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Certain multiple orthogonal polynomials and a discretization of the Bessel equation

By

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Abstract

We extend the Mehler-Heine type formula of Jacobi polynomials to a class of multiple orthogonal polynomials of type II. The Mehler-Heine type formulas show standard orthogonal polynomials or multiple orthogonal polynomials near the endpoints of the interval of orthogonality.

1. Introduction

Hermite considered certain simultaneous approximation to several exponential functions [5]. His simultaneous approximation can be used to approximate the Stieltjes transforms of several positive measures $\mu_1, \mu_2, \dots, \mu_l$. For a vector-index $\boldsymbol{n} = (n_1, n_2, \dots, n_l) \in \mathbb{Z}_+^l$ there exists a polynomial $Q_{\boldsymbol{n}}$, which is not identically 0 and $\deg Q_n \leq |n| = n_1 + \cdots + n_l$, such that

$$Q_{\boldsymbol{n}}(z) \int_{\Delta_j} \frac{d\mu_j(x)}{z-x} - P_{\boldsymbol{n}}^{(j)}(z) = O\left(z^{-n_j-1}\right), \quad j = 1, 2, \dots, l.$$

This approximation is called the Hermite-Padé approximation (of type II). The denominator $Q_{\boldsymbol{n}}$ has the following simultaneous orthogonality properties:

$$\int_{\Delta_j} x^{\nu} Q_{\boldsymbol{n}}(x) d\mu_j(x) = 0, \quad \nu = 0, 1, \dots, n_j - 1.$$

They are called multiple orthogonal polynomials (of type II), the Hermite-Padé polynomials, polyorthogonal polynomials, or l-orthogonal polynomials more precisely.

Let us consider the case $l = 2$ and $\boldsymbol{n} = (k, k)$ or $(k+1, k)$. Furthermore define q_n for a scalar index n by

$$q_n = \begin{cases} Q_{(k,k)}, & \text{if } n = 2k \\ Q_{(k+1,k)}, & \text{if } n = 2k+1 \end{cases}.$$

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Then $\{q_n\}$ satisfies a four-terms recurrence relation of the following form:

$$a_n q_{n+1}(z) + b_n q_n(z) + c_n q_{n-1}(z) + d_n q_{n-2} = z q_n(z).$$

There are many applications of multiple orthogonal polynomials as well as standard orthogonal polynomials, for example to the random matrix theory, the number theory, special functions and spectral analysis of nonsymmetric band operators. More information about multiple orthogonal polynomials is found in [2], [6] and [9]. For classical multiple orthogonal polynomials see [3] or [10].

Asymptotics is of great interest in the theory of multiple orthogonal polynomials as well as standard orthogonal polynomials. The asymptotic behavior of multiple orthogonal polynomials plays an important role in approximation theory. In this paper, we will obtain certain asymptotic formula of polynomials $p_n(z)$ in a class near a point $z = 1$ as $n \to \infty$. The class is defined by certain four-terms recurrence relations and contains certain multiple orthogonal polynomials. In multiple orthogonal polynomials case, the asymptotic formula is called the Mehler-Heine type formula. Thus we can consider it as an extension of the Mehler-Heine type formula in this sense. It tells us a precise information about the asymptotic behavior of the zeros of the polynomials $p_n(z)$ near the point $z = 1$.

Let P_n be the n-th Legendre polynomial with $P_n(1) = 1$. The asymptotic formula

$$\lim_{n\to\infty} P_n\left(1 - \frac{z^2}{2n^2}\right) = J_0(z)$$

has been known since the beginning of the 20th century [11]. Here J_0 is the Bessel function of the first kind. Asymptotic formulas of this type are called the Mehler-Heine type formulas. Note that the Bessel function $J_0(z)$ is the solution of the following initial value problem:

$$\left\{\begin{array}{c} u''(z) + \frac{1}{z}u'(z) + u(z) = 0, \\ u(0) = 1, \\ u'(0) = 0. \end{array}\right. \tag{1.1}$$

We can consider the recurrence relation of the Legendre polynomials as a discretization of this problem. Fix the step h and set

$$z = nh, \quad u_n^h := u(nh), \quad n = 0, 1, \ldots, N.$$

Let us consider the following discretization of the initial value problem:

$$\left\{\begin{array}{c} \frac{u_{n+1}^h - 2u_n^h + u_{n-1}^h}{h^2} + \frac{1}{(n+\frac{1}{2})h}\frac{u_{n+1}^h - u_{n-1}^h}{2h} + u_n^h = 0, \\ u_0^h = 1, \quad u_1^h = 1 - \frac{1}{2}h^2. \end{array}\right. \tag{1.2}$$

We can rewrite this difference scheme in the form

$$\left\{\begin{array}{c} \left(\frac{1}{2} + \frac{1}{4(n+1/2)}\right)u_{n+1}^h + \left(\frac{1}{2} - \frac{1}{4(n+1/2)}\right)u_{n-1}^h = \left(1 - \frac{1}{2}h^2\right)u_n^h, \\ u_0^h = 1, \quad u_1^h = 1 - \frac{1}{2}h^2. \end{array}\right.$$

If we set $u_n^h = P_n\left(1-\frac{h^2}{2}\right)$, this recurrence relation coincides with that of the Legendre polynomials. Therefore the Mehler-Heine type formula of Legendre polynomials is equivalent to the convergence of the solution of (1.2) to the solution of (1.1).

From this relation Aptekarev proved the following theorem [1]:

Theorem 1.1 (Aptekarev, 1992)**.**
Let $\{q_n(x)\}$ be orthonormal system of polynomials with respect to a measure $d\mu(x) \in M$. The class M is defined in terms of the behavior of the coefficients of the recurrence relations of orthonormal polynomials:

$$\begin{cases} b_n q_{n+1}(x) + a_n q_n(x) + b_{n-1}q_{n-1}(x) = xq_n(x), \\ b_n \to \frac{1}{2},\ a_n \to 0,\ \text{as } n \to \infty. \end{cases}$$

Moreover suppose that

$$\frac{q_{n+1}(1)}{q_n(1)} \simeq 1 + \frac{\alpha+\frac{1}{2}}{n} + o(1/n), \quad \alpha > -1,$$

as $n \to \infty$. Then

$$n^{-(\alpha+1/2)} q_n\left(1-\frac{z^2}{2n^2}\right) \simeq \frac{J_\alpha(z)}{z^\alpha} + o(1)$$

as $n \to \infty$ uniformly on compact subsets of the complex z-plane.

By this theorem, we can extend the Mehler-Heine type formulas of Jacobi polynomials. Indeed, Jacobi polynomials satisfy the assumption of this theorem. Recently, a few Mehler-Heine type formulas for multiple orthogonal polynomials of type II were found ([4] and [8]). In particular, we will extend the Mehler-Heine type formula of multiple orthogonal polynomials in [8]. The definition of those polynomials are as follows:

For $\alpha, \beta, \gamma > -1$ and $a > 1$, we define the multiple orthogonal polynomials by the following orthogonality condition:

$$\begin{cases} \displaystyle\int_{-1}^{+1} x^\nu p_{2k}^{(\alpha,\beta,\gamma)}(x)|1-x|^\alpha(1+x)^\beta(a-x)^\gamma dx = 0 \\ \qquad\qquad\qquad\qquad 0 \le \nu \le k-1, \\ \displaystyle\int_{+1}^{a} x^\nu p_{2k}^{(\alpha,\beta,\gamma)}(x)|1-xt|^\alpha(1+x)^\beta(a-x)^\gamma dx = 0 \\ \qquad\qquad\qquad\qquad 0 \le \nu \le k-1, \end{cases}$$

$$\begin{cases} \displaystyle\int_{-1}^{+1} x^\nu p_{2k+1}^{(\alpha,\beta,\gamma)}(x)|1-x|^\alpha(1+x)^\beta(a-x)^\gamma dx = 0 \\ \qquad\qquad\qquad\qquad 0 \le \nu \le k, \\ \displaystyle\int_{+1}^{a} x^\nu p_{2k+1}^{(\alpha,\beta,\gamma)}(x)|1-x|^\alpha(1+x)^\beta(a-x)^\gamma dx = 0 \\ \qquad\qquad\qquad\qquad 0 \le \nu \le k-1. \end{cases}$$

Such polynomials are sometimes called the Jacobi-Jacobi (or Jacobi-Angelesco) type polynomials. Furthermore, we set

$$p_n^{(\alpha,\beta,\gamma)}(1) = 1.$$

Then we can prove by a direct calculation the Mehler-Heine type formula: **(Takata, 2003)** For $a \neq 3$ we have

$$\lim_{n\to\infty} p_n^{(\alpha,\beta,\gamma)}\left(1 - \frac{2z^2}{n^2}\right) = \Gamma(\alpha+1)\left(\sqrt{\frac{3-a}{1-a}}\cdot\frac{z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\frac{3-a}{1-a}}z\right).$$

In this paper, we will extend this formula to a class of polynomials which includes $p_n^{(\alpha,\beta,\gamma)}$. We will define the class and state our main result in the section 2. In the section 3, we will recall the properties of the multiple orthogonal polynomials discussed in [8]. Finally, we will give a proof of our main result in the section 4.

2. Four-terms recurrence relations and a discretization of the Bessel equation

Let p_n be multiple orthogonal polynomials with $p_n(1) = 1$. Assume the polynomials $\{p_n\}$ have the following four-terms recurrence relations:

$$\begin{cases} u_k\ p_{2k+1} + a_k\ p_{2k} + v_k\ p_{2k-1} + w_k p_{2k-2} = zp_{2k}, \\ u'_k\ p_{2k} + a'_k\ p_{2k-1} + v'_k\ p_{2k-2} + w'_k\ p_{2k-3} = zp_{2k-1}. \end{cases}$$

Assume there exist $u_\infty, u'_\infty, V, v, W, w, V', v', W'$ and w' such that

$$\begin{cases} u_\infty = \lim_{k\to\infty} u_k \neq 0 \\ u'_\infty = \lim_{k\to\infty} u'_k \neq 0 \end{cases},$$

$$\begin{cases} \frac{v_k}{u_k} \simeq V + \frac{v}{k} + o\left(\frac{1}{k}\right), \\ \frac{w_k}{u_k} \simeq W + \frac{w}{k} + o\left(\frac{1}{k}\right), \\ \frac{v'_k}{u'_k} \simeq V' + \frac{v'}{k} + o\left(\frac{1}{k}\right), \\ \frac{w'_k}{u'_k} \simeq W' + \frac{w'}{k} + o\left(\frac{1}{k}\right), \end{cases}$$

as $k \to \infty$. Furthermore we consider the following three conditions:

Condition 1.

$$V + W + W' = V' + W + W' = 1.$$

Condition 2.

$$WW' \neq 1.$$

Condition 3.

We fix $Z > 0$ and $h_0 > 0$. If we set $0 < h < h_0$, $N := \left[\frac{Z}{h}\right]$ and $n = 0, \ldots, 2N-1$, there exist M and M' such that

$$\left| p_n \left(1 - \frac{h^2}{2}\right) \right| \leq M, \tag{2.1}$$

$$\left| p_{n+1} \left(1 - \frac{h^2}{2}\right) - p_n \left(1 - \frac{h^2}{2}\right) \right| \leq M'(n+1)h^2 \leq Mh, \tag{2.2}$$

for any h.

Remark.

The inequality (2.1) follows from (2.2) immediately.

If $|WW'| < 1$ and $-\frac{(1-W)(v+w+w')+(1-W')(v'+w+w')}{1-WW'} > -1$, we can prove Condition 3 from Conditions 1 and 2.

Theorem 2.1. *If we assume Conditions* 1, 2, 3 *we have*

$$\lim_{n\to\infty} p_n \left(1 - \frac{2z^2}{n^2}\right) = \Gamma(\alpha+1) \left(\frac{\sqrt{\lambda}z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\lambda}z\right),$$

for some λ where we set

$$2\alpha + 1 := -\frac{(1-W)(v+w+w') + (1-W')(v'+w+w')}{1-WW'}.$$

This convergence is uniform on compact subsets of the complex z-plane.

We will give a proof of this theorem in the Section 4.

3. The "classical" multiple orthogonal polynomials

3.1. Nonsymmetric case

In this subsection, we show that the "classical" multiple orthogonal polynomials $p_n^{(\alpha,\beta,\gamma)}$ satisfy Conditions 1, 2, 3 if the lengths of the intervals of orthogonality are not the same. We write the recurrence formula as

$$\begin{cases} u_k\, p_{2k+1}^{(\alpha,\beta,\gamma)} + a_k\, p_{2k}^{(\alpha,\beta,\gamma)} + v_k\, p_{2k-1}^{(\alpha,\beta,\gamma)} + w_k p_{2k-2}^{(\alpha,\beta,\gamma)} = z p_{2k}^{(\alpha,\beta,\gamma)} \\ u'_k\, p_{2k}^{(\alpha,\beta,\gamma)} + a'_k\, p_{2k-1}^{(\alpha,\beta,\gamma)} + v'_k\, p_{2k-2}^{(\alpha,\beta,\gamma)} + w'_k\, p_{2k-3}^{(\alpha,\beta,\gamma)} = z p_{2k-1}^{(\alpha,\beta,\gamma)} \end{cases}, \tag{3.1}$$

and set

$$
\begin{aligned}
&p_{2k}^{(\alpha,\beta,\gamma)}(x)\\
&\qquad = \lambda_0(k,\alpha,\beta,\gamma)x^{2k} + \lambda_1(k,\alpha,\beta,\gamma)x^{2k-1} + \lambda_2(k,\alpha,\beta,\gamma)x^{2k-2} + \cdots + 1,\\
&p_{2k+1}^{(\alpha,\beta,\gamma)}(x)\\
&\qquad = \mu_0(k,\alpha,\beta,\gamma)x^{2k+1} + \mu_1(k,\alpha,\beta,\gamma)x^{2k} + \mu_2(k,\alpha,\beta,\gamma)x^{2k-1} + \cdots + 1.
\end{aligned}
$$

From these formulas we obtain

$$
\begin{aligned}
u_k &= \frac{\lambda_0(k,\alpha,\beta,\gamma)}{\mu_0(k,\alpha,\beta,\gamma)},\\
u'_k &= \frac{\mu_0(k-1,\alpha,\beta,\gamma)}{\lambda_0(k,\alpha,\beta,\gamma)},\\
a_k &= \frac{\lambda_1(k,\alpha,\beta,\gamma) - u_k\mu_1(k,\alpha,\beta,\gamma)}{\lambda_0(k,\alpha,\beta,\gamma)},\\
a'_k &= \frac{\mu_1(k-1,\alpha,\beta,\gamma) - u'_k\lambda_1(k,\alpha,\beta,\gamma)}{\mu_0(k-1,\alpha,\beta,\gamma)},\\
v_k &= \frac{\lambda_2(k,\alpha,\beta,\gamma) - u_k\mu_2(k,\alpha,\beta,\gamma) - a_k\lambda_1(k,\alpha,\beta,\gamma)}{\mu_0(k-1,\alpha,\beta,\gamma)},\\
v'_k &= \frac{\mu_2(k-1,\alpha,\beta,\gamma) - u'_k\lambda_2(k,\alpha,\beta,\gamma) - a'_k\mu_1(k-1,\alpha,\beta,\gamma)}{\lambda_0(k-1,\alpha,\beta,\gamma)},\\
w_k &= 1 - u_k - a_k - v_k,\\
w'_k &= 1 - u'_k - a'_k - v'_k.
\end{aligned}
$$

Furthermore, we know the following explicit representation of $p_n^{(\alpha,\beta,\gamma)}$ [8]:

$$
\begin{aligned}
p_{2k}^{(\alpha,\beta,\gamma)}(x) &= c_k(1-x)^{-\alpha}(1+x)^{-\beta}(a-x)^{-\gamma}\\
&\qquad \times \left(\frac{d}{dx}\right)^k \left\{(1-x)^{k+\alpha}(1+x)^{k+\beta}(a-x)^{k+\gamma}\right\},
\end{aligned}
$$

$$
p_{2k+1}^{(\alpha,\beta,\gamma)}(x) = \frac{1+x}{2-(a-1)d_k}p_{2k}^{(\alpha,\beta+1,\gamma)}(x) - \frac{d_k(a-x)}{2-(a-1)d_k}p_{2k}^{(\alpha,\beta,\gamma+1)}(x), \tag{3.2}
$$

$$
\begin{aligned}
c_k &= \binom{k+\alpha}{k}^{-1}(a-1)^{-k}\frac{(-1)^k}{2^k k!},\\
d_k &= \frac{\int_{-1}^{+1}(1-x)^{k+\alpha}(1+x)^{k+\beta+1}(a-x)^{k+\gamma}dx}{\int_{-1}^{+1}(1-x)^{k+\alpha}(1+x)^{k+\beta}(a-x)^{k+\gamma+1}dx}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\lambda_0(k,\alpha,\beta,\gamma)/c_k &= \frac{\Gamma(3k+\alpha+\beta+\gamma+1)}{\Gamma(2k+\alpha+\beta+\gamma+1)},\\
\lambda_1(k,\alpha,\beta,\gamma)/c_k &= \frac{\Gamma(3k+\alpha+\beta+\gamma)\{-(k+\gamma)a+\beta-\alpha\}}{\Gamma(2k+\alpha+\beta+\gamma)}\\
&\quad+\frac{\Gamma(3k+\alpha+\beta+\gamma+1)(a\gamma-\beta+\alpha)}{\Gamma(2k+\alpha+\beta+\gamma+1)},\\
\lambda_1(k,\alpha,\beta,\gamma)/c_k &= \frac{\Gamma(3k+\alpha+\beta+\gamma-1)}{\Gamma(2k+\alpha+\beta+\gamma-1)}\left\{\frac{1}{2}a^2(k+\gamma)(k+\gamma-1)\right.\\
&\quad-a(\beta-\alpha)(k+\gamma)\\
&\quad+\frac{1}{2}(k+\alpha)(k+\alpha-1)\\
&\quad\left.+\frac{1}{2}(k+\beta)(k+\beta-1)-(k+\alpha)(k+\beta)\right\}\\
&\quad+\frac{\Gamma(3k+\alpha+\beta+\gamma)\{-(k+\gamma)a+\beta-\alpha\}(a\gamma-\beta+\alpha)}{\Gamma(2k+\alpha+\beta+\gamma)}\\
&\quad+\frac{\Gamma(3k+\alpha+\beta+\gamma+1)}{\Gamma(2k+\alpha+\beta+\gamma+1)}\left\{\frac{a^2}{2}\gamma(\gamma+1)+(-\beta+\alpha)\gamma a\right.\\
&\quad\left.+\frac{1}{2}\alpha(\alpha+1)+\frac{1}{2}\beta(\beta+1)-\alpha\beta\right\},\\
\mu_0(k,\alpha,\beta,\gamma) &= \frac{\lambda_0(k,\alpha,\beta+1,\gamma)}{2-(a-1)d_k}+\frac{d_k\lambda_0(k,\alpha,\beta,\gamma+1)}{2-(a-1)d_k},\\
\mu_1(k,\alpha,\beta,\gamma) &= \frac{\lambda_0(k,\alpha,\beta+1,\gamma)}{2-(a-1)d_k}+\frac{\lambda_1(k,\alpha,\beta+1,\gamma)}{2-(a-1)d_k}\\
&\quad-\frac{ad_k\lambda_0(k,\alpha,\beta,\gamma+1)}{2-(a-1)d_k}+\frac{d_k\lambda_1(k,\alpha,\beta,\gamma+1)}{2-(a-1)d_k},\\
\mu_2(k,\alpha,\beta,\gamma) &= \frac{\lambda_1(k,\alpha,\beta+1,\gamma)}{2-(a-1)d_k}+\frac{\lambda_2(k,\alpha,\beta+1,\gamma)}{2-(a-1)d_k}\\
&\quad-\frac{ad_k\lambda_1(k,\alpha,\beta,\gamma+1)}{2-(a-1)d_k}+\frac{d_k\lambda_2(k,\alpha,\beta,\gamma+1)}{2-(a-1)d_k}.
\end{aligned}
$$

Therefore we can write the coefficients of the recurrence formula 3.1 explicitly. To calculate $u_\infty, u'_\infty, V, v, W, w, V', v', W'$ and w' we need the following lemma.

Lemma 3.1. *We have*

$$
\begin{aligned}
d &= \frac{a+1-\sqrt{a^2+3}}{a-1},\\
2(a-1)\left(a^2+3\right)d' &= (\alpha+\beta-2\gamma-1)a^2\left(a-\sqrt{a^2+3}\right)\\
&\quad+(\alpha-\beta)\left(a^2-2a\sqrt{a^2+3}\right)\\
&\quad+(\alpha+\beta-2\gamma+1)\left(3a-\sqrt{a^2+3}\right)\\
&\quad-2a+3(\alpha-\beta),
\end{aligned}
$$

where $d_k = d + \frac{d'}{k} + O\left(\frac{1}{k^2}\right)$.

Proof. By Laplace's method [7], we obtain

$$d = \frac{1 + x_0}{a - x_0}, \tag{3.3}$$

$$d' = \frac{1}{4} \frac{\left(2q''(x_0) - \frac{p'''(x_0)q'(x_0)}{p''(x_0)}\right)(1 + a)}{r(x_0)\, p''(x_0)\,(a - x_0)}. \tag{3.4}$$

Here we put

$$\begin{aligned} x_0 &= \frac{a - \sqrt{a^2 + 3}}{3}, \\ p(x) &= -\log\{(1 - x)(1 + x)(a - x)\}, \\ q(x) &= (1 - x)^\alpha (1 + x)^\beta (a - x)^\gamma (x - x_0), \\ r(x) &= (1 - x)^\alpha (1 + x)^\beta (a - x)^{\gamma + 1}. \end{aligned}$$

□

By this lemma, we obtain

$$\begin{aligned} u_\infty &= -\frac{2}{9}\left(a - 3 + \sqrt{a^2 + 3}\right), \\ u'_\infty &= -\frac{2}{9}\left(a - 3 - \sqrt{a^2 + 3}\right), \\ V = V' &= \frac{-a^2 - 3}{6(a - 1)}, \\ W &= \frac{a^2 + 6a - 3 - (a - 3)\sqrt{a^2 + 3}}{12(a - 1)}, \\ W' &= \frac{a^2 + 6a - 3 + (a - 3)\sqrt{a^2 + 3}}{12(a - 1)}. \end{aligned}$$

Thus we have

$$\begin{gathered} u_\infty \neq 0, \quad u'_\infty \neq 0, \\ V + W + W' = V' + W + W' = 1. \end{gathered}$$

Namely, $p_n^{(\alpha,\beta,\gamma)}$ satisfies Condition 1. Moreover we have $WW' = \frac{(a+1)^2}{8(a-1)} > 1$ since $a \neq 3$. The lower terms v, w, v' and w' are so complicated. Indeed, we have

$$
\begin{aligned}
v &= \frac{(8\alpha+2\beta-\gamma+3)a^2+6(\alpha-\beta)a+9(2\alpha+\gamma+1)}{36(a-1)},\\
36(a-1)\left(a^2+3\right)v' &= (8\alpha+2\beta-\gamma+3)a^4+2a^3\sqrt{a^2+3}+6(\alpha-\beta)a^3\\
&\quad+6(7\alpha+\beta+\gamma+3)a^2-18a\sqrt{a^2+3}+18(\alpha-\beta)a\\
&\quad+27(2\alpha+\gamma+1),\\
-72(a-1)\left(a^2+3\right)w &= (8\alpha+2\beta-\gamma+3)a^3\left(a-\sqrt{a^2+3}\right)\\
&\quad+3(20\alpha-2\beta+7)a^3+3(10\alpha-4\beta+3\gamma+5)a^2\sqrt{a^2+3}\\
&\quad-3(4\alpha-2\beta-2\gamma+7)a^2-3(16\alpha-2\beta-5\gamma+11)a\sqrt{a^2+3}\\
&\quad+9(20\alpha-2\beta+11)a+9(10\alpha-\gamma+5)\sqrt{a^2+3}\\
&\quad-27(4\alpha-\gamma+2),\\
-72(a-1)\left(a^2+3\right)w' &= (8\alpha+2\beta-\gamma+5)a^3\left(a+\sqrt{a^2+3}\right)\\
&\quad+3(20\alpha-2\beta+13)a^3-3(10\alpha-4\beta+3\gamma+5)a^2\sqrt{a^2+3}\\
&\quad-3(4\alpha-2\beta-2\gamma-3)a^2+3(16\alpha-2\beta-5\gamma+5)a\sqrt{a^2+3}\\
&\quad+9(20\alpha-2\beta+9)a-9(10\alpha-\gamma+5)\sqrt{a^2+3}\\
&\quad-27(4\alpha-\gamma+2).
\end{aligned}
$$

Therefore we have

$$
-\frac{(1-W)\left(v+w+w'\right)+\left(1-W'\right)\left(v'+w+w'\right)}{1-WW'}=2\alpha+1>-1.
$$

We proceed to the proof of the boundedness of $\frac{1}{kh^2}\left|p_{2k+1}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)-p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)\right|$. From (3.2) we have

$$
\begin{aligned}
&p_{2k}^{(\alpha,\beta+1,\gamma)}\left(1-\frac{h^2}{2}\right)-p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)\\
&=\Gamma(\alpha+1)\left(\frac{a-1}{a-1+\frac{h^2}{2}}\right)^{\gamma}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\frac{h^2}{2}}{1-a}\right)^{i}\left(\frac{\frac{h^2}{2}}{2}\right)^{i}\\
&\times\frac{P_k^{(\alpha+i,\beta+1)}\left(1-\frac{h^2}{2}\right)-P_k^{(\alpha+i,\beta)}\left(1-\frac{h^2}{2}\right)}{\Gamma(\alpha+i+1)}\frac{\Gamma(k+\alpha+i+1)}{k^i\Gamma(k+\alpha+1)}\frac{\Gamma(k+\gamma+1)}{k^i\Gamma(k+\gamma-i+1)}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&P_k^{(\alpha+i,\beta+1)}\left(1-\frac{h^2}{2}\right)-P_k^{(\alpha+i,\beta)}\left(1-\frac{h^2}{2}\right)\\
&=\Gamma(\alpha+i+1)\sum_{\nu=0}^{k}\frac{k!}{\nu!(k-\nu)!}\frac{\left(-\frac{h^2}{4}\right)^\nu}{\Gamma(\alpha+i+\nu+1)}\\
&\quad\times\left[\frac{\Gamma(k+\alpha+i+\beta+1+\nu+1)}{\Gamma(k+\alpha+i+\beta+1+1)}-\frac{\Gamma(k+\alpha+i+\beta+\nu+1)}{\Gamma(k+\alpha+i+\beta+1)}\right]\\
&=\Gamma(\alpha+i+1)\sum_{\nu=0}^{k}\frac{k!}{\nu!(k-\nu)!}\frac{\left(-\frac{h^2}{4}\right)^\nu}{\Gamma(\alpha+i+\nu+1)}\\
&\quad\times\frac{\Gamma(k+\alpha+i+\beta+\nu+1)}{\Gamma(k+\alpha+i+\beta+1)}\left[\frac{k+\alpha+i+\beta+\nu+1}{k+\alpha+i+\beta+1}-1\right]\\
&=\Gamma(\alpha+i+1)\sum_{\nu=0}^{k}\frac{k!}{\nu!(k-\nu)!}\frac{\left(-\frac{h^2}{4}\right)^\nu}{\Gamma(\alpha+i+\nu+1)}\\
&\quad\times\frac{\Gamma(k+\alpha+i+\beta+\nu+1)}{\Gamma(k+\alpha+i+\beta+1)}\frac{\nu}{k+\alpha+i+\beta+1}\\
&=\Gamma(\alpha+i+1)\sum_{\nu=0}^{k}\frac{(-1)^\nu\left(\frac{kh}{2}\right)^{2\nu}}{\nu!\Gamma(\alpha+i+\nu+1)}\frac{k!}{k^\nu(k-\nu)!}\\
&\quad\times\frac{\Gamma(k+\alpha+i+\beta+\nu+1)}{k^\nu\Gamma(k+\alpha+i+\beta+1)}\frac{\nu}{k+\alpha+i+\beta+1}\\
&=\frac{kh^2}{4}\Gamma(\alpha+i+1)\sum_{\mu=0}^{k-1}\frac{(-1)^{\mu+1}\left(\frac{kh}{2}\right)^{2\mu}}{\mu!\Gamma(\alpha+i+\mu+2)}\frac{k!}{k^{\mu+1}(k-\mu-1)!}\\
&\quad\times\frac{\Gamma(k+\alpha+i+\beta+\mu+2)}{k^{\mu+1}\Gamma(k+\alpha+i+\beta+1)}\frac{k}{k+\alpha+i+\beta+1}.
\end{aligned}
$$

Thus $\frac{1}{kh^2}\left|P_k^{(\alpha+i,\beta+1)}\left(1-\frac{h^2}{2}\right)-P_k^{(\alpha+i,\beta)}\left(1-\frac{h^2}{2}\right)\right|$ is bounded for $0\le i\le k+\lceil\gamma\rceil$. Therefore we conclude that

$$\frac{1}{kh^2}\left|p_{2k}^{(\alpha,\beta+1,\gamma)}\left(1-\frac{h^2}{2}\right)-p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)\right|$$

is bounded. Strictly speaking, we must argue in the same way as in the proof of the boundedness of $p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)$. Similarly, we can prove that $\frac{1}{kh^2}\left|p_{2k}^{(\alpha,\beta,\gamma+1)}\left(1-\frac{h^2}{2}\right)-p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)\right|$ is bounded. Therefore by (3.2) we obtain $\frac{1}{kh^2}\left|p_{2k+1}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)-p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)\right|$ is also bounded.

By the similar argument above we may show that $\frac{1}{kh^2}\left|p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)\right.$ $\left.-\,p_{2k-1}^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)\right|$ is bounded.

We conclude that "classical" multiple orthogonal polynomials $p_n^{(\alpha,\beta,\gamma)}\left(1-\frac{h^2}{2}\right)$ satisfy the Conditions 1, 2, 3.

Remark.
Set $f(z)=\lim_{k\to\infty} p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)$. Then we can obtain

$$\frac{k}{z}\left(p_{2k+1}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)-p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)\right)\to\frac{a-3+\sqrt{a^2+3}}{2(a-3)}f'(z),$$
$$\frac{k}{z}\left(p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)-p_{2k-1}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)\right)\to\frac{a-3-\sqrt{a^2+3}}{2(a-3)}f'(z).$$

We thus have

$$\frac{k}{z}\left(p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)-p_{2k-2}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)\right)\to f'(z),$$
$$\frac{k}{z}\left(p_{2k+1}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)-p_{2k-1}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)\right)\to f'(z).$$

This is the reason why we use $f_{n+2}^h-f_n^h$ and $f_n^h-f_{n-2}^h$ in our definition of $f_n'^h$ in the Section 4.

3.2. Symmetric case

In this subsection, we consider the case where the intervals of orthogonality are of equal length. In other words, we assume that $a=3$. Then $p_n^{(\alpha,\beta,\gamma)}$ does not have the Mehler-Heine type formula. Indeed, $p_n^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2n^2}\right)\to 1$ as $n\to\infty$. But if we choose the suitable scaling, we can obtain a different asymptotic formula.

Theorem 3.1. *If $a=3$, we have*

$$\lim_{n\to\infty} p_{2n}^{(\alpha,\beta,\gamma)}\left(1-\frac{z}{n^{3/2}}\right)=\Gamma(\alpha+1)\,{}_0F_2\left(\begin{matrix}\emptyset\\ 1+\frac{\alpha}{2},\frac{1}{2}+\frac{\alpha}{2}\end{matrix};-\frac{z^2}{16}\right)$$
$$=\Gamma(\alpha+1)\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(2\nu+\alpha+1)}.$$

Proof. We have

$$(1-x)^\alpha(1+x)^\beta(3-x)^\gamma p_{2n}^{(\alpha,\beta,\gamma)}(x)$$
$$=c_n\left(\frac{d}{dx}\right)^n\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}(3-x)^{n+\gamma}\right\}$$

where $c_n = \binom{n+\alpha}{n}^{-1} \frac{(-1)^n}{2^{2n} n!}$. By setting $y = 1 - x$ we will rewrite the right side as follows

$$\begin{aligned}
& c_n \left(\frac{d}{dx}\right)^n \left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}(3-x)^{n+\gamma}\right\} \\
&= (-1)^n c_n \left(\frac{d}{dy}\right)^n \left\{y^{n+\alpha}(2-y)^{n+\beta}(2+y)^{n+\gamma}\right\} \\
&= (-1)^n 2^{2n+\beta+\gamma} c_n \left(\frac{d}{dy}\right)^n \left\{y^{n+\alpha}\left(1-\frac{y}{2}\right)^{n+\beta}\left(1+\frac{y}{2}\right)^{n+\gamma}\right\} \\
&= \frac{2^{\beta+\gamma}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \left(\frac{d}{dy}\right)^n \left\{y^{n+\alpha}\left(1-\frac{y^2}{4}\right)^{n}\left(1-\frac{y}{2}\right)^{\beta}\left(1+\frac{y}{2}\right)^{\gamma}\right\} \\
&= \frac{2^{\beta+\gamma}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \left(\frac{d}{dy}\right)^n \left\{y^{n+\alpha}\sum_{\nu=0}^{n}\binom{n}{\nu}\left(-\frac{y^2}{4}\right)^{\nu}\phi(y)\right\} \\
&= \frac{2^{\beta+\gamma}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \sum_{\mu=0}^{n}\sum_{\nu=0}^{n}\binom{n}{\mu}\binom{n}{\nu}\left(-\frac{1}{4}\right)^{\nu}\frac{\Gamma(n+2\nu+\alpha+1)}{\Gamma(\mu+2\nu+\alpha+1)} y^{\mu+2\nu+\alpha}\phi^{(\mu)}(y).
\end{aligned}$$

Here we set $\phi(y) = \left(1-\frac{y}{2}\right)^{\beta}\left(1+\frac{y}{2}\right)^{\gamma}$. Since we can easily show that

$$\begin{aligned}
& \frac{2^{\beta+\gamma}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \sum_{\nu=0}^{n}\binom{n}{\nu}\left(-\frac{1}{4}\right)^{\nu}\frac{\Gamma(n+2\nu+\alpha+1)}{\Gamma(2\nu+\alpha+1)}\left(\frac{z}{n^{3/2}}\right)^{2\nu+\alpha}\phi\left(\frac{z}{n^{3/2}}\right) \\
&\quad = 2^{\beta+\gamma}\left(\frac{z}{n^{3/2}}\right)^{\alpha}\Gamma(\alpha+1) \\
&\qquad \times \sum_{\nu=0}^{n}\frac{(-1)^{\nu}\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(2\nu+\alpha+1)}\frac{\Gamma(n+2\nu+\alpha+1)}{n^{2\nu}\Gamma(n+\alpha+1)}\frac{n!}{n^{\nu}(n-\nu)!}\phi\left(\frac{z}{n^{3/2}}\right) \\
&\quad \simeq 2^{\beta+\gamma}\left(\frac{z}{n^{3/2}}\right)^{\alpha}\Gamma(\alpha+1)\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(2\nu+\alpha+1)},
\end{aligned}$$

as $n \to \infty$, we should show that

$$\begin{aligned}
& \frac{1}{\Gamma(n+\alpha+1)} \\
&\qquad \times \sum_{\mu=1}^{n}\sum_{\nu=0}^{n}\binom{n}{\mu}\binom{n}{\nu}\left(-\frac{1}{4}\right)^{\nu}\frac{\Gamma(n+2\nu+\alpha+1)}{\Gamma(\mu+2\nu+\alpha+1)}\left(\frac{z}{n^{3/2}}\right)^{\mu+2\nu}\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)
\end{aligned}$$

converges to 0 as $n \to \infty$ for z in any compact subset of the complex plane. We have

$$
\begin{aligned}
&\left|\frac{1}{\Gamma(n+\alpha+1)}\right| \\
&\qquad \times \left|\sum_{\mu=1}^{n}\sum_{\nu=0}^{n}\binom{n}{\mu}\binom{n}{\nu}\left(-\frac{1}{4}\right)^{\nu}\frac{\Gamma(n+2\nu+\alpha+1)}{\Gamma(\mu+2\nu+\alpha+1)}\left(\frac{z}{n^{3/2}}\right)^{\mu+2\nu}\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)\right| \\
&\quad \leq \left|\sum_{\mu=1}^{n}\left(\frac{z}{n^{1/2}}\right)^{\mu}\frac{n!}{n^{\mu}(n-\mu)!}\frac{\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)}{\mu!}\right| \\
&\qquad \times \left|\sum_{\nu=0}^{n}\frac{(-1)^{\nu}\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(\mu+2\nu+\alpha+1)}\frac{\Gamma(n+2\nu+\alpha+1)}{n^{2\nu}\Gamma(n+\alpha+1)}\frac{n!}{n^{\nu}(n-\nu)!}\right| \\
&\quad \leq \sum_{\mu=1}^{n}\left|\left(\frac{z}{n^{1/2}}\right)^{\mu}\frac{n!}{n^{\mu}(n-\mu)!}\frac{\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)}{\mu!}\right| \\
&\qquad \times \sum_{\nu=0}^{n}\left|\frac{(-1)^{\nu}\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(1+2\nu+\alpha+1)}\frac{\Gamma(n+2\nu+\alpha+1)}{n^{2\nu}\Gamma(n+\alpha+1)}\frac{n!}{n^{\nu}(n-\nu)!}\right|.
\end{aligned}
$$

We can show easily that

$$
\begin{aligned}
&\sum_{\nu=0}^{n}\left|\frac{(-1)^{\nu}\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(1+2\nu+\alpha+1)}\frac{\Gamma(n+2\nu+\alpha+1)}{n^{2\nu}\Gamma(n+\alpha+1)}\frac{n!}{n^{\nu}(n-\nu)!}\right| \\
&\quad \leq \sum_{\nu=0}^{n}\left|\frac{(-1)^{\nu}\left(\frac{z}{2}\right)^{2\nu}}{\nu!\Gamma(2\nu+\alpha+2)}\prod_{k=1}^{2\nu}\left(1+\frac{\alpha+k}{n}\right)\cdot 1\right| \\
&\quad \leq \sum_{\nu=0}^{n}\frac{\left|\frac{z}{2}\right|^{2\nu}}{\nu!\Gamma(2\nu+\alpha+2)}\prod_{k=1}^{2\nu}\left(1+\frac{|\alpha|+k}{n}\right) \\
&\quad \leq \sum_{\nu=0}^{n}\frac{\left|\frac{z}{2}\right|^{2\nu}}{\nu!\Gamma(2\nu+\alpha+2)}\left(3+\frac{|\alpha|+2(\nu-n)}{n}\right)^{2\nu} \\
&\quad \leq \left(1+\frac{|\alpha|}{3n}\right)^{2n}\sum_{\nu=0}^{n}\frac{\left|\frac{3z}{2}\right|^{2\nu}}{\nu!\Gamma(2\nu+\alpha+2)}
\end{aligned}
$$

converges as $n \to \infty$. Thus we will prove

$$
\sum_{\mu=1}^{n}\left|\left(\frac{z}{n^{1/2}}\right)^{\mu}\frac{n!}{n^{\mu}(n-\mu)!}\frac{\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)}{\mu!}\right|
$$

converges to 0. Since

$$
\begin{aligned}
\phi^{(\mu)}(y) = \sum_{j=0}^{\mu}\binom{\mu}{j}&\beta(\beta-1)\cdots(\beta-j+1)\left(1-\frac{y}{2}\right)^{\beta-j}\left(-\frac{1}{2}\right)^{j} \\
&\times \gamma(\gamma-1)\cdots(\gamma-\mu+j+1)\left(1+\frac{y}{2}\right)^{\gamma-\mu+j}\left(\frac{1}{2}\right)^{\mu-j},
\end{aligned}
$$

we have

$$\left|\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)\right| \le \left(\frac{1}{2}\right)^{\mu}\sum_{j=0}^{\mu}\binom{\mu}{j}(|\beta|)_j(|\gamma|)_{\mu-j}\left|1-\frac{z}{2n^{3/2}}\right|^{\beta-j}\left|1+\frac{z}{2n^{3/2}}\right|^{\gamma-\mu+j}.$$

Here if n is sufficiently large we have

$$\begin{aligned}\left|1-\frac{z}{2n^{3/2}}\right|^{\beta-j} &\le \begin{cases}\left(1-\left|\frac{z}{2n^{3/2}}\right|\right)^{\beta-j} & \text{if } \beta-j\le 0\\ \left(1+\left|\frac{z}{2n^{3/2}}\right|\right)^{\beta-j} & \text{if } \beta-j>0\end{cases}\\ &\le \begin{cases}\left(1-\left|\frac{z}{2n^{3/2}}\right|\right)^{\beta-n} & \text{if } \beta-j\le 0\\ \left(1+\left|\frac{z}{2n^{3/2}}\right|\right)^{\beta} & \text{if } \beta-j>0\end{cases}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|1+\frac{z}{2n^{3/2}}\right|^{\gamma-\mu+j} &\le \begin{cases}\left(1-\left|\frac{z}{2n^{3/2}}\right|\right)^{\gamma-\mu+j} & \text{if } \gamma-\mu+j\le 0\\ \left(1+\left|\frac{z}{2n^{3/2}}\right|\right)^{\gamma-\mu+j} & \text{if } \gamma-\mu+j>0\end{cases}\\ &\le \begin{cases}\left(1-\left|\frac{z}{2n^{3/2}}\right|\right)^{\gamma-n} & \text{if } \gamma-\mu+j\le 0\\ \left(1+\left|\frac{z}{2n^{3/2}}\right|\right)^{\gamma} & \text{if } \gamma-\mu+j>0\end{cases}\end{aligned}$$

for sufficiently large n. Thus in all the cases $\left|1-\frac{z}{2n^{3/2}}\right|^{\beta-j}\left|1+\frac{z}{2n^{3/2}}\right|^{\gamma-\mu+j}$ is bounded. Therefore there exists M such that

$$\begin{aligned}\left|\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)\right| &\le \left(\frac{1}{2}\right)^{\mu} M\sum_{j=0}^{\mu}\binom{\mu}{j}(|\beta|)_j(|\gamma|)_{\mu-j}\\ &= \left(\frac{1}{2}\right)^{\mu} M(|\beta|+|\gamma|)_{\mu}\\ &\le \left(\frac{1}{2}\right)^{\mu} M\Gamma(\mu+|\beta|+|\gamma|+1).\end{aligned}$$

Therefore we have

$$\begin{aligned}&\sum_{\mu=1}^{n}\left|\left(\frac{z}{n^{1/2}}\right)^{\mu}\frac{n!}{n^{\mu}(n-\mu)!}\frac{\phi^{(\mu)}\left(\frac{z}{n^{3/2}}\right)}{\mu!}\right|\\ &\le M\sum_{\mu=1}^{n}\left|\mu^{|\beta|+vert\gamma|}\left(\frac{z}{2n^{1/2}}\right)^{\mu}\frac{n!}{n^{\mu}(n-\mu)!}\frac{\Gamma(\mu+|\beta|+|\gamma|+1)}{\mu^{|\beta|+|\gamma|}\mu!}\right|\\ &= M'\sum_{\mu=1}^{n}\mu^{|\beta|+|\gamma|}\left|\frac{z}{2n^{1/2}}\right|^{\mu}\\ &= \left|\frac{z}{2n^{1/2}}\right| M'\sum_{\mu=1}^{n}\mu^{|\beta|+|\gamma|}\left|\frac{z}{2n^{1/2}}\right|^{\mu-1}\to 0 \quad\text{as}\quad n\to\infty\end{aligned}$$

for some constant M', since $\sum_{\mu=1}^{n}\mu^{|\beta|+|\gamma|}\left|\frac{z}{2n^{1/2}}\right|^{\mu-1}$ converges. We have finished the proof. □

Remark.

The function ${}_0F_2\left(\begin{matrix} \emptyset \\ 1+\frac{\alpha}{2}, \frac{1}{2}+\frac{\alpha}{2} \end{matrix}; -\frac{z^2}{16}\right)$ satisfies the following third-order differential equation:

$$u'''(z) + \frac{2(\alpha+1)}{z}u''(z) + \frac{\alpha(\alpha+1)}{z^2}u'(z) + \frac{u(z)}{2z} = 0.$$

Thus in this case $p_n^{(\alpha,\beta,\gamma)}$ behaves asymptotically like the function which satisfies not the second-order differential equation but the third-order one.

4. A proof of the theorem

Set

$$\begin{aligned} f_n^h &:= p_n\left(1 - \frac{h^2}{2}\right), \\ f_n'^h &:= \frac{f_{n+2}^h - f_n^h - WW'\left(f_n^h - f_{n-2}^h\right)}{h\left(1 - WW'\right)}, \end{aligned}$$

and define $f^h, f'^h \in C[0,Z]$ as follows:

$$\begin{aligned} & f^h(z) := f_{2n}^h + \left(f_{2n+2}^h - f_{2n}^h\right)\frac{z - nh}{h} \\ & \text{if } \ nh \le z \le (n+1)h, \\ & f'^h(z) := f_{2n}'^h + \left(f_{2n+2}'^h - f_{2n}'^h\right)\frac{z - nh}{h} \\ & \text{if } \ nh \le z \le (n+1)h. \end{aligned}$$

Here we assume $Z > 0$ and let $C[0,Z]$ is the set of continuous functions on $[0,Z]$.

Lemma 4.1. *We have*

$$f_{2k}'^h - f_{2k-2}'^h = -\frac{2\alpha+1}{kh}f_{2k-2}'^h h + h\lambda f_{2k}^h + o\left(\frac{1}{k}\right)kh,$$

as $k \to \infty$, *where*

$$\lambda = \frac{1}{2}\cdot\frac{2 - W - W'}{1 - WW'}\left(\frac{1}{u_\infty} + \frac{1}{u'_\infty}\right),$$

$u_\infty = \lim_{k\to\infty} u_k$ *and* $u'_\infty = \lim_{k\to\infty} u'_k$.

Proof. If we write $V_k := \frac{v_k}{u_k}$, $W_k := \frac{w_k}{u_k}$, $V'_k := \frac{v'_k}{u'_k}$, $W'_k := \frac{w'_k}{u'_k}$, and $\Delta f_n := f_{n+1}^h - f_n^h$, we have

$$\Delta f_{2k} = (V_k + W_k)\,\Delta f_{2k-1} + W_k \Delta f_{2k-2} - \frac{h^2}{2u_k} f_{2k}^h,$$
$$\Delta f_{2k-1} = (V_k' + W_k')\,\Delta f_{2k-2} + W_k' \Delta f_{2k-3} - \frac{h^2}{2u_k'} f_{2k-1}^h.$$

From these formulas we obtain

$$\begin{aligned}
&\left\{(V_k + W_k)(V_k' + W_k') - 1 + \frac{V_k + W_k}{V_{k-1} + W_{k-1}} W_k' \right. \\
&\quad \left. + W_k - \frac{V_k + W_k}{V_{k-1} + W_{k-1}} W_k' W_{k-1} \right\} \Delta f_{2k-2} \\
&= \Delta f_{2k} - \Delta f_{2k-2} - \frac{V_k + W_k}{V_{k-1} + W_{k-1}} W_k' W_{k-1} (\Delta f_{2k-2} - \Delta f_{2k-4}) \\
&\quad + \frac{h^2}{2u_k} f_{2k}^h - \frac{V_k + W_k}{V_{k-1} + W_{k-1}} W_k' \frac{h^2}{2u_{k-1}} f_{2k-2}^h + (V_k + W_k) \frac{h^2}{2u_k'} f_{2k-1}^h,
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\left(V_{k+1}' + W_{k+1}'\right)(V_k + W_k) - 1 + \frac{V_{k+1}' + W_{k+1}'}{V_k' + W_k'} W_k + W_{k+1}' \right. \\
&\quad \left. - \frac{V_{k+1}' + W_{k+1}'}{V_k' + W_k'} W_k W_k' \right\} \Delta f_{2k-1} \\
&= \Delta f_{2k+1} - \Delta f_{2k-1} - \frac{V_{k+1}' + W_{k+1}'}{V_k' + W_k'} W_k W_k' (\Delta f_{2k-1} - \Delta f_{2k-3}) \\
&\quad + \frac{h^2}{2u_{k+1}'} f_{2k+1}^h - \frac{V_{k+1}' + W_{k+1}'}{V_k' + W_k'} W_k \frac{h^2}{2u_k'} f_{2k-1}^h + (V_k' + W_k') \frac{h^2}{2u_k} f_{2k}^h.
\end{aligned}$$

Since we have

$$\begin{aligned}
&(V_k + W_k)(V_k' + W_k') - 1 + \frac{V_k + W_k}{V_{k-1} + W_{k-1}} W_k' + W_k - \frac{V_k + W_k}{V_{k-1} + W_{k-1}} W_k' W_{k-1} \\
&= \left(V + W + \frac{v + w}{k}\right)\left(V' + W' + \frac{v' + w'}{k}\right) \\
&- 1 + W' + \frac{w'}{k} + W + \frac{w}{k} - W'W - \frac{W'w + w'W}{k} + o\left(\frac{1}{k}\right) \\
&= \frac{(1 - W')(v' + w + w') + (1 - W)(v + w + w')}{k} + o\left(\frac{1}{k}\right) \\
&= -(1 - WW') \frac{2\alpha + 1}{k} + o\left(\frac{1}{k}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(V'_{k+1}+W'_{k+1}\right)\left(V_k+W_k\right)-1+\frac{V'_{k+1}+W'_{k+1}}{V'_k+W'_k}W_k+W'_{k+1}\\
&\quad-\frac{V'_{k+1}+W'_{k+1}}{V'_k+W'_k}W_kW'_k\\
&\quad=\left(V'+W'+\frac{v'+w'}{k}\right)\left(V+W+\frac{v+w}{k}\right)\\
&\quad-1+W+\frac{w}{k}+W'+\frac{w'}{k}-WW'-\frac{Ww'+wW'}{k}+o\left(\frac{1}{k}\right)\\
&\quad=-\left(1-WW'\right)\frac{2\alpha+1}{k}+o\left(\frac{1}{k}\right)
\end{aligned}$$

as $k\to\infty$, we can write as follows:

$$\begin{aligned}
&-\frac{2\alpha+1}{kh}\ frac\Delta f^h_{2k-2}-WW'\Delta f^h_{2k-4}hh\\
&=\frac{\Delta f^h_{2k}-\Delta f^h_{2k-2}}{h}-WW'\frac{\Delta f^h_{2k-2}-\Delta f^h_{2k-4}}{h}\\
&-\frac{h}{2}\left(1-W'\right)\left(\frac{1}{u_\infty}+\frac{1}{u'_\infty}\right)f^h_{2k}\\
&+g^h_{2k}
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{2\alpha+1}{kh}\frac{\Delta f^h_{2k-1}-WW'\Delta f^h_{2k-3}}{h}h\\
&=\frac{\Delta f_{2k+1}-\Delta f_{2k-1}}{h}-WW'\frac{\Delta f_{2k-1}-\Delta f_{2k-3}}{h}\\
&-\frac{h}{2}(1-W)\left(\frac{1}{u_\infty}+\frac{1}{u'_\infty}\right)f^h_{2k}\\
&+g'^h_{2k}.
\end{aligned}$$

Here we have

$$\begin{aligned}
g^h_{2k}&=o\left(\frac{1}{k}\right)kh,\\
g'^h_{2k}&=o\left(\frac{1}{k}\right)kh.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&-\left(1-WW'\right)\frac{2\alpha+1}{kh}f'^{h}_{2k-2}h\\
&=\left(1-WW'\right)\left(f'^{h}_{2k}-f'^{h}_{2k-2}\right)\\
&-\frac{h}{2}\left(2-W-W'\right)\left(\frac{1}{u_\infty}+\frac{1}{u'_\infty}\right)f^{h}_{2k}+o\left(\frac{1}{k}\right)kh\\
&=\left(1-WW'\right)\left(f'^{h}_{2k}-f'^{h}_{2k-2}\right)-h\lambda f^{h}_{2k}+o\left(\frac{1}{k}\right)kh,
\end{aligned}$$

or

$$\begin{aligned}
&f'^{h}_{2k}-f'^{h}_{2k-2}\\
&=-\frac{2\alpha+1}{kh}f'^{h}_{2k-2}h+h\lambda f^{h}_{2k}+o\left(\frac{1}{k}\right)kh.
\end{aligned}$$

□

Lemma 4.2. *The families* $\left\{f^h\right\}_{h<h_0}$ *and* $\left\{f'^h\right\}_{h<h_0}$, $z\in[0,Z]$ *are precompact in* $C[0,Z]$.

Proof. Since $\left|f^h_n\right|<M$, the family $\left\{f^h\right\}_{h<h_0}$ is uniformly bounded. For an arbitrary $\delta>0$, take z_1 and z_2 such that $0\le z_1<z_2\le Z$ and $|z_2-z_1|<\delta$. Then we have

$$\begin{aligned}
\left|f^h(z_2)-f^h(z_1)\right|\simeq\left|f^h_{2[z_2/h]}-f^h_{2[z_1/h]}\right|&\le\sum_{k=2[z_1/h]}^{2[z_2/h]-1}\left|f^h_{k+1}-f^h_k\right|\\
&\le Mh\frac{2\delta}{h}=2M\delta.
\end{aligned}$$

Thus, the family $\left\{f^h\right\}_{h<h_0}$ is equicontinuous. Since we have

$$\left|f'^h_n\right|\le\frac{(1+|WW'|)}{h\,|1-WW'|}2Mh=\frac{(1+|WW'|)}{|1-WW'|}2M,$$

the family $\left\{f'^h\right\}_{h<h_0}$ is uniformly bounded. And from Lemma 4.1, if we take z_1 and z_2 such that $0\le z_1<z_2\le Z$ and $|z_2-z_1|<\delta$, we get

$$\begin{aligned}
\left|f'^h(z_2)-f'^h(z_1)\right|\simeq\left|f'^h_{2[z_2/h]}-f'^h_{2[z_1/h]}\right|&\le\left|\sum_{k=\left[\frac{z_1}{h}\right]+1}^{\left[\frac{z_2}{h}\right]}\left(f'^h_{2k}-f'^h_{2k-2}\right)\right|\\
&<\sum_{k=\left[\frac{z_1}{h}\right]+1}^{\left[\frac{z_2}{h}\right]}\left\{\left|\frac{2\alpha+1}{kh}f'^h_{2k-2}h\right|+\left|h\lambda f^h_{2k}\right|+o\left(\frac{1}{k}\right)kh\right\}\\
&<\sum_{k=\left[\frac{z_1}{h}\right]+1}^{\left[\frac{z_2}{h}\right]}\left\{\frac{C_1}{kh}2kh^2+|\lambda Mh|+C_2h\right\}\\
&<\left(2C_1+|\lambda|\,M+C_2\right)\delta
\end{aligned}$$

for some $C_1, C_2 > 0$. Thus $\left\{f'^h\right\}_{h<h_0}$ is equicontinuous too. The lemma is proved. □

Lemma 4.3. *We have*

$$
\begin{aligned}
&(1-WW')\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)\frac{2\alpha+1}{jh}f'^h_{2j-2}\cdot h\\
&\simeq -\left(1-WW'\right)f'^h_{2N-2}\\
&\quad -\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f^h_2\\
&\quad +WW'\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f^h_0
\end{aligned}
$$

as $h \to 0$.

Proof. From the Lemma 4.1 we obtain

$$
\begin{aligned}
&-\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)\frac{2\alpha+1}{jh}f'^h_{2j-2}h\\
&\simeq\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)\frac{f'^h_{2j}-f'^h_{2j-2}}{h}h-\lambda h\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)f^h_{2j},
\end{aligned}
$$

as $h \to 0$. We apply the Abel transformation to the first sum:

$$
\begin{aligned}
&-\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)\frac{\left(f^h_{2j+2}-f^h_{2j}\right)-\left(f^h_{2j}-f^h_{2j-2}\right)}{h^2}h\\
&=\sum_{j=2}^{N}\left[\left\{\cos\left(\sqrt{\lambda}h(N-j+1)\right)-\cos\left(\sqrt{\lambda}h(N-j)\right)\right\}\frac{f^h_{2j}-f^h_{2j-2}}{h}\right]\\
&\quad-\cos\left(\sqrt{\lambda}h\left(N-1\right)\right)\frac{f^h_4-f^h_2}{h}+\frac{f^h_{2N}-f^h_{2N-2}}{h}\\
&=\frac{f^h_{2N}-f^h_{2N-2}}{h}-\cos\left(\sqrt{\lambda}h(N-1)\right)\frac{f^h_4-f^h_2}{h}\\
&\quad-2\sin\frac{\sqrt{\lambda}h}{2}\sum_{j=2}^{N}\sin\left(\sqrt{\lambda}h\left(N-j+\frac{1}{2}\right)\right)\frac{f^h_{2j}-f^h_{2j-2}}{h}.
\end{aligned}
$$

Furthermore we apply the Abel transformation again to the last sum:

$$
\begin{aligned}
&-\sum_{j=2}^{N}\sin\left(\sqrt{\lambda}h\left(N-j+\frac{1}{2}\right)\right)\left(f_{2j}^h-f_{2j-2}^h\right)\\
&\quad=\sum_{j=2}^{N-1}\left\{\sin\left(\sqrt{\lambda}h\left(N-j+\frac{1}{2}\right)\right)-\sin\left(\sqrt{\lambda}h\left(N-j-\frac{1}{2}\right)\right)\right\}f_{2j}^h\\
&\qquad+\sin\frac{\sqrt{\lambda}h}{2}f_{2N}^h-\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f_2^h\\
&\quad=2\sin\frac{\sqrt{\lambda}h}{2}\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)f_{2j}^h\\
&\qquad-\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f_2^h+\sin\frac{\sqrt{\lambda}h}{2}f_{2N}^h.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&-\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)\\
&\qquad\times\frac{\left(f_{2j+2}^h-f_{2j}^h\right)-\left(f_{2j}^h-f_{2j-2}^h\right)}{h^2}h\\
&\quad=\frac{f_{2N}^h-f_{2N-2}^h}{h}-\cos\left(\sqrt{\lambda}h(N-1)\right)\frac{f_4^h-f_2^h}{h}\\
&\qquad-\left(\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\right)^2\lambda\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)f_{2j}^h h\\
&\qquad+\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f_2^h-\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\frac{\sqrt{\lambda}h}{2}f_{2N}^h\\
&\quad\simeq\frac{f_{2N}^h-f_{2N-2}^h}{h}-\left(\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\right)^2\lambda\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)f_{2j}^h h\\
&\qquad+\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f_2^h,
\end{aligned}
$$

as $h\to 0$. Similarly, we get

$$
\begin{aligned}
&-\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)\frac{\left(f_{2j}^h-f_{2j-2}^h\right)-\left(f_{2j-2}^h-f_{2j-4}^h\right)}{h^2}h\\
&\quad\simeq\frac{f_{2N-2}^h-f_{2N-4}^h}{h}+\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f_0^h\\
&\qquad-\left(\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\right)^2\lambda\sum_{j=2}^{N-2}\cos\left(\sqrt{\lambda}h(N-j)\right)f_{2j}^h h,
\end{aligned}
$$

as $h \to 0$. Therefore we obtain

$$
\begin{aligned}
&(1-WW')\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)\frac{2\alpha+1}{jh}f'^{h}_{2j-2}\cdot h \\
&\simeq -(1-WW')\,f'^{h}_{2N-2} \\
&\quad -\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f^{h}_{2} \\
&\quad +WW'\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f^{h}_{0} \\
&\quad +(1-WW')\left(\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\right)^{2}\lambda\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)f^{h}_{2j}h \\
&\quad -(1-WW')\,\lambda h\sum_{j=2}^{N-1}\cos\left(\sqrt{\lambda}h(N-j)\right)f^{h}_{2j}, \\
&\simeq -(1-WW')\,f'^{h}_{2N-2} \\
&\quad -\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f^{h}_{2} \\
&\quad +WW'\frac{\sin\frac{\sqrt{\lambda}h}{2}}{\frac{\sqrt{\lambda}h}{2}}\sqrt{\lambda}\sin\left(\sqrt{\lambda}h\left(N-\frac{3}{2}\right)\right)f^{h}_{0}.
\end{aligned}
$$

as $h \to 0$. We have finished the proof of this lemma. □

Corollary 4.1. *All the limit points of* $\left\{f'^{h}(z)\right\}_{h}$ *as* $h \to 0$ *satisfy the following integral equation*:

$$
\int_{0}^{z}\cos\left(\sqrt{\lambda}(z-\tau)\right)\frac{2\alpha+1}{\tau}u(\tau)d\tau = -u(z)-\sqrt{\lambda}\sin\left(\sqrt{\lambda}z\right). \tag{4.1}
$$

This integral equation has a unique solution. More precisely, the following lemma is known (Lemma 6 in [1]).

Lemma 4.4. *The homogeneous equation*

$$
\gamma\int_{0}^{z}\cos\left(\sqrt{\lambda}(z-\tau)\right)\frac{F(\tau)}{\tau}d\tau = -F(z)
$$

has only the trivial solution in the class

$$
F(\tau)=\tau\Psi(\tau),\quad \Psi(\tau)\in C[0,z], \tag{4.2}
$$

when $\gamma > -1$.

Since all the functions in the family $\left\{f'^h(z)\right\}$ satisfy the condition (4.2), $f'^h(z)$ converges to $u(z)$ uniformly. Furthermore, the solution of (4.1) is satisfied by the derivative of the solution of the following initial value problem

$$\frac{d^2}{dz^2}v(z) + \frac{2\alpha+1}{z}\frac{d}{dz}v(z) + \lambda v(z) = 0$$
$$v(0) = 1, \quad v'(0) = 0.$$

Therefore,

$$\begin{aligned} f'^h(z) &\to f'(z) \\ &= \frac{d}{dz}\left\{\Gamma(\alpha+1)\left(\frac{\sqrt{\lambda}z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\lambda}z\right)\right\}, \quad \text{as } h \to 0. \end{aligned}$$

Since we have

$$f^h_{2N} = \sum_{j=1}^{N-1} f'^h_{2j}h - \frac{WW'\left(f^h_{2N} - f^h_{2N-2}\right) - f^h_2 + WW'f^h_0}{1 - WW'},$$

we obtain

$$f^h(z) \to \int_0^z f'(\tau)d\tau + 1$$

by taking $h \to 0$. Therefore we obtain

$$\begin{aligned} f^h(z) &\to f(z) \\ &= \Gamma(\alpha+1)\left(\frac{\sqrt{\lambda}z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\lambda}z\right), \quad \text{as } h \to 0 \end{aligned}$$

and

$$p_{2n}\left(1 - \frac{z^2}{2n^2}\right) = f^{z/n}_{2n} \to \Gamma(\alpha+1)\left(\frac{\sqrt{\lambda}z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\lambda}z\right) \quad \text{as } n \to \infty.$$

Finally, let us consider p_{2n+1}. From Condition 3 we obtain

$$\left|p_{2n+1}\left(1 - \frac{z^2}{2n^2}\right) - p_{2n}\left(1 - \frac{z^2}{2n^2}\right)\right| \le M'(2n+1)\frac{z^2}{n^2} \to 0, \quad \text{as } n \to \infty.$$

Thus $p_{2n+1}\left(1 - \frac{z^2}{2n^2}\right)$ converges to the same function. Therefore we have finished the proof of the theorem.

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