M} = (\operatorname{supp}^{-1}\Phi)_0$. We observe by Lemma 2.1(2) that $\mathcal{M}$ is closed under direct sums and summands. Fix an A-module M. According to [9, Lemma 3.3] and Lemma 2.2(2), we have

$$\begin{aligned} M \in \mathcal{M} &\iff \operatorname{supp} M \subseteq \Phi \\ &\iff \operatorname{Ass} E^i(M) \subseteq \Phi \text{ for all } i \geq 0 \\ &\iff \operatorname{supp} E^i(M) \subseteq \Phi \text{ for all } i \geq 0 \\ &\iff E^i(M) \in \mathcal{M} \text{ for all } i \geq 0. \end{aligned}$$

Hence $\mathcal{M}$ is E-stable, which says that the map g is well-defined.

Let $\mathcal{M}$ be an E-stable subcategory of $\operatorname{Mod} A$ closed under direct sums and summands. It is obvious that $\mathcal{M}$ is contained in $(\operatorname{supp}^{-1}(\operatorname{supp}\mathcal{M}))_0$. Let N be an A-module with $\operatorname{supp} N \subseteq \operatorname{supp}\mathcal{M}$. Then we see from [9, Lemma 3.3] that for each $i \geq 0$ and $\mathfrak{p} \in \operatorname{Ass} E^i(N)$ there exists a module $M \in \mathcal{M}$ and an integer $j \geq 0$ such that $\mathfrak{p} \in \operatorname{Ass} E^j(M)$. Hence $E(A/\mathfrak{p})$ is isomorphic to a direct summand of $E^j(M)$. The module $E^j(M)$ is in $\mathcal{M}$ since $\mathcal{M}$ is E-stable, and $E(A/\mathfrak{p})$ is also in $\mathcal{M}$ since $\mathcal{M}$ is closed under direct summands. Therefore by Lemma 2.2(1) the module $E^i(N)$ is in $\mathcal{M}$ for every $i \geq 0$ since $\mathcal{M}$ is closed under direct sums, and N is also in $\mathcal{M}$ since $\mathcal{M}$ is E-stable. Thus we conclude that the composite map gf is the identity map.

Let Φ be a subset of $\operatorname{Spec} A$. It is obvious that $\operatorname{supp}((\operatorname{supp}^{-1}\Phi)_0)$ is contained in Φ. For $\mathfrak{p} \in \Phi$ we have $\operatorname{supp} E(A/\mathfrak{p}) = \operatorname{Ass} E(A/\mathfrak{p}) = \{\mathfrak{p}\} \subseteq \Phi$ by Lemma 2.2(2). This implies that Φ is contained in $\operatorname{supp}((\operatorname{supp}^{-1}\Phi)_0)$, and we conclude that the composite map fg is the identity map. □

We say that a subcategory $\mathcal{M}$ of $\operatorname{Mod} A$ is *closed under short exact sequences* provided that for any short exact sequence $0 \to L \to M \to N \to 0$ of A-modules, if two of L, M and N are in $\mathcal{M}$, then so is the third. We say that $\mathcal{M}$ is *closed under left resolutions* provided that for any exact sequence $\cdots \to M_2 \to M_1 \to M_0 \to N \to 0$ of A-modules, if every M_i is in $\mathcal{M}$ then so is N. Theorem 2.3 yields the following result.

Corollary 2.3. *Let $\mathcal{M}$ be an E-stable subcategory of* $\operatorname{Mod} A$ *closed under direct sums and summands. Then $\mathcal{M}$ is closed under short exact sequences and left resolutions.*

Proof. By virtue of Theorem 2.3 there exists a subset Φ of $\operatorname{Spec} A$ such that $\mathcal{M} = (\operatorname{supp}^{-1}\Phi)_0$.

Let $0 \to L \to M \to N \to 0$ be an exact sequence of A-modules. Assume that two of L, M and N, say L and M, are in $\mathcal{M}$. Then $\operatorname{supp} L$ and $\operatorname{supp} M$ are contained in Φ, and so is $\operatorname{supp} N$ by Lemma 2.1(1). Hence N is also in $\mathcal{M}$, and therefore $\mathcal{M}$ is closed under short exact sequences.

Let $\cdots \to M_2 \to M_1 \to M_0 \to N \to 0$ be an exact sequence of A-modules with $M_i \in \mathcal{M}$ for any $i \geq 0$. Then we have an A-complex $X = (\cdots \to X^{-2} \to X^{-1} \to X^0 \to 0)$ with $X^{-i} = M_i$ for $i \geq 0$ which is quasi- isomorphic to N. Lemma 2.1(3) implies that $\operatorname{supp} N = \operatorname{supp} X \subseteq \bigcup_{i \leq 0} \operatorname{supp} X^i \subseteq \Phi$. Therefore N is in $\mathcal{M}$. □

An A-complex X is called *K-injective* if every morphism from an acyclic A-complex to X is null-homotopic. An A-complex I is called a *minimal K-injective resolution* of an A-complex X if there exists a quasi- isomorphism $X \to I$, each I^i is an injective module, I is a K-injective complex, and the kernel of the differential map $I^i \to I^{i+1}$ is an essential submodule of I^i for all $i \in \mathbb{Z}$. Every A-complex admits a minimal K-injective resolution; see [9, before Proposition 5.1].

For a subcategory $\mathcal{M}$ of $\operatorname{Mod} A$, we denote by $\widetilde{\mathcal{M}}$ the localizing subcategory of $\mathcal{D}(A)$ generated by $\mathcal{M}$, and by $\overline{\mathcal{M}}$ the localizing subcategory of $\mathcal{D}(A)$ consisting of all complexes each of whose homology modules is in $\mathcal{M}$. For an A-complex X and an integer i, let $Z^i(X)$ (respectively, $B^i(X)$) denote the ith cycle (respectively, boundary) of X.

Proposition 2.3.

(1) *Let $\mathcal{M}$ be an E-stable subcategory of* $\operatorname{Mod} A$ *closed under direct sums and summands. Then*

$$\operatorname{supp}^{-1}(\operatorname{supp} \mathcal{M}) = \widetilde{\mathcal{M}}.$$

(2) *Let $\mathcal{M}$ be a thick subcategory of* $\operatorname{Mod} A$ *closed under direct sums. Then*

$$\operatorname{supp}^{-1}(\operatorname{supp} \mathcal{M}) = \overline{\mathcal{M}}.$$

Proof. (1) Set $\mathcal{X} = \operatorname{supp}^{-1}(\operatorname{supp} \mathcal{M})$. We see from Proposition 2.1 that $\mathcal{X}$ is a localizing subcategory of $\mathcal{D}(A)$ containing $\mathcal{M}$. Hence $\mathcal{X}$ contains $\widetilde{\mathcal{M}}$. Corollary 2.1(1) says that $\mathcal{X}$ is the localizing subcategory of $\mathcal{D}(A)$ generated by $\{k(\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{supp} \mathcal{M}}$. Hence we have only to show that $k(\mathfrak{p})$ belongs to $\widetilde{\mathcal{M}}$ for every $\mathfrak{p} \in \operatorname{supp} \mathcal{M}$.

Fix a prime ideal $\mathfrak{p}$ in $\operatorname{supp} \mathcal{M}$. Then $k(\mathfrak{p}) \otimes^{\mathbf{L}}_A M$ is nonzero for some $M \in \mathcal{M}$. Since M is in $\widetilde{\mathcal{M}}$, the complex $k(\mathfrak{p}) \otimes^{\mathbf{L}}_A M$ is in $\widetilde{\mathcal{M}}$ by [10, (2.1.7)]. Note that $k(\mathfrak{p}) \otimes^{\mathbf{L}}_A M$ is isomorphic to a nonzero direct sum of shifts of $k(\mathfrak{p})$. Since $\widetilde{\mathcal{M}}$ is closed under shifts and direct summands by Corollary 2.1(2), $k(\mathfrak{p})$ is in $\widetilde{\mathcal{M}}$, as required.

(2) Fix an A-complex X. We want to prove that $\operatorname{supp} X \subseteq \operatorname{supp} \mathcal{M}$ if and only if $H^i(X) \in \mathcal{M}$ for all integers i.

Suppose that the inclusion relation $\operatorname{supp} X \subseteq \operatorname{supp} \mathcal{M}$ holds. Let I be a minimal K-injective resolution of X. Then we see from [9, Lemma 3.3 and Proposition 5.1] that for each $i \in \mathbb{Z}$ and $\mathfrak{p} \in \operatorname{Ass} I^i$ there exists a module $M \in \mathcal{M}$ and an integer $j \geq 0$ such that $\mathfrak{p} \in \operatorname{Ass} E^j(M)$. Hence $E(A/\mathfrak{p})$ is isomorphic to a direct summand of $E^j(M)$. We observe from [9, Lemma 3.5] and the closedness of $\mathcal{M}$ under cokernels that $E^j(M)$ is in $\mathcal{M}$, and from the closedness of $\mathcal{M}$ under direct summands that $E(A/\mathfrak{p})$ is also in $\mathcal{M}$. Therefore each I^i is in $\mathcal{M}$ by Lemma 2.2(1) as $\mathcal{M}$ is closed under direct sums. For every

$i \in \mathbb{Z}$ there are exact sequences of A-modules:

$$0 \to Z^i(I) \to I^i \to I^{i+1},$$
$$0 \to Z^i(I) \to I^i \to B^{i+1}(I) \to 0,$$
$$0 \to B^i(I) \to Z^i(I) \to H^i(X) \to 0.$$

Since $\mathcal{M}$ is closed under kernels and cokernels, from these exact sequences we observe that $H^i(X)$ is in $\mathcal{M}$ for every $i \in \mathbb{Z}$.

Conversely, suppose that all $H^i(X)$ belong to $\mathcal{M}$. Then $\operatorname{supp} H^i(X)$ is contained in $\operatorname{supp} \mathcal{M}$ for all $i \in \mathbb{Z}$. Here note from [9, Theorem 3.1] that $\operatorname{supp} \mathcal{M}$ is a coherent subset of $\operatorname{Spec} A$. Therefore it follows by [9, Theorem 5.2] that $\operatorname{supp} X$ is contained in $\operatorname{supp} \mathcal{M}$, as desired. □

Now we are in a position to prove our main theorem which we stated in Introduction.

Proof of Main Theorem. The first commutative diagram of bijections in Main Theorem is obtained from Theorems 2.1(1), 2.3 and Proposition 2.3(1). Theorem 2.2, [9, Theorem 3.1] and Proposition 2.3(2) make the second diagram in Main Theorem. Theorem 2.1(2), [9, Corollary 3.6] and Proposition 2.3(2) give the third one. All these three commutative diagrams together with Proposition 2.2, Corollary 2.2(2) and [9, Proposition 4.1(2)] yield the last diagram in Main Theorem. □

Acknowledgements. The author is indebted to the referee for his/her valuable comments. The author also thanks Hiroki Miyahara very much for his helpful comments.

Department of Mathematical Sciences
Faculty of Science, Shinshu University
3-1-1 Asahi, Matsumoto, Nagano 390-8621
Japan
e-mail: takahasi@math.shinshu-u.ac.jp

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