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Decaying solution of a Navier-Stokes flow without surface tension

By

Yasushi Hataya

1. Introduction

We consider an initial boundary value problem for the motion of a Navier-Stokes flow in a 3D domain with free boundary. Under the periodic boundary condition in horizontal directions, we discuss the global existence of small smooth solution with decaying property in time. The surface tension effect is not taken into account throughout this paper.

The domain which the fluid occupies is bounded below by a rigid flat floor and above by an atmosphere of constant pressure; the upper free surface moves with the change of motion of the fluid. We introduce a spatial coordinate system $\boldsymbol{x}' = (x_1, x_2)$ and $\boldsymbol{x} = (\boldsymbol{x}', x_3)$, and assume that the domain of fluid at time t is described by $\Omega(t) = \{\boldsymbol{x} \in \mathbb{T}^2 \times \mathbb{R} : -1 < x_3 < \eta(t, \boldsymbol{x}')\}$, which is bounded below by a fixed bottom $S_B = \{(\boldsymbol{x}', -1) : \boldsymbol{x}' \in \mathbb{T}^2\}$, and above by a free surface $\Gamma(t) = \{(\boldsymbol{x}', \eta(t, \boldsymbol{x}')) : \boldsymbol{x}' \in \mathbb{T}^2\}$. We denote the velocity of fluid by $\boldsymbol{v} = {}^t(v_1, v_2, v_3)$, and by p a correction to the hydrostatic pressure $\bar{p}$ as $p = \bar{p}(t, \boldsymbol{x}) - P_0 + \rho g x_3$, where P_0 is the atmospheric pressure above the fluid, ρ is a constant density and g is the acceleration of gravity. The equations describing the motion of fluid are given as follows:

$$\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p - \nu \Delta \boldsymbol{v} = \boldsymbol{0} \qquad \text{in } \Omega(t) \tag{1.1}$$

$$\nabla \cdot \boldsymbol{v} = 0 \qquad \text{in } \Omega(t) \tag{1.2}$$

$$\eta_t + \boldsymbol{v} \cdot \nabla' \eta - v_3 = 0 \qquad \text{on } \Gamma(t) \tag{1.3}$$

$$p n_j - \sum_{k=1}^{3} \nu (v_{j,x_k} + v_{k,x_j}) n_k = g \eta\, n_j \quad \text{on } \Gamma(t),\ j = 1, 2, 3 \tag{1.4}$$

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } S_B. \tag{1.5}$$

Here ν denotes a positive viscous constant, and (n_1, n_2, n_3) the outward normal to the free surface. We denote spatial derivatives by $\nabla' = {}^t(\partial/\partial x_1, \partial/\partial x_2, 0)$, $\nabla = {}^t(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, and $\Delta = \nabla \cdot \nabla$. Subscripts after comma denote derivatives, and bold type letters mean vectors. The equilibrium state to this

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system is $(\eta, \boldsymbol{v}, p) = (0, \mathbf{0}, 0)$, and the corresponding domain and free surface are given by $\Omega = \mathbb{T}^2 \times (-1, 0)$ and $\Gamma = \{(\boldsymbol{x}', 0) : \boldsymbol{x}' \in \mathbb{T}^2\}$ respectively.

At $t = 0$, we suppose that η_0 is given, as well as $\boldsymbol{v}_0$ defined in $\Omega(0) = \{(\boldsymbol{x}', x_3) : -1 < x_3 < \eta_0(\boldsymbol{x}'),\ \boldsymbol{x}' \in \mathbb{T}^2\}$, so we set

$$(\eta, \boldsymbol{v}) = (\eta_0, \boldsymbol{v}_0) \quad \text{at } t = 0. \tag{1.6}$$

We should note that if $(\eta, \boldsymbol{v}, p)$ satisfies (1.2) and (1.3), it follows

$$\frac{d}{dt} \int_{\mathbb{T}^2} \eta \, d\boldsymbol{x}' = \int_{\Gamma(t)} (v_3 - \boldsymbol{v} \cdot \nabla' \eta) \, d\boldsymbol{x}' = \int_{\Omega(t)} \nabla \cdot \boldsymbol{v} \, d\boldsymbol{x} = 0, \tag{1.7}$$

so that the solution $(\eta, \boldsymbol{v}, p)$ satisfies the condition $\int_{\mathbb{T}^2} \eta d\boldsymbol{x}' = 0$ for the initial data satisfying $\int_{\mathbb{T}^2} \eta_0 d\boldsymbol{x}' = 0$. This gives one of the compatibility conditions on the initial data, which will be given precisely in Section 5.

Before stating our results, we introduce function spaces. We define $H^s(\Omega)$ (resp. $H^s(\mathbb{T}^2)$) to be the space of functions f which are defined in Ω (resp.$\mathbb{T}^2$), and belong to L^2-Sobolev space of order s. For $I = (0, T)$ or $(0, +\infty)$, we set $K^r(I \times \Omega) = H^{r/2}(I; L^2(\Omega)) \cap L^2(I; H^r(\Omega))$, and also define $K^r(I \times \mathbb{T}^2)$ similarly.

Our main result is stated as follows:

Theorem 1.1. *Let $r \in (5, 11/2)$. Suppose the initial data $(\eta_0, \boldsymbol{v}_0)$ to satisfy*

$$\|\boldsymbol{v}_0\|_{H^{r-1}(\Omega(0))} + \|\eta_0\|_{H^{r-1/2}(\mathbb{T}^2)} \le \delta$$

for suitably small $\delta > 0$ and the compatibility conditions (5.1)–(5.5). *Then, there exist $\eta(t, \boldsymbol{x}')$ defined in $(t, \boldsymbol{x}') \in \mathbb{R}_+ \times \mathbb{T}^2$ and $(\boldsymbol{v}, p)(t, \boldsymbol{x})$ defined in $(t, \boldsymbol{x}) \in \mathbb{R}_+ \times (\mathbb{T}^2 \times \mathbb{R})$ such that η and the restriction of $(\boldsymbol{v}, p)$ on the fluid domain $\{-1 < x_3 < \eta(t, \boldsymbol{x}')\}$ satisfy* (1.1)–(1.6).

Moreover, $(\eta, \boldsymbol{v}, p)$ belongs to a function space X^r, where

$$\begin{aligned} X^r = \Big\{ &(\eta, \boldsymbol{v}, p) : \langle t \rangle^{\gamma_0} \boldsymbol{v} \in K^{r-\gamma_0}(\mathbb{R}_+ \times \Omega),\ \langle t \rangle^{\gamma_0} \nabla p \in K^{r-2-\gamma_0}(\mathbb{R}_+ \times \Omega), \\ &\langle t \rangle^{\gamma_0} \eta \in L^2(\mathbb{R}_+; H^{r-3/2-\gamma_0}(\mathbb{T}^2)) \cap H^{(r-\gamma_0)/2}(\mathbb{R}_+; L^2(\mathbb{T}^2)),\ \gamma_0 \in [0, 1], \\ &\langle t \rangle^{\gamma_1} \eta \in B(\mathbb{R}_+; H^{r-1-\gamma_1}(\mathbb{T}^2)),\ \gamma_1 \in \left[-\frac{1}{2}, 1\right],\ \int_{\mathbb{T}^2} \eta \, d\boldsymbol{x}' = 0,\ \boldsymbol{v} = \mathbf{0} \text{ on } S_B \Big\}. \end{aligned}$$

Here B is a space of bounded continuous functions, and $\langle t \rangle = 1 + t$.

A key result to prove Theorem 1.1 is a linear existence theorem of decaying solution to the following equations:

$$\eta_t + \boldsymbol{w}\cdot\nabla'\eta - u_3 + \fint_\Gamma \eta\nabla'\cdot\boldsymbol{w}\,d\boldsymbol{x}' = f_0 \qquad \text{on } \Gamma \tag{1.8}$$

$$\boldsymbol{u}_t + \nabla q - \nu\Delta\boldsymbol{u} = \boldsymbol{f} \qquad \text{in } \Omega \tag{1.9}$$

$$\nabla\cdot\boldsymbol{u} = \sigma \qquad \text{in } \Omega \tag{1.10}$$

$$-\nu(u_{i,x_3} + u_{3,x_i}) = f_{i+3} \qquad \text{on } \Gamma,\ i=1,2 \tag{1.11}$$

$$-2\nu u_{3,x_3} + q - g\eta = f_6 \qquad \text{on } \Gamma \tag{1.12}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } S_B \tag{1.13}$$

$$(\eta,\boldsymbol{u}) = (\eta_0,\boldsymbol{u}_0) \qquad \text{at } t=0, \tag{1.14}$$

where $\fint_\Gamma f d\boldsymbol{x}' = \int_\Gamma f d\boldsymbol{x}'/|\Gamma|$. The function $(0,\boldsymbol{w},0)$ is given in X^r, and $f_0, \boldsymbol{f} = {}^t(f_1,f_2,f_3)$, σ, f_4, f_5, f_6 are in $Y^r_{\varepsilon_0}$ for $\varepsilon_0 \in (0,1/2)$, where

$$\begin{aligned} Y^r_{\varepsilon_0} = \Big\{ & (f_0,\boldsymbol{f},\sigma,f_4,f_5,f_6) :\ \langle t\rangle^{\gamma_0}\boldsymbol{f} \in K^{r-\gamma_0-2}(\mathbb{R}_+\times\Omega), \\ & \langle t\rangle^{\gamma_0} f_0 \in H^k(\mathbb{R}_+; H^{r-2k-\gamma_0-1/2}(\mathbb{T}^2)) \cap W^{k,1}(\mathbb{R}_+; H^{r-2k-1-\gamma_0}(\mathbb{T}^2)) \\ & \langle t\rangle^{\gamma_0} f_0 \in H^{(r-\gamma_0)/2-1}(\mathbb{R}_+; L^2(\mathbb{T}^2)),\ \langle t\rangle^{\gamma_2+\varepsilon_0}\sigma \in DK^{r-\gamma_2}(\mathbb{R}_+\times\Omega), \\ & \int_{\mathbb{T}^2} f_0 d\boldsymbol{x}' + \int_\Omega \sigma d\boldsymbol{x} = 0,\ \langle t\rangle^{\gamma_2+\varepsilon_0} f_{j+3} \in K^{r-\gamma_2-3/2}(\mathbb{R}_+\times\mathbb{T}^2) \\ & \text{for } j=1,2,3,\ k=0,1,\ \gamma_0\in[0,1], \text{and } \gamma_2 \in [0,3/2] \Big\}. \end{aligned}$$

Here, we have set $DK^r(\mathbb{R}_+\times\Omega) = K^{r-1}(\mathbb{R}_+\times\Omega) \cap H^{r/2}(\mathbb{R}_+; {}_0H^{-1}(\Omega))$, and ${}_0H^{-1}(\Omega)$ the dual space of ${}^0H^1(\Omega) = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma\}$ characterized as the completion of $L^2(\Omega)$ with respect to the norm $\|f\|_{{}_0H^{-1}(\Omega)} = \sup\{(f,u)_{L^2(\Omega)} : \|u\|_{{}^0H^1(\Omega)} \le 1\}$.

Remark 1.2. We will mention the linearized version of the compatibility condition (1.7). If $(\eta,\boldsymbol{u})$ satisfies (1.8) and (1.10), it follows

$$\frac{d}{dt}\int_{\mathbb{T}^2} \eta\,d\boldsymbol{x}' = \int_\Gamma (f_0 + u_3 - \nabla'(\boldsymbol{w}\eta))\,d\boldsymbol{x}' = \int_{\mathbb{T}^2} f_0 d\boldsymbol{x}' + \int_\Omega \sigma\,d\boldsymbol{x}.$$

This relation implies that the space $Y^r_{\varepsilon_0}$ of inhomogeneous data should equip the compatibility condition $\int_{\mathbb{T}^2} f_0 d\boldsymbol{x}' + \int_\Omega \sigma d\boldsymbol{x} = 0$. In Remark 5.5, we describe the reason why we should invoke the correction term $\fint_\Gamma \eta\nabla'\cdot\boldsymbol{w}d\boldsymbol{x}'$ in (1.8).

Now we state our linear existence theorem of decaying solution for the case $(\sigma, f_4, f_5) = (0,0,0)$.

Theorem 1.3. *Let $r \in (5, 11/2)$ and $\varepsilon_0 > 0$. For $(f_0,\boldsymbol{f},0,0,0,f_6) \in Y^r_{\varepsilon_0}$, there exists $\delta > 0$ so that for $(0,\boldsymbol{w},0) \in X^r$ satisfying $\|(0,\boldsymbol{w},0)\|_{X^r} \le \delta$, and for $(\eta_0,\boldsymbol{u}_0) \in H^{r-1/2}(\mathbb{T}^2)\times H^{r-1}(\Omega)$ satisfying the compatibility conditions*

(3.1) *and* (3.2)*, a solution to the linear system* (1.8)–(1.14) *exists in* X^r *and satisfies*

$$\|(\eta, \boldsymbol{u}, q)\|_{X^r} \leq c_\delta(\|(f_0, \boldsymbol{f}, 0, 0, 0, f_6)\|_{Y^r_{\varepsilon_0}} + \|\eta_0\|_{H^{r-1/2}(\mathbb{T}^2)} + \|\boldsymbol{u}_0\|_{H^{r-1}(\Omega)}).$$

Here, the constant c_δ *does not depend on* $\boldsymbol{w}$*.*

Now we shall mention some of related results. In [7] and [8], Greenlee discussed a linearized system in an open container. He studied the stability and the decay property for the solution of a constant coefficient linear system. Sylvester [15] also investigated a linear constant coefficient system, and showed that some eigen values locate on the negative real axis and accumulate at the origin. Beale [2] obtained local existence results for a free surface problem without surface tension, and he [3] also obtained a global solution to a free surface problem with surface tension effect in the Euler coordinates. Beale and Nishida [4] investigated its asymptotic behavior. According to their analysis, if the surface tension effect is taken into account under the periodic boundary condition, no spectra exist in the neighborhood of origin ([4]). However, once this effect is neglected, countable eigen values appear there ([15]). Due to this presence of the slow mode, our solution temporarily decays not in exponential order but in polynomial one. Tanaka and Tani [16] investigated global existence of a solution without surface tension effect. However, it seems to the author that their discussion on the uniform boundedness $\sup_{t\geq 1} \|\eta(t)\|_{H^{r-1/2}(\mathbb{T}^2)}$ in the continuation process is unconvincing.

Our discussion is as follows: we use a mapping Θ depending on unknown function η introduced by Sylvester [14], and convert the problem to one on the equilibrium domain. The Jacobian of this transform requires that η should be as smooth as $\boldsymbol{u}$ in some sense. However, the surface tension effect is not taken into account here, so we obtain the regularities of η by the decaying property of $\boldsymbol{u}$, and this is the key estimation in this paper. To be more precise, both the function $\eta(t)$ and the trace of $\boldsymbol{u}(t)$ on the free surface belong to the same space $H^{r-1/2}(\mathbb{T}^2)$ for a.e. $t > 0$ if we neglect their temporal behavior.

In Sections 3 and 4, we give a solvability of the linear operator with a correction term which is due to the compatibility of solenoidal condition and the normal traces of velocity on the boundaries. Though the transform Θ may not preserve the solenoidal property (1.2), we work in Sections 2, 3, and 4 for vectors of such property for technical reasons. Hence, the linearized operator may violate the compatibility condition as a consequence of (1.8), (1.10), (1.13), and (1.14), if we do not adopt the additional correction term $⨍_\Gamma \eta \cdot \nabla' \boldsymbol{u}\, d\boldsymbol{x}'$. In Section 5, we discuss the full nonlinear problem. There, we give an existence theorem on the equilibrium domain and show that Θ is a C^1-diffeomorphism, and as a result, obtain the proof of Theorem 1.1.

2. A priori estimates

In this section, we prepare several inequalities which will be referred to later. Hereafter, we denote $L^2(\Omega)$ or $L^2(\mathbb{T}^2)$ norm by $\|\cdot\|$.

For $\boldsymbol{u} \in {}_0H^1(\Omega) = \{\boldsymbol{u} \in H^1(\Omega) : \boldsymbol{u}|_{S_B} = \boldsymbol{0}\}$, the Poincaré inequality $\|\boldsymbol{u}\| \leq c\|\nabla \boldsymbol{u}\|$ and the Korn inequality $\|\nabla \boldsymbol{u}\| \leq c\|\mathbb{S}(\boldsymbol{u})\|$ for $\mathbb{S}(\boldsymbol{u}) = \nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u}$ are known, so that

$$c_0\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \|\mathbb{S}(\boldsymbol{u})\|_{L^2(\Omega)}$$

holds for $\boldsymbol{u} \in {}_0H^1(\Omega)$. The proofs for these inequalities can be found in [2] and [6]. The Kato-Ponce type estimates are also well-known: for c a constant independent of f and g,

$$\|[\Lambda^s, f]g\| \leq c(\|\nabla' f\|_{L^\infty} \cdot \|\Lambda^{s-1}g\| + \|\Lambda^s f\| \cdot \|g\|_{L^\infty}) \tag{2.1}$$

$$\|fg\|_{H^s} \leq c(\|f\|_{W^{s,p_1}} \cdot \|g\|_{L^{q_1}} + \|f\|_{L^{q_2}} \cdot \|g\|_{W^{s,p_2}}), \tag{2.2}$$

where the constants are chosen to be $1/p_1+1/q_1 = 1/p_2+1/q_2 = 1/2$, $p_1,\ p_2 \neq \infty$ and $s \geq 0$. Here, $[f, g]$ denotes the commutator $f\,g - g\,f$, and Λ denotes $(1 - \nabla' \cdot \nabla')^{1/2}$. The proofs for these inequalities are given in [10], [18], and [19]. By slightly modifying the proof of [3, Lemma 5.1], we can also prove the inequality:

$$\begin{aligned} \|\langle t\rangle^\gamma fg\|_{H^\alpha(\mathbb{R}_+;L^2(\mathbb{T}^2))} \leq c(&\|f\|_{H^\alpha(\mathbb{R}_+;L^2(\mathbb{T}^2))}\|\langle t\rangle^\gamma g\|_{K^s}+ \\ &+ \|\langle t\rangle^\gamma f\|_{K^s}\|g\|_{H^\alpha(\mathbb{R}_+;L^2(\mathbb{T}^2))}) \end{aligned} \tag{2.3}$$

for $\alpha, \gamma \geq 0$, and $s > 2$.

We next introduce an estimate for the steady Stokes equations.

Lemma 2.1. *Let $\alpha \geq 2$, $\boldsymbol{f} \in H^{\alpha-2}(\Omega), f_{j+3} \in H^{\alpha-3/2}(\mathbb{T}^2)$ $(j = 1, 2)$ and $f_7 \in H^{\alpha-1/2}(\mathbb{T}^2)$. Suppose that $(\boldsymbol{u}, q) \in H^2(\Omega) \times H^1(\Omega)$ satisfies the following Stokes system*:

$$-\nu\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{in } \Omega \tag{2.4}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \tag{2.5}$$

$$-\nu(u_{i,x_3} + u_{3,x_i}) = f_{i+3} \quad \text{on } \Gamma,\ i = 1, 2 \tag{2.6}$$

$$u_3 = f_7 \quad \text{on } \Gamma \tag{2.7}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } S_B. \tag{2.8}$$

Then, it holds

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{H^\alpha(\Omega)} + \|q\|^2_{H^{\alpha-1}(\Omega)} \leq \\ &\leq c_1\left(\|\boldsymbol{f}\|^2_{H^{\alpha-2}(\Omega)} + \sum_{j=1,2}\|f_{j+3}\|^2_{H^{\alpha-3/2}(\mathbb{T}^2)} + \|f_7\|^2_{H^{\alpha-1/2}(\mathbb{T}^2)}\right). \end{aligned} \tag{2.9}$$

We omit the proof of this lemma since the Stokes equations (2.4), (2.5) with boundary conditions (2.6)–(2.8) are well-known elliptic boundary problem ([1]). In what follows, we assume that the constants $r \in (5, 11/2)$ and $\varepsilon_1 \in (0, 1/2)$ are fixed, and that a function $(0, \boldsymbol{w}, 0) \in X^r$ with $\|(0, \boldsymbol{w}, 0)\|_{X^r} \leq \delta$ and non-negative constants $\alpha, \beta, \gamma, \varepsilon_0$ are to be given. The constants $c_j(1 \leq j \leq 8)$

do not depend on $\boldsymbol{w}$ nor δ. We shall discuss several a priori estimates with temporal weights for the solution of linear system (1.8)–(1.13) associated with, instead of (1.14), the homogeneous initial data:

$$(\eta_0, \boldsymbol{u}_0) = (0, \boldsymbol{0}). \tag{2.10}$$

The associated function spaces of inhomogeneous data are given by

$$X_0^r = \{\boldsymbol{z} \in X^r : \boldsymbol{z} \text{ can be smoothly extended by } \boldsymbol{0} \text{ for } t < 0\}$$
$$Y_{\varepsilon_0,0}^r = \{\boldsymbol{F} \in Y_{\varepsilon_0}^r : \boldsymbol{F} \text{ can be smoothly extended by } \boldsymbol{0} \text{ for } t < 0\}.$$

For simplicity, we shall denote $\|(\eta, \boldsymbol{u})(t)\|_{H^k}^2 = \|\sqrt{g}\eta(t)\|_{H^k(\mathbb{T}^2)}^2 + \|\boldsymbol{u}(t)\|_{H^k(\Omega)}^2$ and $F_0 = \|(f_0, \boldsymbol{f}, 0, 0, 0, f_6)\|_{Y_{\varepsilon_0,0}^r}^2$. The following lemma is a modification of a standard energy estimate.

Lemma 2.2. *Let $0 \le \gamma \le 1$ and $(\sigma, f_4, f_5) = (0, 0, 0)$.*
(i) *Let $2 \le \alpha \le r - \gamma$. We assume that F_1 defined as*

$$F_1 = F_0 + \delta \|\eta\|_{L^2(\mathbb{R}_+; H^{2+\varepsilon_1}(\mathbb{T}^2))}^2$$

is finite. Then the solution of the linear system (1.8)–(1.13) *with* (2.10) *satisfies*

$$\begin{aligned} \|\langle t\rangle^\gamma \Lambda^{\alpha-1}(\eta, \boldsymbol{u})\|_{L^\infty(\mathbb{R}_+; L^2)}^2 + \nu \|\langle t\rangle^\gamma \Lambda^{\alpha-1}\mathbb{S}(\boldsymbol{u})\|_{L^2(\mathbb{R}_+; L^2(\Omega))}^2 \le \\ \le 4\gamma \|\langle t\rangle^{\gamma-1/2}\Lambda^{\alpha-1}(\eta, \boldsymbol{u})\|_{L^2(\mathbb{R}_+; L^2)}^2 + c_2 F_1. \end{aligned} \tag{2.11}$$

(ii) *Let $4 \le \alpha \le r - \gamma + 1/2$. We assume that $F_2 = F_2(\alpha, \gamma)$ defined as*

$$\begin{aligned} F_2 = F_0 + \delta(\|\eta\|_{H^1(\mathbb{R}_+; H^{2+\varepsilon_1}(\mathbb{T}^2))}^2 + \|\eta\|_{L^\infty(\mathbb{R}_+; H^{2+\varepsilon_1}(\mathbb{T}^2))}^2 + \\ + \|\langle t\rangle^\gamma \eta\|_{L^\infty(\mathbb{R}_+; H^{\alpha-2}(\mathbb{T}^2))}^2) \end{aligned}$$

is finite. Then the solution of the linear system (1.8)–(1.13) *with* (2.10) *satisfies*

$$\begin{aligned} \|\langle t\rangle^\gamma \Lambda^{\alpha-3}(\eta_t, \boldsymbol{u}_t)\|_{L^\infty(\mathbb{R}_+; L^2)}^2 + \nu \|\langle t\rangle^\gamma \Lambda^{\alpha-3}\mathbb{S}(\boldsymbol{u}_t)\|_{L^2(\mathbb{R}_+; L^2(\Omega))}^2 \le \\ \le 4\gamma \|\langle t\rangle^{\gamma-1/2}\Lambda^{\alpha-3}(\eta_t, \boldsymbol{u}_t)\|_{L^2(\mathbb{R}_+; L^2)}^2 + c_3 F_2(\alpha, \gamma). \end{aligned} \tag{2.12}$$

(iii) *Let $\beta \ge 1$. We assume that $F_3 = F_3(\beta, \gamma)$ defined as*

$$\begin{aligned} F_3 = \|\langle t\rangle^\gamma \boldsymbol{f}\|_{H^{\beta-1}(\mathbb{R}_+; L^2(\Omega))} \|\langle t\rangle^\gamma \boldsymbol{u}\|_{H^\beta(\mathbb{R}_+; L^2(\Omega))} + \\ + \|\langle t\rangle^\gamma (f_6 + g\eta)\|_{H^{\beta-1/2}(\mathbb{R}_+; L^2(\mathbb{T}^2))}^2 \end{aligned}$$

is finite. Then the solution of the linear system (1.8)–(1.13) *with* (2.10) *satisfies*

$$c_0 \frac{\nu}{4} \|\langle t\rangle^\gamma \boldsymbol{u}\|_{H^{\beta-1/2}(\mathbb{R}_+; H^1(\Omega))}^2 \le \gamma \|\langle t\rangle^{\gamma-1/2}\boldsymbol{u}\|_{H^{\beta-1/2}(\mathbb{R}_+; L^2(\Omega))}^2 + c_4 F_3(\beta, \gamma).$$

Proof. (i) Multiplying (1.9) by $\langle t\rangle^{2\gamma}\Lambda^{2(\alpha-1)}\boldsymbol{u}$ and integrating over Ω yield

$$(2.13)\quad \frac{1}{2}\frac{d}{dt}\langle t\rangle^{2\gamma}\|\Lambda^{\alpha-1}(\eta,\boldsymbol{u})\|^2+\frac{\nu}{2}\langle t\rangle^{2\gamma}\|\Lambda^{\alpha-1}\mathbb{S}(\boldsymbol{u})\|^2=\sum_{j=1}^{4}\langle t\rangle^{2\gamma}\Phi_j+\langle t\rangle^{2\gamma-1}\Phi_5,$$

where

$$\Phi_1=-g\sum_{i=1,2}\left(\Lambda^{\alpha-1}\eta,\Lambda^{\alpha-1}(w_i\eta_{x_i})+\fint_\Gamma w_{i,x_i}\eta d\boldsymbol{x}'\right)_{L^2(\mathbb{T}^2)}$$
$$\Phi_2=-(\Lambda^{\alpha-1}f_6,\Lambda^{\alpha-1}u_3|_\Gamma)_{L^2(\mathbb{T}^2)},\qquad \Phi_3=(\Lambda^{\alpha-1}\boldsymbol{f},\Lambda^{\alpha-1}\boldsymbol{u})_{L^2(\Omega)}$$
$$\Phi_4=g(\Lambda^{\alpha-1}\eta,\Lambda^{\alpha-1}f_0)_{L^2(\mathbb{T}^2)},\qquad \Phi_5=\gamma\|\Lambda^{\alpha-1}(\eta,\boldsymbol{u})\|^2_{L^2}.$$

Here and what follows, we denote the inner product in $L^2(\Omega)$ (resp. $L^2(\mathbb{T}^2)$) by $(\cdot,\cdot)_{L^2(\Omega)}$ (resp. $(\cdot,\cdot)_{L^2(\mathbb{T}^2)}$), and drop $L^2(\Omega)$ (resp. $L^2(\mathbb{T}^2)$) if it is not ambiguous. Using (2.1), we obtain

$$\begin{aligned}|\Phi_1|&\le c(\|\nabla'\boldsymbol{w}|_\Gamma\|_{L^\infty}\|\Lambda^{\alpha-1}\eta\|^2+\|\Lambda^{\alpha-1}\eta\|\|[\Lambda^{\alpha-1},w_i]\eta_{x_i}\|)\\&\le c(\|\boldsymbol{w}|_\Gamma\|_{H^{2+\varepsilon_1}}\|\eta\|_{H^{\alpha-1}}+\|\boldsymbol{w}|_\Gamma\|_{H^{\alpha-1}}\|\eta\|_{H^{2+\varepsilon_1}})\|\eta\|_{H^{\alpha-1}},\end{aligned}$$

and hence

$$\begin{aligned}\int_{\mathbb{R}_+}\langle t\rangle^{2\gamma}|\Phi_1|dt\le c(&\|\langle t\rangle^\gamma\eta\|^2_{L^\infty(\mathbb{R}_+;H^{\alpha-1})}\|\boldsymbol{w}|_\Gamma\|_{L^1(\mathbb{R}_+;H^{2+\varepsilon_1})}+\\&+\|\langle t\rangle^\gamma\eta\|_{L^\infty(\mathbb{R}_+;H^{\alpha-1})}\|\langle t\rangle^\gamma\boldsymbol{w}|_\Gamma\|_{L^2(\mathbb{R}_+;H^{\alpha-1})}\|\eta\|_{L^2(\mathbb{R}_+;H^{2+\varepsilon_1})}).\end{aligned}$$

If we note

$$(2.14)\qquad \|\boldsymbol{w}|_\Gamma\|_{L^1(\mathbb{R}_+;H^{2+\varepsilon_1}(\mathbb{T}^2))}\le c\|\langle t\rangle^{1/2+\varepsilon_0}\boldsymbol{w}\|_{L^2(\mathbb{R}_+;H^{5/2+\varepsilon_1}(\Omega))}\le c_5\delta,$$

and integrate (2.13) in t, we obtain (2.11).

(ii) We differentiate (1.9) in t and take inner product with $\langle t\rangle^{2\gamma}\Lambda^{2(\alpha-3)}\boldsymbol{u}_t$, so that we have for $\alpha>4$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\langle t\rangle^\gamma\Lambda^{\alpha-3}(\eta_t,\boldsymbol{u}_t)\|^2+\frac{\nu}{4}\|\langle t\rangle^\gamma\Lambda^{\alpha-3}\mathbb{S}(\boldsymbol{u}_t)\|^2\le c\langle t\rangle^{2\gamma}\{&\|\Lambda^{\alpha-4}\boldsymbol{f}_t\|^2+\\&+\|f_{6,t}\|^2_{H^{\alpha-7/2}}+\|\eta_t\|_{H^{\alpha-3}}\|f_{0,t}\|_{H^{\alpha-3}}+\Phi_6\},\end{aligned}$$

where $\Phi_6=g|(\Lambda^{\alpha-3}\eta_t,\Lambda^{\alpha-3}\partial_t(\boldsymbol{w}\cdot\nabla'\eta)+\partial_t\fint_\Gamma\eta\nabla'\cdot\boldsymbol{w}\,d\boldsymbol{x}')|$. By (2.14), we have

$$\begin{aligned}\int_{\mathbb{R}_+}\langle t\rangle^{2\gamma}\Phi_6dt\le c\delta\{&\|\langle t\rangle^\gamma\Lambda^{\alpha-3}\eta_t\|^2_{L^\infty(\mathbb{R}_+;L^2(\mathbb{T}^2))}+\\&+\|\eta\|^2_{H^1(\mathbb{R}_+;H^{2+\varepsilon_1}(\mathbb{T}^2))}+\|\langle t\rangle^\gamma\eta\|^2_{L^\infty(\mathbb{R}_+;H^{\alpha-2}(\mathbb{T}^2))}\},\end{aligned}$$

which gives (2.12).

(iii) Hereafter, we shall denote the Laplace transform of f by $\hat{f}(\lambda)=\int_0^\infty e^{\lambda t}f(t)dt$. Since we can extend $(\eta,\boldsymbol{u},p)$ smoothly by $\mathbf{0}$ for $t<0$, we apply the Laplace transform to (1.9), and take inner product with $\hat{\boldsymbol{u}}(\lambda)$ in $L^2(\Omega)$, so that we obtain for $\mathrm{Re}\lambda \geq 0$,

$$\mathrm{Re}\lambda\,\|\hat{\boldsymbol{u}}\|^2 + \frac{\nu}{2}\|\mathbb{S}(\hat{\boldsymbol{u}})\|^2 \leq \|\hat{f}\|\|\hat{\boldsymbol{u}}\| + c\|\hat{\boldsymbol{u}}\|_{H^1(\Omega)}\|g\hat{\eta} + \widehat{f_6}\|.$$

If we multiply $(1+|\lambda|)^{2\beta-1}$ to the both hand sides, integrate on the line $\mathrm{Re}\ \lambda = \lambda_0$ in the complex plane and let $\lambda_0 \to +0$, then we obtain the desired estimate. The weighted version of this estimate can be similarly obtained. □

Before proceeding the next lemma, we introduce an extension of a function η to $\mathbb{R}_+ \times \Omega$.

Proposition 2.3. *Suppose that* $q \in H^{\alpha_0-1}(\Omega)$ *and* $b \in H^{\alpha_0-1/2}(\mathbb{T}^2)$ *satisfy* $\int_\Omega q\,d\boldsymbol{x} = \int_{\mathbb{T}^2} b\,d\boldsymbol{x}'$ *for some* $\alpha_0 \geq 1$. *Then there exists at least one function* $\boldsymbol{a} = {}^t(a_1, a_2, a_3) \in H^{\alpha_0}(\Omega)$ *which satisfies*

$$\nabla\cdot\boldsymbol{a} = q \quad \text{in } \Omega$$
$$\boldsymbol{a} = \boldsymbol{0} \quad \text{on } S_B, \quad a_3 = b \quad \text{on } \Gamma$$
$$\|\boldsymbol{a}\|_{H^{\alpha_0}(\Omega)} \leq c(\|q\|_{H^{\alpha_0-1}(\Omega)} + \|b\|_{H^{\alpha_0-1/2}(\mathbb{T}^2)}).$$

For a proof of this proposition, see [11, Sec.1.2.1] or [9, Lemma 1]. Now we state an estimate of $\|\langle t\rangle^\gamma\eta\|_{L^2(\mathbb{R}_+;H^{\alpha-3/2}(\mathbb{T}^2))}$ following [9].

Lemma 2.4. *Let* $(\sigma, f_4, f_5) = (0, 0, 0)$. *Suppose that* F_4 *defined for* $\alpha \geq 2$ *as*

$$F_4(\alpha,\gamma) = \|\langle t\rangle^\gamma\Lambda^{\alpha-2}\boldsymbol{u}_t\|^2_{L^2(\mathbb{R}_+;L^2(\Omega))} + \|\langle t\rangle^\gamma\Lambda^{\alpha-1}\boldsymbol{u}\|^2_{L^2(\mathbb{R}_+;H^1(\Omega))} + F_0$$

is finite. Then the solution of (1.8)–(1.13) *with* (2.10) *satisfies*

$$g\|\langle t\rangle^\gamma\eta\|^2_{L^2(\mathbb{R}_+;H^{\alpha-3/2}(\mathbb{T}^2))} \leq c_6F_4. \tag{2.15}$$

Proof. As in Remark 1.2, we obtain $\int_{\mathbb{T}^2}\eta(t,\boldsymbol{x}')\,d\boldsymbol{x}' = 0$ by virtue of (1.8), (1.10), (2.10) and $\int_{\mathbb{T}^2} f_0\,d\boldsymbol{x}' = 0$. According to Proposition 2.3, we can choose a function $\boldsymbol{a}$ satisfying $\nabla\cdot\boldsymbol{a} = 0$ in Ω, $\boldsymbol{a}\cdot\boldsymbol{n} = \eta$ on Γ, $\boldsymbol{a} = \boldsymbol{0}$ on S_B and $\|\Lambda^{\alpha-2}\boldsymbol{a}\|_{H^1(\Omega)} \leq c\|\eta\|_{H^{\alpha-3/2}(\mathbb{T}^2)}$. Applying $\Lambda^{\alpha-3/2}$ to (1.9) and taking inner product with $\Lambda^{\alpha-3/2}\boldsymbol{a}$, we have

$$\frac{\nu}{2}(\Lambda^{\alpha-1}\mathbb{S}(\boldsymbol{u}), \Lambda^{\alpha-2}\mathbb{S}(\boldsymbol{a}))_{L^2(\Omega)} + (\Lambda^{\alpha-2}(\boldsymbol{u}_t - \boldsymbol{f}), \Lambda^{\alpha-1}\boldsymbol{a})_{L^2(\Omega)} +$$
$$+\, g\|\Lambda^{\alpha-3/2}\eta\|^2 + (\Lambda^{\alpha-3/2}f_6, \Lambda^{\alpha-3/2}\eta)_{L^2(\mathbb{T}^2)} = 0,$$

so that

$$g\|\Lambda^{\alpha-3/2}\eta\|^2 \leq c\{(\|\Lambda^{\alpha-2}(\boldsymbol{u}_t - \boldsymbol{f})\| + \|\Lambda^{\alpha-1}\mathbb{S}(\boldsymbol{u})\|)\|\Lambda^{\alpha-2}\boldsymbol{a}\|_{H^1(\Omega)} +$$
$$+\, \|\Lambda^{\alpha-3/2}f_6\|\|\Lambda^{\alpha-3/2}\eta\|\}.$$

Integrating this inequality in t after multiplying $\langle t\rangle^{2\gamma}$ gives (2.15). □

We next prepare an estimate for time derivatives of η.

Lemma 2.5. *Let $k = 1$ or 2. Suppose that $F_5 = F_5(\alpha, \gamma, k)$ defined for $1/2 \le \alpha \le r - 3/2$ as*

$$F_5 = \|\langle t\rangle^\gamma \eta\|^2_{L^\infty(\mathbb{R}_+;H^{\alpha+k}(\mathbb{T}^2))} + \\ + \sum_{j=0}^{k-1} (\|\langle t\rangle^\gamma \Lambda^{\alpha+k-j-3/2}\partial_t^j \boldsymbol{u}\|^2_{L^2(\mathbb{R}_+;H^1(\Omega))} + \|\langle t\rangle^\gamma \partial_t^j f_0\|^2_{L^2(\mathbb{R}_+;H^{\alpha+k-j-1}(\mathbb{T}^2))})$$

is finite. Then the solution of (1.8) with (2.10) satisfies

$$\|\langle t\rangle^\gamma \partial_t^k \eta\|^2_{L^2(\mathbb{R}_+;H^\alpha(\mathbb{T}^2))} \le c_7(1+\delta^2)F_5. \tag{2.16}$$

Proof. By (1.8), it is easy to check (2.16) for $k = 1$. To see the case of $k = 2$, the t-derivative of (1.8) leads

$$\|\langle t\rangle^\gamma \eta_{tt}\|_{L^2(\mathbb{R}_+;H^\alpha(\mathbb{T}^2))} \le c\delta\{\|\langle t\rangle^\gamma \eta\|_{L^2(\mathbb{R}_+;H^{\alpha+1}(\mathbb{T}^2))} + \|\langle t\rangle^\gamma \eta_t\|_{L^2(\mathbb{R}_+;H^{\alpha+1}(\mathbb{T}^2))}\} + \\ + c\|\langle t\rangle^\gamma \Lambda^{\alpha-1/2}\boldsymbol{u}_t\|_{L^2(\mathbb{R}_+;H^1(\Omega))} + \|\langle t\rangle^\gamma f_{0,t}\|_{L^2(\mathbb{R}_+;H^\alpha(\mathbb{T}^2))}. \tag{2.17}$$

Applying (2.16) for $k = 1$ to the second term in (2.17), we have the desired result. □

We next give an estimate for time derivatives of $\boldsymbol{u}$.

Lemma 2.6. *Let $\alpha \ge 2$ and $(\sigma, f_4, f_5) = (0, 0, 0)$. Suppose that $F_6 = F_6(\alpha, \beta, \gamma)$ defined as*

$$F_6 = F_0 + g\|\langle t\rangle^\gamma \eta\|^2_{H^\beta(\mathbb{R}_+;H^{\alpha-3/2}(\mathbb{T}^2))}$$

is finite. Then the solution of (1.8)–(1.13) with (2.10) satisfies

$$\|\langle t\rangle^\gamma \Lambda^{\alpha-2}\boldsymbol{u}\|^2_{H^{\beta+1}(\mathbb{R}_+;L^2(\Omega))} \le c_8(\gamma\|\langle t\rangle^{\gamma-1/2}\Lambda^{\alpha-2}\boldsymbol{u}\|^2_{H^\beta(\mathbb{R}_+;H^1(\Omega))} + F_6). \tag{2.18}$$

Proof. Since $f_0, \boldsymbol{f}, f_6$ can be smoothly extended by 0 for $t < 0$, the solution $(\eta, \boldsymbol{u}, p)$ can be also extended by $\mathbf{0}$ for $t < 0$. If we apply $\langle t\rangle^\gamma \Lambda^{\alpha-2}$ and the Laplace transform in t to (1.9), and take inner product with $\Lambda^{\alpha-2}(\langle t\rangle^\gamma \boldsymbol{u}_t)^\wedge$, then we obtain

$$\|\Lambda^{\alpha-2}\widehat{\langle t\rangle^\gamma \boldsymbol{u}_t}\|^2 + \frac{\nu}{2}\mathrm{Re}\lambda\|\Lambda^{\alpha-2}\widehat{\langle t\rangle^\gamma \mathbb{S}(\boldsymbol{u})}\|^2 \le |(\Lambda^{\alpha-2}\widehat{\langle t\rangle^\gamma \boldsymbol{f}}, \Lambda^{\alpha-2}\widehat{\langle t\rangle^\gamma \boldsymbol{u}_t})| + \\ + |(\Lambda^{\alpha-2}(\langle t\rangle^\gamma(g\eta + f_6))^\wedge, \Lambda^{\alpha-2}\widehat{\langle t\rangle^\gamma u_{3,t}})| + \frac{\nu\gamma}{2}\|\Lambda^{\alpha-2}(\langle t\rangle^{\gamma-1/2}\mathbb{S}(\boldsymbol{u}))^\wedge\|^2.$$

If we multiply $(1 + |\lambda|)^{2\beta}$ to the both hands sides and integrate them on the line $\mathrm{Re}\lambda = \lambda_0 > 0$ in the complex plane, we have the desired inequality (2.18) by letting $\lambda_0 \to 0$. □

We prepare the following proposition, which will be used in the next lemma.

Proposition 2.7. *Let $T > 0$. Assume that $a(t), b(t), c(t) \geq 0$, $a \in B[0,T)$, $b \in L^1(0,T)$, $c \in L^2(0,T)$ and $da(t)/dt \leq b(t) \cdot a(t) + c(t)$. Then it holds for $t \in (0,T)$,*

$$a(t) \leq e^{\int_0^t b(s)ds}(a(0) + \sqrt{t}\|c\|_{L^2(0,t)}).$$

Proof. By Gronwall's lemma, we have

$$a(t) \leq e^{\int_0^t b(s)ds}a(0) + \int_0^t e^{\int_s^t b(\tau)d\tau}c(s)ds.$$

The Schwarz' inequality implies that the second term in the right hand side is bounded by $e^{\int_0^t b(s)ds}\sqrt{t}\|c\|_{L^2(0,T)}$. □

We are in a position to give a key estimate of $\|\langle t\rangle^{-1/2}\eta\|_{L^2(\mathbb{R}_+;H^{\alpha-1/2}(\mathbb{T}^2))}$.

Lemma 2.8. *Let $\alpha \geq 1$ and $(\sigma, f_4, f_5) = (0,0,0)$. Suppose $\Lambda^{\alpha-1}\boldsymbol{u} \in L^2(\mathbb{R}_+; H^1(\Omega))$ and that*

$$F_7 = \|f_0\|^2_{L^2(\mathbb{R}_+;H^{\alpha-1/2}(\mathbb{T}^2))} + \delta\|\eta\|^2_{L^\infty(\mathbb{R}_+;H^{2+\varepsilon}(\mathbb{T}^2))}$$

is finite. Then the solution to (1.8)–(1.13) *with* (2.10) *satisfies*

$$\|\langle t\rangle^{-1/2}\eta\|^2_{L^\infty(\mathbb{R}_+;H^{\alpha-1/2}(\mathbb{T}^2))} \leq c_9(\|\Lambda^{\alpha-1}\boldsymbol{u}\|^2_{L^2(\mathbb{R}_+;H^1(\Omega))} + F_7). \tag{2.19}$$

Proof. If we take inner product of (1.8) and $\Lambda^{2\alpha-1}\eta$, we have

$$\frac{d}{dt}\|\Lambda^{\alpha-1/2}\eta\| \leq c\biggl(\|\nabla'\boldsymbol{w}|_\Gamma\|_{L^\infty}\|\Lambda^{\alpha-1/2}\eta\| + \\ + \|\Lambda^{\alpha-1/2}\boldsymbol{w}|_\Gamma\|\|\nabla'\eta\|_{L^\infty} + \biggl\|\Lambda^{\alpha-1/2}\biggl(u_3 + f_0 - \fint_\Gamma w_{i,x_i}\eta d\boldsymbol{x}'\biggr)\biggr\|\biggr).$$

Proposition 2.7 implies

$$\|\Lambda^{\alpha-1/2}\eta(t)\| \leq ce^{c\int_0^t \|\nabla'\boldsymbol{w}(s)|_\Gamma\|_{L^\infty(\mathbb{T}^2)}ds}\sqrt{t}\ (\|\Lambda^{\alpha-1}\boldsymbol{u}\|_{L^2(\mathbb{R}_+;H^1(\Omega))} + F_7)^{1/2}.$$

Hence, (2.14) gives (2.19). □

3. Global behavior of a solution of the linear system

In this section, we give an existence result for a solution of the linear system (1.8)–(1.14). As in the previous section, we assume the constant $r \in (5, 11/2)$, the function $(0, \boldsymbol{w}, 0) \in X^r$ and the inhomogeneous data $(f_0, \boldsymbol{f}, 0, 0, 0, f_6) \in Y^r_{\varepsilon_0}$ are appropriately given later. Following [3], we first discuss the compatibility conditions on the initial data $\eta_0 \in H^{r-1/2}(\mathbb{T}^2)$ and $\boldsymbol{u}_0 = {}^t(u_{01}, u_{02}, u_{03}) \in H^{r-1}(\Omega)$. We require $(\eta_0, \boldsymbol{u}_0)$ to satisfy the conditions:

$$\int_{\mathbb{T}^2} \eta_0 d\boldsymbol{x}' = \int_{\mathbb{T}^2} \eta_0^{(1)} d\boldsymbol{x}' = \int_{\mathbb{T}^2} \eta_0^{(2)} d\boldsymbol{x}' = 0 \tag{3.1}$$

$$\boldsymbol{u}_0 \text{ and } \boldsymbol{u}_0^{(1)} \text{ satisfy (1.10), (1.11), and (1.13),} \tag{3.2}$$

where $\eta_0^{(1)}, \eta_0^{(2)}$, and $\boldsymbol{u}_0^{(1)}$ are traces on $t = 0$ of η_t, η_{tt}, and $\boldsymbol{u}_t$ respectively, which will be determined here. Using (1.8), we set $\eta_0^{(1)} \in H^{r-3/2}(\mathbb{T}^2)$ as

$$\eta_0^{(1)} = -\boldsymbol{w} \cdot \nabla' \eta_0 + u_{03} - ⨍_\Gamma \eta_0 \nabla' \cdot \boldsymbol{w}\, d\boldsymbol{x}' + f_0 \quad \text{at } t = 0. \tag{3.3}$$

Before determining $\boldsymbol{u}_0^{(1)}$, we shall choose an initial pressure $q_0 \in H^{r-2}(\Omega)$ as

$$\begin{aligned} \Delta q_0 &= \nabla \cdot \boldsymbol{f} \quad \text{in } \Omega \\ q_{0,x_3} &= \nu \Delta u_{03} + f_3 \quad \text{on } S_B \\ q_0 &= 2\nu u_{03,x_3} + g\eta_0 + f_6 \quad \text{on } \Gamma \end{aligned} \tag{3.4}$$

at $t = 0$. Using (1.9), we set $\boldsymbol{u}_0^{(1)} \in H^{r-3}(\Omega)$ as

$$\boldsymbol{u}_0^{(1)} = \nu \Delta \boldsymbol{u}_0 - \nabla q_0 + \boldsymbol{f} \text{ at } t = 0, \tag{3.5}$$

and $\eta_0^{(2)} \in H^{r-7/2}(\mathbb{T}^2)$ as

$$\begin{aligned} \eta_0^{(2)} = &-(\boldsymbol{w}_t \cdot \nabla' \eta_0 + \boldsymbol{w} \cdot \nabla' \eta_0^{(1)}) + u_{03}^{(1)} - \\ &- ⨍_\Gamma (\eta_0 \nabla' \cdot \boldsymbol{w}_t + \eta_0^{(1)} \nabla' \cdot \boldsymbol{w})\, d\boldsymbol{x}' + f_{0,t} \quad \text{at } t = 0. \end{aligned} \tag{3.6}$$

We do not need further compatibility conditions, since we work for $r < 11/2$. However, we shall determine the traces of $\boldsymbol{u}_{tt}$ and ∇q_t at $t = 0$, since they are also continuous in $L^2(\Omega)$ and are needed when we construct a smooth extension of initial data. We determine $q_0^{(1)} \in H^{r-4}(\Omega)$ as

$$\begin{aligned} \Delta q_0^{(1)} &= \nabla \cdot \boldsymbol{f}_t \quad \text{in } \Omega \\ q_{0,x_3}^{(1)} &= \nu \Delta u_{03}^{(1)} + f_{3,t} \quad \text{on } S_B \\ q_0^{(1)} &= 2\nu u_{03,x_3}^{(1)} + g\eta_0^{(1)} + f_{6,t} \quad \text{on } \Gamma, \end{aligned} \tag{3.7}$$

at $t = 0$, and $\boldsymbol{u}_0^{(2)} \in H^{r-5}(\Omega)$ as

$$\boldsymbol{u}_0^{(2)} = \nu \Delta \boldsymbol{u}_0^{(1)} - \nabla q_0^{(1)} + \boldsymbol{f}_t \quad \text{at } t = 0. \tag{3.8}$$

As a result, we obtain

$$\begin{aligned} &\|\boldsymbol{u}_0^{(1)}, \nabla q_0\|_{H^{r-3}(\Omega)} + \|\boldsymbol{u}_0^{(2)}, \nabla q_0^{(1)}\|_{H^{r-5}(\Omega)} + \|\eta_0^{(1)}\|_{H^{r-3/2}(\mathbb{T}^2)} + \\ &+ \|\eta_0^{(2)}\|_{H^{r-7/2}(\mathbb{T}^2)} \le c(\|\eta_0\|_{H^{r-1/2}(\mathbb{T}^2)} + \|\boldsymbol{u}_0\|_{H^{r-1}(\Omega)} + \|(f_0, \boldsymbol{f}, 0, 0, 0, f_6)\|_{Y^r_{\varepsilon_0}}). \end{aligned} \tag{3.9}$$

Lemma 3.1. *Let* $(\eta_0, \boldsymbol{u}_0) \in H^{r-1/2}(\mathbb{T}^2) \times H^{r-1}(\Omega)$. *Assume that* $\eta_0^{(1)}$, $\eta_0^{(2)}, \boldsymbol{u}_0^{(1)}$ *determined as in* (3.3), (3.5), *and* (3.6) *satisfy the compatibility conditions* (3.1) *and* (3.2). *Then there exists* $(\overline{\eta_0}, \overline{\boldsymbol{u}_0}, \overline{q_0}) \in X^r$ *with* $(\overline{\eta_0}, \overline{\boldsymbol{u}_0})(0) =$

$(\eta_0, \boldsymbol{u}_0)$, *which satisfies for* $j = 0, 1$,

$$\partial_t^j \left(\overline{\eta_0}_{,t} + \boldsymbol{w} \cdot \nabla' \overline{\eta_0} - \overline{u_{03}} + \fint_\Gamma \overline{\eta_0} \nabla' \cdot \boldsymbol{w} \, d\boldsymbol{x}' - f_0 \right)(0) = 0 \quad on\ \Gamma$$
$$\partial_t^j (\overline{\boldsymbol{u}_0}_{,t} - \nu \Delta \overline{\boldsymbol{u}_0} + \nabla \overline{q_0} - \boldsymbol{f})(0) = 0 \quad in\ \Omega$$
$$\partial_t^j (\nabla \cdot \overline{\boldsymbol{u}_0})(0) = 0 \quad in\ \Omega$$
$$\partial_t^j (\overline{u_{0i}}_{,x_3} + \overline{u_{03}}_{,x_i})(0) = 0 \quad on\ \Gamma$$
$$\partial_t^j (-2\nu \overline{u_{03}}_{,x_3} + \overline{q_0} - g\overline{\eta_0} - f_6)(0) = 0 \quad on\ \Gamma,$$

and

$$\|(\overline{\eta_0}, \overline{\boldsymbol{u}_0}, \overline{q_0})\|_{X^r} \le c(\|\eta_0\|_{H^{r-1/2}(\mathbb{T}^2)} + \|\boldsymbol{u}_0\|_{H^{r-1}(\Omega)} + \|(f_0, \boldsymbol{f}, 0, 0, 0, f_6)\|_{Y^r_{\varepsilon_0}}).$$

Proof. Since the initial data $(\eta_0, \boldsymbol{u}_0)$ satisfies the compatibility conditions, we can construct $(\overline{\eta_0}, \overline{\boldsymbol{u}_0}, \overline{q_0})$ which satisfies

$$\begin{aligned} &\overline{\eta_0} \in L^2(\mathbb{R}_+; H^r(\mathbb{T}^2)),\ \overline{\eta_0}_{,t} \in K^{r-1}(\mathbb{R}_+ \times \mathbb{T}^2) \\ &\overline{\boldsymbol{u}_0} \in K^r(\mathbb{R}_+ \times \Omega), \quad \nabla \overline{q_0} \in K^{r-2}(\mathbb{R}_+ \times \Omega) \\ &(\partial_t^j \overline{\eta_0}, \partial_t^j \overline{\boldsymbol{u}_0}, \partial_t^k \overline{q_0}) = (\eta_0^{(j)}, \boldsymbol{u}_0^{(j)}, q_0^{(k)}) \quad (j = 0, 1, 2,\ k = 0, 1) \end{aligned} \tag{3.10}$$

where $\eta_0^{(j)}, \boldsymbol{u}_0^{(j)}, q_0^{(k)}$ are given in (3.7) and $(\eta_0^{(0)}, \boldsymbol{u}_0^{(0)}) = (\eta_0, \boldsymbol{u}_0)$. In fact, we can obtain such $(\overline{\boldsymbol{u}_0}, \overline{q_0})$ due to [12, Thm 4.2.3] or [3], and such $\overline{\eta_0}$ due to [12, Thm 1.3.1]. Moreover, we can obtain $(\overline{\eta_0}(t), \overline{\boldsymbol{u}_0}(t), \overline{q_0}(t)) = \boldsymbol{0}$ for $t > 1$, by cutting off appropriately and applying the projection to $(\overline{\eta_0}, \overline{\boldsymbol{u}_0}, \overline{q_0})$ to satisfy (1.10), (1.11), (1.13) and $\int_{\mathbb{T}^2} \overline{\eta_0}(t)\, d\boldsymbol{x}' = 0$. □

We shall construct a solution $(\eta, \boldsymbol{u}, q)$ of (1.8)–(1.14) by solving the following mollified equations

$$\boldsymbol{u}_t^\varepsilon + \nabla q^\varepsilon - \nu \Delta \boldsymbol{u}^\varepsilon = \boldsymbol{f} \quad \text{in } \Omega \tag{3.11}$$
$$\nabla \cdot \boldsymbol{u}^\varepsilon = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u}^\varepsilon = \boldsymbol{0} \quad \text{on } S_B \tag{3.12}$$
$$u^\varepsilon_{i,x_3} + u^\varepsilon_{3,x_i} = 0, \quad -2\nu u^\varepsilon_{3,x_3} + q^\varepsilon - g\eta^\varepsilon = f_6 \quad \text{on } \Gamma \tag{3.13}$$
$$(\eta^\varepsilon, \boldsymbol{u}^\varepsilon) = (\eta_0, \boldsymbol{u}_0) \quad \text{at } t = 0 \tag{3.14}$$
$$\eta_t^\varepsilon - u_3^\varepsilon + \sum_{i=1,2} \varphi^\varepsilon * (w_i \varphi^\varepsilon * \eta^\varepsilon_{x_i}) + \fint_\Gamma \eta^\varepsilon \nabla' \cdot \boldsymbol{w}\, d\boldsymbol{x}' = f_0 \quad \text{on } \Gamma, \tag{3.15}$$

where $\varphi^\varepsilon *$ denotes a mollifier w.r.t. (x_1, x_2); that is, $\varphi^\varepsilon * f(\boldsymbol{x}') = \int_{\mathbb{T}^2} f(\boldsymbol{x}' - \boldsymbol{y}')\varphi^\varepsilon(\boldsymbol{y}')d\boldsymbol{y}'$ where $\varphi^\varepsilon(\cdot) = \varepsilon^{-2}\varphi(\cdot/\varepsilon)$, and $\varphi(\cdot)$ is a non-negative $C_0^\infty(\mathbb{R}^2)$ function satisfying $\int_{\mathbb{T}^2} \varphi(\boldsymbol{x}')d\boldsymbol{x}' = 1$ and $\varphi(-\boldsymbol{x}') = \varphi(\boldsymbol{x}')$. The local existence result to this system is as follows:

Lemma 3.2. *Let* $\|(0, \boldsymbol{w}, 0)\|_{X^r} \le \delta$. *Suppose the initial data* $(\eta_0, \boldsymbol{u}_0) \in H^{3/2}(\mathbb{T}^2) \times H^1(\Omega)$ *satisfies the compatibility conditions* $\fint_\Gamma \eta_0\, d\boldsymbol{x}' = 0$ *and* (3.12). *Then for any* $T > 0$, *there exists a solution of* (3.11)–(3.15) *which satisfies* $\eta^\varepsilon \in C([0, T]; H^{3/2}(\mathbb{T}^2))$, $\boldsymbol{u}^\varepsilon \in K^2((0, T) \times \Omega)$ *and* $\nabla q^\varepsilon \in L^2((0, T) \times \Omega)$.

Proof. We shall construct a sequence $\{(\eta^{(n)}, \boldsymbol{u}^{(n)}, q^{(n)})\}_n$ by the following iteration: $(\eta^{(0)}, \boldsymbol{u}^{(0)}, q^{(0)}) \equiv \boldsymbol{0}$ and for $n \geq 0$,

$$\boldsymbol{u}_t^{(n+1)} - \nu\Delta\boldsymbol{u}^{(n+1)} + \nabla q^{(n+1)} = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u}^{(n+1)} = 0 \quad \text{in } \Omega \tag{3.16}$$

$$u_{i,x_3}^{(n+1)} + u_{3,x_i}^{(n+1)} = 0, \quad -2\nu u_{3,x_3}^{(n+1)} + q^{(n+1)} = f_6 + g\eta^{(n)} \quad \text{on } \Gamma \tag{3.17}$$

$$\boldsymbol{u}^{(n+1)}|_{S_B} = \boldsymbol{0}, \quad \boldsymbol{u}^{(n+1)}|_{t=0} = \boldsymbol{u}_0 \tag{3.18}$$

$$\eta_t^{(n+1)} + \sum_{i=1,2}\varphi^\varepsilon * (w_i\varphi^\varepsilon * \eta_{x_i}^{(n+1)}) = f_0 + u_3^{(n)} - \fint_\Gamma \eta^{(n)}\nabla'\cdot\boldsymbol{w}\,d\boldsymbol{x}' \quad \text{on } \Gamma \tag{3.19}$$

$$\eta^{(n+1)}|_{t=0} = \eta_0. \tag{3.20}$$

Let $I = [0, 2T_0]$ for some $T_0 > 0$, and $\eta^{(n)}$ be given in $L^2(I; H^{1/2}(\mathbb{T}^2))$. Then the solution $(\boldsymbol{u}^{(n+1)}, q^{(n+1)})$ to (3.16)–(3.18) exists to satisfy $\boldsymbol{u}^{(n+1)} \in K^2(I\times\Omega)$ and $q^{(n+1)} \in L^2(I; H^1(\Omega))$ by [13, Thm 3.2], and

$$\begin{aligned}\|\boldsymbol{u}^{(n+1)} - \boldsymbol{u}^{(n)}\|_{K^2(I\times\Omega)} + \|\nabla q^{(n+1)} - \nabla q^{(n)}\|_{L^2(I\times\Omega)} &\leq \\ \leq c_{10}\, g\, e^{c_{11}T_0}\, \|\eta^{(n)} - \eta^{(n-1)}\|_{L^2(I;H^{1/2}(\mathbb{T}^2))}& \\ \leq c_{10}\, g\, e^{c_{11}T_0}\, \sqrt{T_0}\|\eta^{(n)} - \eta^{(n-1)}\|_{B(I;H^{1/2}(\mathbb{T}^2))}&.\end{aligned}$$

Here, we note that $\fint_{\mathbb{T}^2} u_3^{(n)}|_\Gamma d\boldsymbol{x}' = 0$ and $u_3^{(n)}|_\Gamma \in K^{3/2}(I\times\mathbb{T}^2)$ hold for $n \geq 1$. On the other hand, since $\varphi^\varepsilon * (w_i\varphi^\varepsilon * \nabla'\cdot)$ is a bounded linear operator on $H^{3/2}(\mathbb{T}^2)$, the solution $\eta^{(n+1)}$ of (3.19) and (3.20) uniquely exists in $B(I; H^{3/2}(\mathbb{T}^2))$ for $f_0 + u_3^{(n)} - \fint_\Gamma \eta^{(n)}\nabla'\cdot\boldsymbol{w}\,d\boldsymbol{x}' \in K^{3/2}(I\times\Gamma)$ and $w_i|_\Gamma \in B(I; H^{r-3/2}(\mathbb{T}^2))$. If we take (2.1) into account, we have for $n \geq 1$,

$$\begin{aligned}&\|\eta^{(n+1)} - \eta^{(n)}\|_{B(I;H^{3/2}(\mathbb{T}^2))} \leq \\ &\leq e^{c_0\int_I \|\boldsymbol{w}(\tau)|_\Gamma\|_{H^{5/2}(\mathbb{T}^2)}d\tau}\big(\|\boldsymbol{u}^{(n)} - \boldsymbol{u}^{(n-1)}\|_{L^1(I;H^2(\Omega))} + \\ &\qquad + \|\boldsymbol{w}|_\Gamma\|_{L^2(I;H^{5/2}(\mathbb{T}^2))}\|\eta^{(n)} - \eta^{(n-1)}\|_{L^2(I;H^{3/2}(\mathbb{T}^2))}\big) \\ &\leq c_{12}(T_0)\big(\|\boldsymbol{u}^{(n)} - \boldsymbol{u}^{(n-1)}\|_{L^2(I;H^2(\Omega))} + \|\eta^{(n)} - \eta^{(n-1)}\|_{B(I;H^{3/2}(\mathbb{T}^2))}\big),\end{aligned}$$

where $c_{12}(T_0) = \sqrt{T_0}(1 + c_{13}\delta)\exp(c_0c_5\delta)$. This estimates yield the convergence of the sequence $\{(\eta^{(n)}, \boldsymbol{u}^{(n)}, \nabla q^{(n)})\}_n$ in the space $B(I; H^{3/2}(\mathbb{T}^2)) \times K^2(I\times\Omega) \times L^2(I\times\Omega)$ to a limit $(\eta^\varepsilon, \boldsymbol{u}^\varepsilon, q^\varepsilon)$ if we choose small $T_0 > 0$. We note that the limit $(\eta^\varepsilon, \boldsymbol{u}^\varepsilon, q^\varepsilon)$ satisfies (3.11)–(3.15), and hence, the condition $\fint_{\mathbb{T}^2} \eta^\varepsilon(t)d\boldsymbol{x}' = 0$.

Now we shall extend the time interval. We can choose $T_1 \in (T_0, 2T_0)$ so that the trace $(\eta^\varepsilon, \boldsymbol{u}^\varepsilon)$ on $t = T_1$ belongs to $H^{3/2}(\mathbb{T}^2) \times H^1(\Omega)$ and satisfies the compatibility conditions $\fint_{\mathbb{T}^2} \eta(T_1)d\boldsymbol{x}' = 0$ and (3.12). We thus repeat the same arguments to obtain the solution on the new interval $[T_1, T_1 + 2T_0]$. And repetition of this process gives our desired result. □

Now we discuss the existence of strong solution of (1.8)–(1.14).

Lemma 3.3. *Suppose* $F_8 = \|\Lambda^{r-1}(\eta_0, \boldsymbol{u}_0)\|^2_{L^2} + \|(f_0, \boldsymbol{f}, 0, 0, 0, f_6)\|^2_{Y^r_{\varepsilon_0}}$ *is finite, and that the initial data* $(\eta_0, \boldsymbol{u}_0)$ *satisfies the compatibility conditions* (3.1) *and* (3.2). *Then, there exists* $\delta > 0$ *so that if* $\|(0, \boldsymbol{w}, 0)\|_{X^r} \le \delta$, *a solution* $(\eta, \boldsymbol{u}, q)$ *of* (1.8)–(1.14) *exists and satisfies the following bounds*:

$$\|\boldsymbol{u}\|^2_{K^r(\mathbb{R}_+\times\Omega)} + \|\nabla q\|^2_{K^{r-2}(\mathbb{R}_+\times\Omega)} + \|\eta\|^2_{L^2(\mathbb{R}_+;H^{r-3/2}(\mathbb{T}^2))} + \\ + \|\eta\|^2_{H^{r/2}(\mathbb{R}_+;L^2(\mathbb{T}^2))} + \|\eta\|^2_{B(\mathbb{R}_+;H^{r-1}(\mathbb{T}^2))} \le c_{14}F_8,$$

where c_{14} *is independent of* $\boldsymbol{w}$.

Proof. We shall first discuss uniform estimates with respect to T of the solution $(\eta^\varepsilon, \boldsymbol{u}^\varepsilon, q^\varepsilon)$ constructed in Lemma 3.2, and then discuss its regularities. Since $\varphi^\varepsilon *$ commutes with Λ, Lemma 2.2, 2.4, and 2.5 can be also applicable to the solution of (3.11)–(3.15). If we apply Lemma 2.2 (i) (ii), Lemma 2.4, and Lemma 2.5 on the interval $I = [0, T]$, by adding (2.11), (2.12), ε_2 times (2.15) for $(\alpha, \gamma) = (r, 0)$, and ε_2 times (2.16) for $(k, \alpha, \gamma) = (1, r-2, 0)$, we have

$$c_{15}\|\Lambda^{r-1}\boldsymbol{u}^\varepsilon, \Lambda^{r-3}\boldsymbol{u}^\varepsilon_t\|^2_{L^2(I;H^1(\Omega))} + c_{16}\|\eta^\varepsilon\|^2_{B(I;H^{r-1}(\mathbb{T}^2))} + \\ + c_{17}\|\eta^\varepsilon\|^2_{L^2(I;H^{r-3/2}(\mathbb{T}^2))} + c_{18}\|\eta^\varepsilon_t\|^2_{L^2(I;H^{r-2}(\mathbb{T}^2))} \le c_{19}F_8,$$

where $c_{15} = c_0^2\nu - \varepsilon_2(c_6 + (1+\delta^2)c_7)$, $c_{16} = g - 2\delta c_3 - \varepsilon_2(1+\delta^2)c_7$, $c_{17} = g\varepsilon_2 - \delta c_2$ and $c_{18} = \varepsilon_2 - \delta c_3$ are independent of ε and T, and are positive provided that δ is appropriately small for small ε_2. Letting $T \to \infty$, we have

$$\|\Lambda^{r-1}\boldsymbol{u}^\varepsilon, \Lambda^{r-3}\boldsymbol{u}^\varepsilon_t\|^2_{L^2(\mathbb{R}_+;H^1(\Omega))} + \|\Lambda^{r-3/2}\eta^\varepsilon, \Lambda^{r-2}\eta^\varepsilon_t\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{T}^2))} + \\ + \|\eta^\varepsilon\|^2_{B(\mathbb{R}_+;H^{r-1}(\mathbb{T}^2))} \le c_{14}F_8. \tag{3.21}$$

Hence, the bounded family $\{(\eta^\varepsilon, \boldsymbol{u}^\varepsilon)\}$ has a weak limit $(\eta, \boldsymbol{u})$ satisfying

$$\begin{gathered}\Lambda^{r-1}\boldsymbol{u}, \Lambda^{r-3}\boldsymbol{u}_t \in L^2(\mathbb{R}_+; H^1(\Omega)) \\ \Lambda^{r-3/2}\eta, \Lambda^{r-2}\eta_t \in L^2(\mathbb{R}_+; L^2(\mathbb{T}^2)), \text{ and } \eta \in L^\infty(\mathbb{R}_+; H^{r-1}(\mathbb{T}^2)).\end{gathered} \tag{3.22}$$

Since this $(\eta, \boldsymbol{u})$ satisfies (1.8)–(1.14), we have $\langle t\rangle^{-1/2}\eta \in L^\infty(\mathbb{R}_+; H^{r-1/2}(\mathbb{T}^2))$ by Lemma 2.8.

We shall next discuss the regularities in time. To reduce the problem to the one with the homogeneous initial data, we set $(\tilde\eta, \tilde{\boldsymbol{u}}, \tilde q) = (\eta - \overline{\eta_0}, \boldsymbol{u} - \overline{\boldsymbol{u}_0}, q - \overline{q_0})$, which belongs to the same space (3.22) as $(\eta, \boldsymbol{u})$, and satisfies the linear system (1.8)–(1.13) with the homogeneous initial condition (2.10) for an inhomogeneous data $(\tilde f_0, \tilde{\boldsymbol{f}}, 0, 0, 0, \tilde f_6)$ satisfying

$$\begin{aligned}\tilde f_0 &= f_0 - \overline{\eta_0}_{,t} + \overline{u_0}_3 - \boldsymbol{w}\cdot\nabla'\overline{\eta_0} - \fint_\Gamma \overline{\eta_0}\nabla'\cdot\boldsymbol{w}\,d\boldsymbol{x}' \\ \tilde{\boldsymbol{f}} &= \boldsymbol{f} - \overline{\boldsymbol{u}_0}_{,t} + \nu\Delta\overline{\boldsymbol{u}_0} - \nabla\overline{q_0} \\ \tilde f_6 &= f_6 + 2\nu\overline{u_0}_{3,x_3} - \overline{q_0} + g\overline{\eta_0}\end{aligned}$$

$$\|(\tilde f_0, \tilde{\boldsymbol{f}}, 0, 0, 0, \tilde f_6)\|_{Y^r_{\varepsilon_0,0}} \le c(\|(\overline{\eta_0}, \overline{\boldsymbol{u}_0}, \overline{q_0})\|_{X^r} + \|(f_0, \boldsymbol{f}, 0, 0, 0, f_6)\|_{Y^r_{\varepsilon_0}}).$$

Recalling that $\eta \in L^\infty(\mathbb{R}_+; H^{r-1}(\mathbb{T}^2))$ and $\Lambda^{r-2j-1}\boldsymbol{u} \in H^j(\mathbb{R}_+; H^1(\Omega))$ $(j = 0, 1)$, we have $\tilde{\eta} \in H^1(\mathbb{R}_+; H^{r-2}(\mathbb{T}^2)) \cap H^2(\mathbb{R}_+; H^{r-3}(\mathbb{T}^2))$ by Lemma 2.5 for $(\alpha, \gamma, k) = (r-2, 0, 1)$ and $(r-3, 0, 2)$, and hence $\Lambda^{r-5/2}\tilde{\boldsymbol{u}} \in H^2(\mathbb{R}_+; L^2(\Omega))$ by Lemma 2.6 for $(\alpha, \beta, \gamma) = (r-1/2, 1, 0)$. By (2.3), we have for $s > 2$

$$\|\boldsymbol{w} \cdot \nabla'\tilde{\eta}\|_{H^{r/2-1}(\mathbb{R}_+; L^2(\Gamma))} \le c(\|\boldsymbol{w}|_\Gamma\|_{H^{r/2-1}(\mathbb{R}_+; L^2(\mathbb{T}^2))}\|\nabla'\tilde{\eta}\|_{K^s} + \\ + \|\boldsymbol{w}|_\Gamma\|_{K^s}\|\nabla'\tilde{\eta}\|_{H^{r/2-1}(\mathbb{R}_+; L^2(\mathbb{T}^2))}). \tag{3.23}$$

This inequality together with (1.8) and the fact $\tilde{\eta} \in H^2(\mathbb{R}_+; H^{r-3}(\mathbb{T}^2)) \subset H^{r/2-1}(\mathbb{R}_+; H^1(\mathbb{T}^2))$ indicate $\tilde{\eta} \in H^{r/2}(\mathbb{R}_+; L^2(\mathbb{T}^2))$. We can also obtain $\tilde{\boldsymbol{u}} \in H^{r/2}(\mathbb{R}_+; L^2(\Omega))$ by Lemma 2.6 for $(\alpha, \beta, \gamma) = (2, r/2-1, 0)$.

Now we shall discuss the vertical regularities of $(\tilde{\boldsymbol{u}}, \tilde{q})$. Recalling that a solenoidal vector $\boldsymbol{u}$ satisfies

$$\|u_3|_\Gamma\|_{H^{s-1/2}(\mathbb{T}^2)} \le c\|\Lambda^s \boldsymbol{u}\|_{L^2(\Omega)} \quad \text{for } s \ge 1/2,$$

and applying the Laplace transform to (1.9), we have by Lemma 2.1,

$$(1+|\lambda|)^{r-2}(\|\hat{\boldsymbol{u}}\|^2_{H^2(\Omega)} + \|\nabla\hat{q}\|^2_{L^2(\Omega)}) \le c_{20}(1+|\lambda|)^{r-2}(\|\hat{\boldsymbol{f}} - \widehat{\boldsymbol{u}_t}\|^2_{L^2(\Omega)} + \|\Lambda^2\hat{\boldsymbol{u}}\|^2_{L^2(\Omega)}).$$

Here the Laplace transform of $(\tilde{\boldsymbol{u}}, \tilde{q})$ is denoted by $(\hat{\boldsymbol{u}}, \hat{q})$. Integrating on the line $\mathrm{Re}\lambda = \lambda_0 > 0$ in the complex plane and letting $\lambda_0 \to 0$, we have $\tilde{\boldsymbol{u}} \in H^{r/2-1}(\mathbb{R}_+; H^2(\Omega))$, and $\nabla q \in H^{r/2-1}(\mathbb{R}_+; L^2(\Omega))$. By interpolating with $\tilde{\boldsymbol{u}} \in H^{r/2}(\mathbb{R}_+; L^2(\Omega))$, we also have $\tilde{\boldsymbol{u}} \in H^2(\mathbb{R}_+; H^{r-4}(\Omega))$. By Lemma 2.1, we have for a.e. $t > 0$,

$$\begin{aligned} \|\tilde{\boldsymbol{u}}\|^2_{H^r(\Omega)} + \|\nabla\tilde{q}\|^2_{H^{r-2}(\Omega)} &\le c_{20}(\|\tilde{\boldsymbol{f}} - \tilde{\boldsymbol{u}}_t\|^2_{H^{r-2}(\Omega)} + \|\Lambda^r\tilde{\boldsymbol{u}}\|^2_{L^2(\Omega)}) \\ \|\tilde{\boldsymbol{u}}_t\|^2_{H^{r-2}(\Omega)} + \|\nabla\tilde{q}_t\|^2_{H^{r-4}(\Omega)} &\le c_{20}(\|\tilde{\boldsymbol{f}}_t - \tilde{\boldsymbol{u}}_{tt}\|^2_{H^{r-4}(\Omega)} + \|\Lambda^{r-2}\tilde{\boldsymbol{u}}_t\|^2_{L^2(\Omega)}), \end{aligned} \tag{3.24}$$

which indicate $\tilde{\boldsymbol{u}} \in L^2(\mathbb{R}_+; H^r(\Omega))$ and $\nabla\tilde{q} \in L^2(\mathbb{R}_+; H^{r-2}(\Omega))$. And hence, we have obtained

$$\begin{gathered} \eta \in L^2(\mathbb{R}_+; H^{r-3/2}) \cap H^{r/2}(\mathbb{R}_+; L^2) \cap B(\mathbb{R}_+; H^{r-1}) \\ \langle t\rangle^{-1/2}\eta \in L^\infty(\mathbb{R}_+; H^{r-1/2}), \quad \boldsymbol{u} \in K^r, \qquad \nabla q \in K^{r-2}. \end{gathered}$$

□

4. Proof of Theorem 1.3

According to Lemma 3.3, the solution $(\eta, \boldsymbol{u}, q)$ to (1.8)–(1.14) exists in a class wider than X^r if we choose small $\delta > 0$. So in this section, we shall show that this solution also satisfies $\|(\eta, \boldsymbol{u}, q)\|_{X^r} < \infty$ for the same $\delta > 0$. As in the proof of Lemma 3.3, we restrict our discussion on the problem with an inhomogeneous data $(f_0, \boldsymbol{f}, 0, 0, 0, f_6) \in Y^r_{\varepsilon_0,0}$ together with the homogeneous initial data (2.10).

(i) We begin with decay estimates of order $t^{-1/2}$. By Lemma 2.2 (i) for $(\alpha,\gamma)=(r-1/2,1/2)$, we have

$$\sup_t \langle t\rangle \|\Lambda^{r-3/2}(\eta,\boldsymbol{u})\|^2 + c_0^2\nu \int_{\mathbb{R}_+} \langle t\rangle \|\Lambda^{r-3/2}\boldsymbol{u}\|^2_{H^1(\Omega)}dt \le c_{21}F_0. \tag{4.1}$$

By Lemma 3.3 and (4.1), we can see that $F_2(r-1/2,1/2)$ in Lemma 2.2 (ii) is bounded, so that we have

$$\sup_t \langle t\rangle \|\Lambda^{r-7/2}(\eta_t,\boldsymbol{u}_t)\|^2 + \int_{\mathbb{R}_+} \langle t\rangle \|\Lambda^{r-7/2}\boldsymbol{u}_t\|^2_{H^1(\Omega)}dt \le c_{22}F_0. \tag{4.2}$$

By (4.1), (4.2), and Lemma 2.4 for $(\alpha,\gamma)=(r-1/2,1/2)$, we have

$$\int_{\mathbb{R}_+} \langle t\rangle \|\Lambda^{r-2}\eta\|^2 dt \le c_{23}F_0.$$

Here, the constants c_{21}, c_{22}, and c_{23} do not depend on $\boldsymbol{w}$ for $(0,\boldsymbol{w},0)\in B_\delta(\boldsymbol{0})$.

By the arguments similar to the one on the temporal regularities in the proof of Lemma 3.3, we have for $k=1,2$,

$$\begin{aligned}\langle t\rangle^{1/2}\eta &\in H^k(\mathbb{R}_+; H^{r-1-3k/2}(\mathbb{T}^2)),\\ \langle t\rangle^{1/2}\Lambda^{r-2k-1/2}\boldsymbol{u} &\in H^k(\mathbb{R}_+; L^2(\Omega)).\end{aligned}$$

Hence by (2.3), we have $\langle t\rangle^{1/2}\eta \in H^{(2r-1)/4}(\mathbb{R}_+; L^2(\mathbb{T}^2))$, as well as

$$\langle t\rangle^{1/2}\boldsymbol{u} \in H^{(2r-1)/4}(\mathbb{R}_+; L^2(\Omega)). \tag{4.3}$$

We shall next estimate the vertical regularities of $(\boldsymbol{u},q)$. By Lemma 2.2 (iii) for $(\beta,\gamma)=((2r-1)/4,1/2)$, we have

$$\langle t\rangle^{1/2}\boldsymbol{u} \in H^{(2r-3)/4}(\mathbb{R}_+; H^1(\Omega)). \tag{4.4}$$

Also by the same discussion as (3.24), we obtain $\langle t\rangle^{1/2}\boldsymbol{u} \in L^2(\mathbb{R}_+; H^{r-1/2}(\Omega))$ and $\langle t\rangle^{1/2}\nabla q \in L^2(\mathbb{R}_+; H^{r-5/2}(\Omega))$.

(ii) We shall next discuss the decay estimates of order t^{-1}, but all the discussion given here is similar to the one given in (i). By Lemma 2.2 (i) for $(\alpha,\gamma)=(r-1,1)$, we have

$$\sup_t \langle t\rangle^2 \|\Lambda^{r-2}(\eta,\boldsymbol{u})\|^2 + \int_{\mathbb{R}_+} \langle t\rangle^2 \|\Lambda^{r-2}\boldsymbol{u}\|^2_{H^1(\Omega)}dt \le c_{21}F_0.$$

Since $F_2(r-1,1)$ given in Lemma 2.2 (ii) is bounded, we have

$$\int_{\mathbb{R}_+} \langle t\rangle^2 \|\Lambda^{r-4}\boldsymbol{u}_t\|^2_{H^1(\Omega)}dt + \sup_t \langle t\rangle^2 \|\Lambda^{r-4}(\eta_t,\boldsymbol{u}_t)\|^2 \le c_{22}F_0,$$

and by Lemma 2.4 for $(\alpha,\gamma)=(r-1,1)$, we have $\int_{\mathbb{R}_+} \langle t\rangle^2 \|\Lambda^{r-5/2}\eta\|^2 dt \le c_{23}F_0$. As in the previous discussion on the temporal regularities, we have $\langle t\rangle\eta \in H^{(r-1)/2}(\mathbb{R}_+; L^2(\mathbb{T}^2))$ and $\langle t\rangle\boldsymbol{u} \in H^{(r-1)/2}(\mathbb{R}_+; L^2(\Omega))$.

Concerning the vertical regularities of $(\boldsymbol{u}, q)$, by Lemma 2.2 (iii) for $(\beta,\gamma) = ((r-1)/2, 1)$, we have $\langle t\rangle \boldsymbol{u} \in H^{r/2-1}(\mathbb{R}_+; H^1(\Omega))$. Also by Lemma 2.1, we have

$$
\begin{aligned}
(4.5)\qquad & \|\langle t\rangle \boldsymbol{u}_t\|^2_{L^2(\mathbb{R}_+;H^{r-3}(\Omega))} + \|\langle t\rangle \nabla q_t\|^2_{L^2(\mathbb{R}_+;H^{r-5}(\Omega))} \le \\
&\qquad \le c_{20}(\|\langle t\rangle(\boldsymbol{u}_{tt} - \boldsymbol{f}_t)\|^2_{L^2(\mathbb{R}_+;H^{r-5}(\Omega))} + \|\langle t\rangle \Lambda^{r-3}\boldsymbol{u}_t\|^2_{L^2(\mathbb{R}_+;L^2(\Omega))}),\\
(4.6)\qquad & \|\langle t\rangle \boldsymbol{u}\|^2_{L^2(\mathbb{R}_+;H^{r-1}(\Omega))} + \|\langle t\rangle \nabla q\|^2_{L^2(\mathbb{R}_+;H^{r-3}(\Omega))} \le \\
&\qquad \le c_{20}(\|\langle t\rangle(\boldsymbol{u}_t - \boldsymbol{f})\|^2_{L^2(\mathbb{R}_+;H^{r-3}(\Omega))} + \|\langle t\rangle \Lambda^{r-1}\boldsymbol{u}\|^2_{L^2(\mathbb{R}_+;L^2(\Omega))}).
\end{aligned}
$$

By virtue of (4.5), (4.6) and the fact

$$\langle t\rangle \boldsymbol{u} \in H^{(r-1)/2}(\mathbb{R}_+; L^2(\Omega)) \cap H^{r/2-1}(\mathbb{R}_+; H^1(\Omega)) \subset H^2(\mathbb{R}_+; H^{r-5}(\Omega)),$$

we obtain $\langle t\rangle \boldsymbol{u} \in L^2(\mathbb{R}_+; H^{r-1}(\Omega))$ and $\langle t\rangle \nabla q \in L^2(\mathbb{R}_+; H^{r-3}(\Omega))$. We thus complete the proof of Theorem 1.3.

5. Nonlinear problem

In this section, we discuss the full nonlinear problem (1.1)–(1.6). Following [3], we begin by giving the compatibility conditions on the initial data η_0 and $\boldsymbol{v}_0 = {}^t(v_{01}, v_{02}, v_{03})$, though their linear versions are already given in (3.1) and (3.2). We require

$$(5.1)\qquad \int_{\mathbb{T}^2} \eta_0 d\boldsymbol{x}' = \int_{\mathbb{T}^2} \eta_0^{(1)} d\boldsymbol{x}' = \int_{\mathbb{T}^2} \eta_0^{(2)} d\boldsymbol{x}' = 0$$

$$(5.2)\qquad \nabla\cdot \boldsymbol{v}_0 = 0, \quad \nabla\cdot \boldsymbol{v}_0^{(1)} = 0 \quad \text{in } \Omega(0)$$

$$(5.3)\qquad \sum_{j,k=1}^{3} (v_{0k,x_j} + v_{0j,x_k}) n_{0j}\tau_{0ik} = 0 \quad \text{on } \Gamma(0),\ i = 1,2$$

$$
\begin{aligned}
(5.4)\qquad & \sum_{j,k=1}^{3} (v_{0k,x_j} + v_{0j,x_k})(n_{0j}^{(1)}\tau_{0ik} + n_{0j}\tau_{0ik}^{(1)}) + \\
&\qquad + \sum_{j,k=1}^{3} (v^{(1)}_{0k,x_j} + v^{(1)}_{0j,x_k}) n_{0j}\tau_{0ik} = 0 \quad \text{on } \Gamma(0),\ i = 1,2
\end{aligned}
$$

$$(5.5)\qquad \boldsymbol{v}_0 = \boldsymbol{0}, \quad \boldsymbol{v}_0^{(1)} = \boldsymbol{0} \quad \text{on } S_B.$$

Here

$$
\begin{aligned}
&\boldsymbol{\tau}_{01} = (1, 0, \eta_{0,x_1}), \quad \boldsymbol{\tau}_{02} = (0, 1, \eta_{0,x_2}), \qquad \boldsymbol{\tau}_{01}^{(1)} = (0, 0, \eta^{(1)}_{0,x_1})\\
&\boldsymbol{\tau}_{02}^{(1)} = (0, 0, \eta^{(1)}_{0,x_2}), \quad \boldsymbol{n}_0 = (-\eta_{0,x_1}, -\eta_{0,x_2}, 1), \quad \boldsymbol{n}_0^{(1)} = (-\eta^{(1)}_{0,x_1}, -\eta^{(1)}_{0,x_2}, 0),
\end{aligned}
$$

using the functions $\eta_0^{(1)}, \eta_0^{(2)}$ and $\boldsymbol{v}_0^{(1)}$, which we shall determine from now. As in (3.3), $\eta_0^{(1)}$ is determined as $\eta_0^{(1)} = -\boldsymbol{v}_0\cdot\nabla'\eta_0 + v_{03}$ on $\Gamma(0)$. As in (3.4), setting

$$\Delta p_0 = -\sum_{j,k=1}^{3} v_{j,x_k} v_{k,x_j} \quad \text{in } \Omega(0)$$

$$p_{0,x_3} = \nu \Delta v_{03} \quad \text{on } S_B$$

$$p_0 = \nu \sum_{j,k=1}^{3} (v_{0j,x_k} + v_{0k,x_j}) n_{0j} n_{0k} + g\eta_0 \quad \text{on } \Gamma(0)$$

at $t = 0$, and $\boldsymbol{v}_0^{(1)} = \nu \Delta \boldsymbol{v}_0 - (\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{v}_0 - \nabla p_0$, we have $\nabla p_0 \in H^{r-3}(\Omega(0))$ and $\boldsymbol{v}_0^{(1)} \in H^{r-3}(\Omega(0))$. We set $\eta_0^{(2)} \in H^{r-7/2}(\mathbb{T}^2)$ as $\eta_0^{(2)} = -(\boldsymbol{v}_0^{(1)} \cdot \nabla' \eta_0 + \boldsymbol{v}_0 \cdot \nabla' \eta_0^{(1)}) + v_{03}^{(1)}$ on $\Gamma(0)$. We should note that $p_0, \eta_0^{(1)}, \eta_0^{(2)}$ and $\boldsymbol{v}_0^{(1)}$, are determined by η_0 and $\boldsymbol{v}_0$ alone, and we do not need further compatibility conditions, since we work for $r < 11/2$.

We next transform the problem (1.1)–(1.6) to the one on the equilibrium domain using the unknown function $\eta(t)$. For that purpose, we extend the domain of $\eta(t,\cdot) \in H^{r-1/2}(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} \eta(t,\boldsymbol{x}')\, d\boldsymbol{x}' = 0$ to $\mathbb{T}^2 \times \mathbb{R}_-$ as

$$\overline{\eta}(t,\boldsymbol{y}',y_3) = |\Gamma|^{-1} \sum_{\boldsymbol{\xi}' \in \mathbb{Z}^2} e^{i\boldsymbol{y}' \cdot \boldsymbol{\xi}'} e^{|\boldsymbol{\xi}'| y_3} \int_{\mathbb{T}^2} \eta(t,\boldsymbol{x}') e^{-i\boldsymbol{x}' \cdot \boldsymbol{\xi}'} d\boldsymbol{x}', \tag{5.6}$$

to satisfy $\overline{\eta}(t,\boldsymbol{y}',0) = \eta(t,\boldsymbol{y}')$ and $\overline{\eta}(t,\cdot) \in H^r(\mathbb{T}^2 \times \mathbb{R}_-)$. We define a mapping $\Theta[t] : \mathbb{T}^2 \times (-1,0) \ni (y_1,y_2,y_3) \mapsto (x_1,x_2,x_3) \in \{-1 < x_3 < \eta(t,\boldsymbol{x}')\}$ as

$$\Theta[t](y_1,y_2,y_3) = (y_1, y_2, \overline{\eta}(\boldsymbol{y},t) + y_3(1 + \overline{\eta}(\boldsymbol{y},t))), \tag{5.7}$$

which stretches or compresses on vertical line segment. We also define the velocity $\boldsymbol{u}$ and the pressure q in the equilibrium domain Ω by the composition with $\Theta[t]$ as $\boldsymbol{u} = \boldsymbol{v} \circ \Theta,\ q = p \circ \Theta$. The inverse $\Theta^{-1}[t] : \{-1 < x_3 < \eta(t,\boldsymbol{x}')\} \to \mathbb{T}^2 \times (-1,0)$ and its spatial Jacobian matrix $d\Theta^{-1} = \partial(\Theta^{-1})/\partial(\boldsymbol{x})$:

$$d\Theta^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\Theta_{3,y_1} J^{-1} & -\Theta_{3,y_2} J^{-1} & J^{-1} \end{pmatrix}$$

are also well-defined provided that the Jacobian $J = 1 + \overline{\eta} + \overline{\eta}_{y_3}(1 + y_3)$ does not vanish. Utilizing this transformation Θ^{-1}, we rewrite the problem (1.1)–(1.6) on the equilibrium domain as

$$\eta_t + \boldsymbol{u} \cdot \nabla'_y \eta - u_3 + \fint_\Gamma \eta \nabla'_y \cdot \boldsymbol{u}\, d\boldsymbol{y}' = f_0(\eta,\boldsymbol{u}) \qquad \text{on } \Gamma \tag{5.8}$$

$$\boldsymbol{u}_t - \nu \Delta_y \boldsymbol{u} + \nabla_y q = \boldsymbol{f}(\eta,\boldsymbol{u},q) \qquad \text{in } \Omega \tag{5.9}$$

$$\nabla_y \cdot \boldsymbol{u} = \sigma(\eta,\boldsymbol{u}) \qquad \text{in } \Omega \tag{5.10}$$

$$-\nu(u_{i,y_3} + u_{3,y_i}) = f_{i+3}(\eta,\boldsymbol{u},q) \ \text{on } \Gamma, \quad i = 1,2 \tag{5.11}$$

$$-2\nu u_{3,y_3} + q - g\eta = f_6(\eta,\boldsymbol{u}) \qquad \text{on } \Gamma \tag{5.12}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } S_B \tag{5.13}$$

$$(\eta,\boldsymbol{u}) = (\eta_0,\boldsymbol{u}_0) \qquad \text{at } t = 0, \tag{5.14}$$

where $\Gamma = \{(\boldsymbol{y}', 0) : \boldsymbol{y}' \in \mathbb{T}^2\}$ and $\boldsymbol{u}_0 = \boldsymbol{v}_0 \circ \Theta[0]$. The functions f_0, $\boldsymbol{f}$, f_j $(4 \leq j \leq 6)$ and σ are quadratic or higher terms with respect to the unknown functions given as

(5.15)

$$
\begin{aligned}
f_0 &= \fint_\Gamma \eta \nabla'_y \cdot \boldsymbol{u}\, d\boldsymbol{y}' \\
\boldsymbol{f} &= -(\boldsymbol{u} \cdot {}^t d\Theta^{-1} \nabla_y)\boldsymbol{u} + (I - {}^t d\Theta^{-1})\nabla_y q + J^{-1}\Theta_{3,t}\boldsymbol{u}_{y_3} + \\
&\quad - \nu(\Delta_y - ({}^t d\Theta^{-1}\nabla_y) \cdot ({}^t d\Theta^{-1}\nabla_y))\boldsymbol{u} \\
\sigma &= \nabla_y \cdot \boldsymbol{u} - J {}^t d\Theta^{-1} \nabla_y \cdot \boldsymbol{u} \\
{}^t(f_4, f_5, f_6) &= \nu\{({}^t d\Theta^{-1}\nabla_y)\boldsymbol{u} + {}^t(({}^t d\Theta^{-1}\nabla_y)\boldsymbol{u})\}\boldsymbol{n} - \\
&\quad - \nu\{\nabla_y \boldsymbol{u} + {}^t(\nabla_y \boldsymbol{u})\}\, {}^t(0,0,1) - (q - g\eta)\, {}^t(-\eta_{y_1}, -\eta_{y_2}, 0),
\end{aligned}
$$

where ∇_y denotes the gradient in $\boldsymbol{y}$, and $\boldsymbol{n} = {}^t(-\eta_{y_1}, -\eta_{y_2}, 1)$ the normal. Since the initial data $(\eta_0, \boldsymbol{v}_0)$ satisfies the compatibility conditions (5.1)–(5.5), the transformed initial data $\boldsymbol{u}_0$ together with η_0 satisfies (3.1)–(3.2).

For given $(0, \boldsymbol{w}, 0) \in X^r$, we shall introduce a linear operator $\mathcal{L}[\boldsymbol{w}]$:

$$
\mathcal{L}[\boldsymbol{w}](\boldsymbol{z}) := \begin{cases}
\eta_t - u_3 + \boldsymbol{w} \cdot \nabla'_y \eta + \fint_\Gamma \eta \nabla'_y \cdot \boldsymbol{w}\, d\boldsymbol{y}' & \text{on } \Gamma \\
\boldsymbol{u}_t - \nu \Delta_y \boldsymbol{u} + \nabla_y q & \text{in } \Omega \\
\nabla_y \cdot \boldsymbol{u} & \text{in } \Omega \\
-\nu(u_{i,y_3} + u_{3,y_i}) & \text{on } \Gamma, \quad i = 1, 2 \\
-2\nu u_{3,y_3} + q - g\eta & \text{on } \Gamma,
\end{cases}
$$

where $\boldsymbol{z} = (\eta, \boldsymbol{u}, q) \in X^r$.

Lemma 5.1. *The extension $\overline{\boldsymbol{z}_0} = (\overline{\eta_0}, \overline{\boldsymbol{u}_0}, \overline{q_0})$ constructed in Lemma 3.1 satisfies $\mathcal{L}[\boldsymbol{w}](\overline{\boldsymbol{z}_0}) \in Y^r_{\varepsilon_0}$.*

Proof. It is easy to see that the conditions $\overline{\boldsymbol{z}_0} \in X^r$ and $\overline{\boldsymbol{z}_0}(t) \equiv \boldsymbol{0}$ for $t > 1$ assure $\mathcal{L}[\boldsymbol{w}](\overline{\boldsymbol{z}_0}) \in Y^r_{\varepsilon_0}$. In fact, we can check $\boldsymbol{w} \cdot \nabla'_y \overline{\eta_0} \in L^1(\mathbb{R}_+; H^{r-1}(\Gamma))$ by (2.2) and Lemma 3.1 as

$$
\begin{aligned}
&\|\boldsymbol{w} \cdot \nabla'_y \overline{\eta_0}\|_{L^1(\mathbb{R}_+; H^{r-1}(\Gamma))} \leq c(\|\boldsymbol{w}\|_{L^2(\mathbb{R}_+; L^\infty(\Gamma))} \|\nabla'_y \overline{\eta_0}\|_{L^2(\mathbb{R}_+; H^{r-1}(\mathbb{T}^2))} + \\
&\quad + \|\boldsymbol{w}\|_{L^2(\mathbb{R}_+; H^{r-1}(\Gamma))} \|\nabla'_y \overline{\eta_0}\|_{L^2(\mathbb{R}_+; L^\infty(\mathbb{T}^2))}) \\
&\quad \leq c\|(0, \boldsymbol{w}, 0)\|_{X^r} \|\eta_0\|_{H^{r-1/2}(\mathbb{T}^2)}.
\end{aligned}
$$

Other terms can be checked in a similar manner. □

Setting $\mathcal{L}_0[\boldsymbol{w}]$ to be the restriction of $\mathcal{L}[\boldsymbol{w}]$ on X^r_0, we obtain

Lemma 5.2. *The linear operator $\mathcal{L}_0[\boldsymbol{w}]$ continuously maps X^r_0 to $Y^{r-1}_{\varepsilon_0, 0}$. Moreover there exists $\delta > 0$ so that for $\boldsymbol{w} \in X^r$ with $\|(0, \boldsymbol{w}, 0)\|_{X^r} \leq \delta$, the operator $\mathcal{L}_0[\boldsymbol{w}]$ has a continuous inverse $\mathcal{L}_0^{-1}[\boldsymbol{w}] : Y^r_{\varepsilon_0, 0} \to X^r_0$ satisfying*

$$
\|\mathcal{L}_0^{-1}[\boldsymbol{w}]\|_{B(Y^r_{\varepsilon_0, 0}, X^r_0)} \leq c_\delta.
$$

Here the constant c_δ does not depend on $\boldsymbol{w}$.

Proof. Since $\mathcal{L}_0[\boldsymbol{w}]$ is in $B(X_0^r, Y_{\varepsilon_0,0}^{r-1})$ for $\varepsilon_0 \in (0,1/2)$, we shall discuss the continuity of $\mathcal{L}_0^{-1}[\boldsymbol{w}]$. For given $\boldsymbol{F} = (f_0, \boldsymbol{f}, \sigma, f_4, f_5, f_6) \in Y_{\varepsilon_0,0}^r$, we begin by eliminating σ, f_4 and f_5 following [2, Theorem 4.1 and Lemma 4.2]. For that purpose, we set a function $\tilde{\boldsymbol{w}}$ to satisfy $\langle t\rangle^{\gamma_2+\varepsilon_0}\tilde{\boldsymbol{w}} \in K^{r-\gamma_2}$ for $\gamma_2 = 0, 3/2$ and

$$\begin{aligned} \nabla_y \cdot \tilde{\boldsymbol{w}} &= \sigma && \text{in } \Omega \\ -\nu(\tilde{w}_{i,y_3} + \tilde{w}_{3,y_i}) &= f_{i+3} && \text{on } \Gamma,\ i=1,2 \\ \tilde{\boldsymbol{w}} &= \boldsymbol{0} && \text{on } S_B \end{aligned}$$
$$\|\langle t\rangle^{\gamma_2+\varepsilon_0}\tilde{\boldsymbol{w}}\|_{K^{r-\gamma_2}} \le c(\|\langle t\rangle^{\gamma_2+\varepsilon_0}\sigma\|_{DK^{r-\gamma_2}} + \|\langle t\rangle^{\gamma_2+\varepsilon_0} f_{i+3}\|_{K^{r-\gamma_2-3/2}}).$$

By virtue of the condition $\varepsilon_0 > 0$, the trace of $\tilde{w}_3$ on Γ belongs to the same space as f_0; in fact, $\langle t\rangle^{\gamma_2+\varepsilon_0}\tilde{w}_3 \in K^{r-\gamma_2}$ for $\gamma_2 \in [0,3/2]$ implies $\langle t\rangle^{\gamma_0}\tilde{w}_3|_\Gamma \in L^1(\mathbb{R}_+; H^{r-1-\gamma_0}) \cap W^{1,1}(\mathbb{R}_+; H^{r-3-\gamma_0})$ for $\gamma_0 \in [0,1]$. We obtain such a function $\tilde{\boldsymbol{w}}$ by setting $\tilde{\boldsymbol{w}} = \nabla_y\phi + \nabla_y \times \boldsymbol{d}$, where $\Delta_y\phi = \sigma$ in Ω, $\phi|_\Gamma = 0$, $\phi_{y_3}|_{S_B} = 0$, and $\boldsymbol{d} = \boldsymbol{d}_{y_3} = \boldsymbol{0}$, $-\nu\boldsymbol{d}_{y_3y_3} = (f_5, -f_4, 0)$ on Γ, $\boldsymbol{d} = \boldsymbol{0}$, $\boldsymbol{d}_{y_3} = (-\phi_{y_2}, \phi_{y_1}, 0)$ on S_B. For the former, we have $\|\nabla\phi\|_{L^2(\Omega)} \le c\|\sigma\|_{{}_0H^{-1}(\Omega)}$ ([2, Theorem 4.1]). And for the latter, $\|\langle t\rangle^{\gamma_2+\varepsilon_0}\nabla_y\boldsymbol{d}\|_{K^s} \le c(\|\langle t\rangle^{\gamma_2+\varepsilon_0} f_{i+3}\|_{K^{s-3/2}} + \|\langle t\rangle^{\gamma_2+\varepsilon_0}\nabla_y\phi\|_{K^s})$ holds for $s > 3/2$ ([2, Lemma 4.2]). Hence we have

$$\mathcal{L}_0[\boldsymbol{w}](0, \tilde{\boldsymbol{w}}, 0) = (-\tilde{w}_3, (\partial_t - \nu\Delta_y)\tilde{\boldsymbol{w}}, \sigma, f_4, f_5, -2\nu\tilde{w}_{3,y_3}) =: \tilde{\boldsymbol{F}},$$

and the estimate $\|\tilde{\boldsymbol{F}}\|_{Y_{\varepsilon_0,0}^r} + \|\tilde{\boldsymbol{w}}\|_{X_0^r} \le c\|(0, \boldsymbol{0}, \sigma, f_4, f_5, 0)\|_{Y_{\varepsilon_0,0}^r}$ since $\varepsilon_0 > 0$. By Theorem 1.3 for $\boldsymbol{F} - \tilde{\boldsymbol{F}} = (f_0 + \tilde{w}_3, \boldsymbol{f} - (\partial_t - \nu\Delta_y)\tilde{\boldsymbol{w}}, 0, 0, 0, f_6 + 2\nu\tilde{w}_{3,y_3}) \in Y_{\varepsilon_0,0}^r$, the solution of $\mathcal{L}_0[\boldsymbol{w}](\tilde{\boldsymbol{z}}) = \boldsymbol{F} - \tilde{\boldsymbol{F}}$ exists and satisfies $\|\tilde{\boldsymbol{z}}\|_{X_0^r} \le c\|\boldsymbol{F} - \tilde{\boldsymbol{F}}\|_{Y_{\varepsilon_0,0}^r}$. Setting $\boldsymbol{z} = \tilde{\boldsymbol{z}} + (0, \tilde{\boldsymbol{w}}, 0)$, the function $\boldsymbol{z}$ satisfies $\mathcal{L}_0[\boldsymbol{w}](\boldsymbol{z}) = \boldsymbol{F}$ and $\|\boldsymbol{z}\|_{X_0^r} \le c\|\boldsymbol{F}\|_{Y_{\varepsilon_0,0}^r}$. □

We shall next discuss the nonlinear terms.

Lemma 5.3. *Let $B_\delta(\boldsymbol{0})$ be a δ-neighborhood of $\boldsymbol{0}$ in X^r. For $\boldsymbol{z} \in B_\delta(\boldsymbol{0})$, the nonlinear terms $\boldsymbol{F}(\boldsymbol{z}) = (f_0, \boldsymbol{f}, \sigma, f_4, f_5, f_6)(\boldsymbol{z})$ given in* (5.15) *satisfy*

$$\|\boldsymbol{F}(\boldsymbol{z})\|_{Y_{\varepsilon_0}^r} \le c_\delta\|\boldsymbol{z}\|_{X^r}^2.$$

Here c_δ depends only on $\delta > 0$.

Proof. We give estimates only for $\sigma(\boldsymbol{z})$ and the term $\nu(\Delta_y - ({}^t d\Theta^{-1}\nabla_y) \cdot ({}^t d\Theta^{-1}\nabla_y))\boldsymbol{u}$ in $\boldsymbol{f}(\boldsymbol{z})$, because in the former estimate, we need the condition $r > 5$ for the product estimates in Sobolev spaces of negative order, and in the latter one, we can see why we need the condition $\langle t\rangle^{-1/2}\eta \in L^\infty(\mathbb{R}_+; H^{r-1/2}(\mathbb{T}^2))$.

We have from (5.15),

$$\sigma(\eta, \boldsymbol{u}) = (1 - J)(u_{1,y_1} + u_{2,y_2}) + \Theta_{3,y_1}u_{1,y_3} + \Theta_{3,y_2}u_{2,y_3},$$

which can be written in a form $(c_{24}\nabla_y\overline{\eta} + c_{25}\overline{\eta})\nabla_y\boldsymbol{u}$, yielding $\langle t\rangle^{\gamma_2+\varepsilon_0}\sigma \in L^2(\mathbb{R}_+; H^{r-1-\gamma_2}(\Omega))$ $(\gamma_2 \in [0,3/2])$. So, we shall show $\langle t\rangle^{\varepsilon_0}\partial_t^2(\nabla_y\overline{\eta}\nabla_y\boldsymbol{u}) \in$

$H^{(r-4)/2}(\mathbb{R}_+; {}_0H^{-1}(\Omega))$ and $\langle t\rangle^{3/2+\varepsilon_0}(\nabla_y\overline{\eta})\nabla_y\boldsymbol{u}_t \in H^{r/2-7/4}(\mathbb{R}_+; {}_0H^{-1}(\Omega))$. As for the former, by Proposition 6.2 (ii) and Remark 6.4, we have for $s_0 > 3/2$ and $s_1 > 1/2$,

$$\|\langle t\rangle^{\varepsilon_0}(\nabla_y\overline{\eta})\nabla_y\boldsymbol{u}_{tt}\|_{H^{(r-4)/2}(\mathbb{R}_+;{}_0H^{-1}(\Omega))} \le c(\|\langle t\rangle^{\varepsilon_0}\eta\|_{H^{s_1}(\mathbb{R}_+;H^{s_0+1/2}(\mathbb{T}^2))}\times \\ \times \|\boldsymbol{u}\|_{H^{r/2}(\mathbb{R}_+;L^2(\Omega))} + \|\langle t\rangle^{\varepsilon_0}\eta\|_{H^{(r-4)/2+s_1}(\mathbb{R}_+;H^{s_0+1/2}(\mathbb{T}^2))}\|\boldsymbol{u}\|_{H^2(\mathbb{R}_+;L^2(\Omega))}).$$

Recalling that $\eta \in H^{r/2-1}(\mathbb{R}_+; H^1(\mathbb{T}^2))$ and $\langle t\rangle^{1/2}\boldsymbol{u} \in H^{r/2-1}(\mathbb{R}_+; H^{3/2}(\Omega))$, we have by Proposition 6.2 (iii) and Proposition 6.3,

$$\|\langle t\rangle^{\varepsilon_0}(\nabla_y\overline{\eta}_t)\nabla_y\boldsymbol{u}_t\|_{H^{(r-4)/2}(\mathbb{R}_+;{}_0H^{-1}(\Omega))} \le c(\|\eta\|_{H^{(r-2)/2}(\mathbb{R}_+;H^{3/4}(\mathbb{T}^2))}\times \\ \times \|\langle t\rangle^{\varepsilon_0}\boldsymbol{u}_t\|_{H^{s_1}(\mathbb{R}_+;H^{5/4}(\Omega))} + \|\eta_t\|_{H^{s_1}(\mathbb{R}_+;H^{3/4}(\mathbb{T}^2))}\|\langle t\rangle^{\varepsilon_0}\boldsymbol{u}\|_{H^{(r-2)/2}(\mathbb{R}_+;H^{5/4}(\Omega))}).$$

Also by Proposition 6.2 (i) and Remark 6.4, we have for $s_0 > 3/2$ and $s_1 > 1/2$,

$$\|\langle t\rangle^{\varepsilon_0}\nabla_y\overline{\eta}_{tt}\nabla_y\boldsymbol{u}\|_{H^{(r-4)/2}(\mathbb{R}_+;{}_0H^{-1}(\Omega))} \le c(\|\langle t\rangle^{\varepsilon_0}\boldsymbol{u}\|_{H^{s_1}(\mathbb{R}_+;H^{s_0+1}(\Omega))}\times \\ \times \|\eta\|_{H^{r/2}(\mathbb{R}_+;L^2(\mathbb{T}^2))} + \|\eta_{tt}\|_{L^2(\mathbb{R}_+;L^2(\mathbb{T}^2))}\|\langle t\rangle^{\varepsilon_0}\boldsymbol{u}\|_{H^{(r-4)/2+s_1}(\mathbb{R}_+;H^{s_0+1}(\Omega))}).$$

We thus have $\|\langle t\rangle^{\varepsilon_0}\partial_t^2(\nabla_y\overline{\eta}\nabla_y\boldsymbol{u})\|_{H^{(r-4)/2}(\mathbb{R}_+;{}_0H^{-1}(\Omega))} \le c\|\boldsymbol{z}\|^2_{X^r}$. Concerning the estimate for $\langle t\rangle^{3/2+\varepsilon_0}(\nabla_y\overline{\eta})\nabla_y\boldsymbol{u}_t \in H^{r/2-7/4}(\mathbb{R}_+; {}_0H^{-1}(\Omega))$, we have

$$\|\langle t\rangle^{3/2+\varepsilon_0}(\nabla_y\overline{\eta})\nabla_y\boldsymbol{u}_t\|_{H^{r/2-7/4}(\mathbb{R}_+;{}_0H^{-1}(\Omega))} \le \\ \le c\{\|\langle t\rangle^{\varepsilon_0+1/2}\eta\|_{H^{s_1}(\mathbb{R}_+;H^{s_0+1/2}(\mathbb{T}^2))}\|\langle t\rangle\boldsymbol{u}\|_{H^{r/2-3/4}(\mathbb{R}_+;L^2(\Omega))}+ \\ + \|\langle t\rangle^{\varepsilon_0+1/2}\eta\|_{H^{r/2-7/4}(\mathbb{R}_+;H^{s_0+1/2}(\mathbb{T}^2))}\|\langle t\rangle\boldsymbol{u}\|_{H^{s_1+1}(\mathbb{R}_+;L^2(\Omega))}\}.$$

For $\varepsilon_0 \in (0, 1/2)$ and $r > 5$, we can choose $s_0 > 3/2$ such that

$$L^2(\mathbb{R}_+; H^{r-2-\varepsilon_0}(\mathbb{T}^2)) \cap H^{(r-1/2-\varepsilon_0)/2}(\mathbb{R}_+; L^2(\mathbb{T}^2)) \subset \\ \subset H^{r/2-7/4}(\mathbb{R}_+; H^{s_0+1/2}(\mathbb{T}^2)).$$

Hence $\|\langle t\rangle^{\varepsilon_0+1/2}\eta\|_{H^{r/2-7/4}(\mathbb{R}_+;H^{1/2+s_0})(\mathbb{T}^2)}$ is bounded if $r > 5$.

We next turn to the estimation of $(\Delta_y - ({}^t d\Theta^{-1}\nabla_y)\cdot({}^t d\Theta^{-1}\nabla_y))\boldsymbol{u}$ in $\boldsymbol{f}(\boldsymbol{z})$, whose m-th component is written as

$$(5.16) \qquad \sum_{j,k,\ell} d\Theta^{-1}_{kj}(\partial_{y_k} d\Theta^{-1}_{\ell j})\partial_{y_\ell}u_m + \sum_{j,k,\ell}(\delta_{k\ell} - d\Theta^{-1}_{kj}d\Theta^{-1}_{\ell j})\partial_{y_k}\partial_{y_\ell}u_m.$$

It is easy to see that

$$(5.17) \quad \|d\Theta^{-1}_{kj}(\partial_{y_k} d\Theta^{-1}_{\ell j})\partial_{y_\ell}u_m\|_{K^{r-2}} \le c_\delta(\|\nabla_y^2\overline{\eta}\nabla_y\boldsymbol{u}\|_{H^{r/2-1}(\mathbb{R}_+;L^2(\Omega))}+ \\ + \|\nabla_y^2\overline{\eta}\nabla_y\boldsymbol{u}\|_{L^2(\mathbb{R}_+;H^{r-2}(\Omega))} + \|\nabla_y\overline{\eta}\nabla_y\boldsymbol{u}\|_{K^{r-2}}).$$

Concerning the first term in (5.17), we have by Proposition 6.3 for $s_0 > 3/2$ and $s_1 > 1/2$,

$$\|\nabla_y^2\overline{\eta}_t\nabla_y\boldsymbol{u}\|_{H^{r/2-2}(\mathbb{R}_+;L^2(\Omega))} \le c(\|\nabla_y^2\overline{\eta}_t\|_{H^{r/2-2}(\mathbb{R}_+;L^2(\Omega))}\|\nabla_y\boldsymbol{u}\|_{H^{s_1}(\mathbb{R}_+;H^{s_0}(\Omega))}+ \\ + \|\nabla_y^2\overline{\eta}_t\|_{H^{s_1}(\mathbb{R}_+;L^2(\Omega))}\|\nabla_y\boldsymbol{u}\|_{H^{r/2-2}(\mathbb{R}_+;H^{s_0}(\Omega))}),$$

$$\|\nabla_y^2\overline{\eta}\nabla_y\boldsymbol{u}_t\|_{H^{r/2-2}(\mathbb{R}_+;L^2(\Omega))} \le c(\|\nabla_y^2\overline{\eta}\|_{H^{r/2-2}(\mathbb{R}_+;H^{1/2}(\Omega))}\|\nabla_y\boldsymbol{u}_t\|_{H^{s_1}(\mathbb{R}_+;H^1(\Omega))}+ \\ + \|\nabla_y^2\overline{\eta}\|_{H^{s_1}(\mathbb{R}_+;H^{1/2}(\Omega))}\|\nabla_y\boldsymbol{u}_t\|_{H^{r/2-2}(\mathbb{R}_+;H^1(\Omega))}).$$

We have for the second term in (5.17),

$$\|\nabla_y^2\overline{\eta}\nabla_y\boldsymbol{u}\|_{L^2(\mathbb{R}_+;H^{r-2}(\Omega))} \le c(\|\langle t\rangle^{-1/2}\eta\|_{L^\infty(\mathbb{R}_+;H^{r-1/2}(\mathbb{T}^2))}\times \\ \times \|\langle t\rangle^{1/2}\boldsymbol{u}\|_{L^2(\mathbb{R}_+;H^{s_0+1}(\Omega))} + \|\eta\|_{L^\infty(\mathbb{R}_+;H^{s_0+3/2}(\mathbb{T}^2))}\|\boldsymbol{u}\|_{L^2(\mathbb{R}_+;H^{r-1}(\Omega))}),$$

so that we have the bounds for K^{r-2}-norm of the first term in (5.16). Other estimates can be done similarly, so we obtain $\|(f_0,\boldsymbol{f},\sigma,f_4,f_5,f_6)(\boldsymbol{z})\|_{Y^r_{\varepsilon_0}} \le c\|\boldsymbol{z}\|^2_{X^r}$ for $\boldsymbol{z}\in B_\delta(\boldsymbol{0})$ and $r\in(5,11/2)$. □

We now define a sequence $\{\boldsymbol{z}^n\}_n = \{(\eta^n,\boldsymbol{u}^n,q^n)\}_n \subset X^r_0$ by $\boldsymbol{z}^0\equiv\boldsymbol{0}$ and

$$\mathcal{L}[\boldsymbol{u}^n+\overline{\boldsymbol{u}_0}](\boldsymbol{z}^{n+1}+\overline{\boldsymbol{z}_0}) = \boldsymbol{F}(\boldsymbol{z}^n+\overline{\boldsymbol{z}_0}).$$

Then, it holds $\boldsymbol{F}(\boldsymbol{z}^n+\overline{\boldsymbol{z}_0})-\mathcal{L}[\boldsymbol{u}^n+\overline{\boldsymbol{u}_0}](\overline{\boldsymbol{z}_0})\in Y^r_{\varepsilon_0,0}$ so that the sequence $\{\boldsymbol{z}^n\}_n$ is well-defined in X^r_0. Therefore by combining Lemma 5.2 and 5.3, we obtain

Lemma 5.4. *There exists $\delta>0$ so that for $\overline{\boldsymbol{z}_0}\in X^r$ with $\|\overline{\boldsymbol{z}_0}\|_{X^r}<\delta$, it holds $\|\boldsymbol{z}^n\|_{X^r_0}<\delta$.*

Remark 5.5. If we do not invoke the term $⨍_\Gamma \eta\nabla'_y\cdot\boldsymbol{w}d\boldsymbol{x}'$ in (1.8), the compatibility relation as in Remark 1.2 requires

$$\int_\Gamma \sigma(\bar{\eta}^n,\boldsymbol{u}^n)d\boldsymbol{y}' = \int_\Gamma \boldsymbol{u}^n\cdot\nabla'_y\eta^{n+1}d\boldsymbol{y}'.$$

So invoking additional terms $⨍_\Gamma \eta^{n+1}\nabla'_y\cdot\boldsymbol{u}^n d\boldsymbol{y}'$ in the left hand side of (1.8) and $⨍_\Gamma \eta^n\nabla'_y\cdot\boldsymbol{u}^n d\boldsymbol{y}'$ in the right helps to preserve the compatibility relation at each iteration step.

We now proceed our main theorem.

Theorem 5.6. *There exists $\delta>0$ so that if $(\eta_0,\boldsymbol{u}_0)$ satisfies*

$$\|\eta_0\|_{H^{r-1/2}(\mathbb{T}^2)} + \|\boldsymbol{u}_0\|_{H^{r-1}(\Omega)} < \delta$$

and if $(\eta_0,\boldsymbol{u}_0\circ\Theta^{-1})$ satisfies (5.1)–(5.5), then the initial boundary value problem (5.8)–(5.15) has a unique solution $\boldsymbol{z}=(\eta,\boldsymbol{u},q)$ in $B_\delta(\boldsymbol{0})\subset X^r$.

Proof. We begin by showing that the approximate sequence $\{\boldsymbol{z}^n\}_n \subset X_0^r$ converges strongly in X_0^{r-1}. In fact, we have

$$\mathcal{L}[\boldsymbol{u}^n + \overline{\boldsymbol{u}_0}](\boldsymbol{z}^{n+1} - \boldsymbol{z}^n) = \boldsymbol{F}(\boldsymbol{z}^n + \overline{\boldsymbol{z}_0}) - \boldsymbol{F}(\boldsymbol{z}^{n-1} + \overline{\boldsymbol{z}_0}) + \boldsymbol{R}(\boldsymbol{z}^n, \boldsymbol{z}^{n-1}),$$

where $\boldsymbol{R}(\boldsymbol{z}^n, \boldsymbol{z}^{n-1}) \in Y_{\varepsilon_0}^{r-1}$ is given by

$$\left((u_i^{n-1} - u_i^n)(\eta^n + \overline{\eta_0})_{y_i} + \fint_\Gamma (u_i^{n-1} - u_i^n)_{y_i}(\eta^n + \overline{\eta_0})d\boldsymbol{y}', \boldsymbol{0}, 0, 0, 0, 0\right).$$

By Lemma 5.7 we have

$$\|\boldsymbol{F}(\boldsymbol{z}^n + \overline{\boldsymbol{z}_0}) - \boldsymbol{F}(\boldsymbol{z}^{n-1} + \overline{\boldsymbol{z}_0}) + \boldsymbol{R}(\boldsymbol{z}^n, \boldsymbol{z}^{n-1})\|_{Y_{\varepsilon_0}^{r-1}} \le c\delta \|\boldsymbol{z}^n - \boldsymbol{z}^{n-1}\|_{X_0^{r-1}}.$$

Choosing $\delta > 0$ so small, we have $\{\boldsymbol{z}^n\}_n$ converges to $\boldsymbol{z}$ strongly in X_0^{r-1}.

We next show that the sequence $\{\boldsymbol{z}^n\}_n$ converges weakly in X_0^r. Since the sequence $\{\boldsymbol{z}^n\}_n$ is uniformly bounded by Lemma 5.4, we have

$$\langle t\rangle^{\gamma_0}\boldsymbol{u}^n \rightharpoonup \langle t\rangle^{\gamma_0}\boldsymbol{u} \text{ weakly in } K^{r-\gamma_0}(\mathbb{R}_+ \times \Omega),$$

and for almost every $t \in \mathbb{R}_+$,

$$\langle t\rangle^{-1/2}\eta^n(t) \rightharpoonup \langle t\rangle^{-1/2}\eta(t) \text{ weakly in } H^{r-1/2}(\mathbb{T}^2),$$

so we obtain $\langle t\rangle^{-1/2}\eta \in L^\infty(\mathbb{R}_+; H^{r-1/2}(\mathbb{T}^2))$. To improve this to $\langle t\rangle^{-1/2}\eta \in B(\mathbb{R}_+; H^{r-1/2}(\mathbb{T}^2))$, we are suffice to check that $\|\eta(t)\|_{H^{r-1/2}(\mathbb{T}^2)}$ is a continuous function of t. Such arguments are found in [17, Prop. 5.1D], so we will skip them here. Other convergences can be also obtained in similar discussions. □

Lemma 5.7. *Let $B_\delta(\boldsymbol{0})$ be a δ-neighborhood of $\boldsymbol{0}$ in X^r, then we have*

$$\|\boldsymbol{F}(\boldsymbol{z}) - \boldsymbol{F}(\boldsymbol{z}')\|_{Y_{\varepsilon_0}^{r-1}} \le c\delta\|\boldsymbol{z} - \boldsymbol{z}'\|_{X^{r-1}}, \tag{5.18}$$

$$\|\boldsymbol{R}(\boldsymbol{z}, \boldsymbol{z}')\|_{Y_{\varepsilon_0}^{r-1}} \le c\delta\|\boldsymbol{z} - \boldsymbol{z}'\|_{X^{r-1}} \tag{5.19}$$

for $\boldsymbol{z}, \boldsymbol{z}' \in B_\delta(\boldsymbol{0})$. Here c does not depend on $\delta > 0$.

We omit this proof because it is similar to the one of Lemma 5.3.

We now wish to obtain the solution in the physical domain by reversing our steps. We first extend the functions $(\boldsymbol{u}, q)$ defined in the equilibrium domain to $\mathbb{T}^2 \times \mathbb{R}$ in a standard way preserving the regularities and the decay rates. Then, the velocity and the pressure in the physical domain are the restriction of $\boldsymbol{u} \circ \Theta^{-1}$, and $q \circ \Theta^{-1}$ to $\{\Theta[t](\boldsymbol{y}) : \boldsymbol{y} \in \Omega\}$. Since η is in $C^0(\mathbb{R}_+; H^{r-1}(\mathbb{T}^2))$, its extension $\overline{\eta}$ is in $C^0(\mathbb{R}_+; C^1(\mathbb{T}^2 \times \mathbb{R}))$ so that Θ is a C^1-diffeomorphism provided that η is small in its norm. The following proposition ensures the regularities of the composition of mapping Θ and its inverse Θ^{-1} on $\mathbb{T}^2 \times \mathbb{R}$.

Proposition 5.8. *Let $(0, \boldsymbol{v}, p)$ and $(0, \boldsymbol{u}, q)$ belong to X^r. With Θ as* (5.7)*, the compositions of $(0, \boldsymbol{v}, p)$ and Θ, or $(0, \boldsymbol{u}, q)$ and Θ^{-1} are bounded:*

$$\|(0, \boldsymbol{v}, p) \circ \Theta\|_{X^r} \le c\|(0, \boldsymbol{v}, p)\|_{X^r}, \ \|(0, \boldsymbol{u}, q) \circ \Theta^{-1}\|_{X^r} \le c\|(0, \boldsymbol{u}, q)\|_{X^r},$$

where the constant c depends on $\|(\eta, 0, 0)\|_{X^r}$ and is bounded provided that η is small in its norm.

Proof. We begin by showing $\|\boldsymbol{u}\circ\Theta\|_{L^2(\mathbb{R};H^r(\mathbb{T}^2\times\mathbb{R}))} \le c\|\boldsymbol{u}\|_{L^2(\mathbb{R};H^r(\mathbb{T}^2\times\mathbb{R}))}$. It is easy to see

$$\|\boldsymbol{u}\circ\Theta\|^2_{L^2(\mathbb{T}^2\times\mathbb{R})} \le \|J^{-1}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R})}\|\boldsymbol{u}\|^2_{L^2(\mathbb{T}^2\times\mathbb{R})}$$
$$\|\boldsymbol{u}\circ\Theta\|^2_{H^1(\mathbb{T}^2\times\mathbb{R})} \le \|J^{-1}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R})}(1+\|\Theta\|^2_{W^{1,\infty}(\mathbb{T}^2\times\mathbb{R})})\|\boldsymbol{u}\|^2_{H^1(\mathbb{T}^2\times\mathbb{R})}.$$

Since the mapping $\boldsymbol{u}\mapsto\boldsymbol{u}\circ\Theta$ is bounded linear on $L^2(\mathbb{T}^2\times\mathbb{R})$ and on $H^1(\mathbb{T}^2\times\mathbb{R})$, and the complex interpolation between them is exact ([5, Theorem 4.1.2.]), we have for $s\in(0,1)$

$$\|\boldsymbol{u}\circ\Theta\|^2_{H^s(\mathbb{T}^2\times\mathbb{R})} \le \|J^{-1}\|_{L^\infty(\mathbb{T}^2\times\mathbb{R})}(1+\|\Theta\|^2_{W^{1,\infty}(\mathbb{T}^2\times\mathbb{R})})^s\|\boldsymbol{u}\|^2_{H^s(\mathbb{T}^2\times\mathbb{R})}. \tag{5.20}$$

If we expand $\nabla_x^j(\boldsymbol{u}\circ\theta)$ $(j=4,5)$, and plug them into (5.20), we obtain for $\gamma_0\in[0,1]$

$$\|\langle t\rangle^{\gamma_0}\boldsymbol{u}\circ\Theta\|_{L^2(\mathbb{R}_+;H^{r-\gamma_0})} \le c\|\langle t\rangle^{\gamma_0}\boldsymbol{u}\|_{L^2(\mathbb{R}_+;H^{r-\gamma_0})}.$$

As for temporal regularities, differentiating $\boldsymbol{u}\circ\Theta$ in t, we have $\|\boldsymbol{u}\circ\Theta\|_{H^j(\mathbb{R}_+;L^2)} \le c\|\boldsymbol{u}\|_{K^{2j}}, (j=0,1,2)$ and $\|\boldsymbol{u}\circ\Theta\|_{H^{1+s}(\mathbb{R}_+;L^2)} \le c\|\boldsymbol{u}\|_{K^{2(1+s)}}$ for $s\in(0,1)$ by interpolation. We also obtain $\|\partial_t(\boldsymbol{u}\circ\Theta)\|_{H^{1+s}(\mathbb{R}_+;L^2)} \le c\|\boldsymbol{u}\|_{K^{2(2+s)}}$. Weighted estimates and estimates of compositions with Θ^{-1} can be also obtained in a similar manner. □

6. Appendix

In this section, we give propositions and a lemma which are not given before the previous section. We begin with the following lemma concerning complex interpolation results. For the definition of complex interpolation, we refer to [5, Chapter 4] or [20, 1.6.6].

Lemma 6.1. *Let $s_j\ge 0$ $(j=0,1)$, $\theta\in(0,1)$, and $s=(1-\theta)s_0+\theta s_1$.*

(i) *The complex interpolation $[K^{s_0}(\mathbb{R}_+\times\Omega), K^{s_1}(\mathbb{R}_+\times\Omega)]_\theta = K^s(\mathbb{R}_+\times\Omega)$ holds.*

(ii) *Let $r_j\in\mathbb{R}$ $(j=0,1)$. Then it holds $[\langle t\rangle^{r_0}K^{s_0}, \langle t\rangle^{r_1}K^{s_1}]_\theta = \langle t\rangle^r K^s$ for $r=(1-\theta)r_0+\theta r_1$.*

Proof. (i) We give a proof replacing Ω by $\mathbb{T}^2\times\mathbb{R}$ because we can easily rewrite its counterpart for Ω by extending and restricting arguments. By setting $\hat{\boldsymbol{u}}(\lambda,\boldsymbol{\xi}',\xi_3) = \int_0^\infty dt\int_{\mathbb{R}} dx_3\int_{\mathbb{T}^2} e^{\lambda t + i\boldsymbol{x}\cdot\boldsymbol{\xi}}\boldsymbol{u}(t,\boldsymbol{x})d\boldsymbol{x}'$ for Re $\lambda\le 0$ and $\boldsymbol{\xi} = (\boldsymbol{\xi}',\xi_3)\in\mathbb{Z}^2\times\mathbb{R}$, we assume the function space $K^s(\mathbb{R}_+\times(\mathbb{T}^2\times\mathbb{R}))$ to be equipped with the norm $\|\boldsymbol{u}\|^2_{K^s} = \int_{-i\infty}^{i\infty} d\lambda\int_{\mathbb{R}} d\xi_3\sum_{\boldsymbol{\xi}'\in\mathbb{Z}^2}(1+|\boldsymbol{\xi}|^2+|\lambda|)^s|\hat{\boldsymbol{u}}(\lambda,\boldsymbol{\xi}',\xi_3)|^2$. For a Banach couple (A_0,A_1), a strip $S=\{z\in\mathbb{C} : \text{Re } z\in[0,1]\}$ and an open strip $S_0=\{z\in\mathbb{C} : \text{Re } z\in(0,1)\}$, we define

$$\mathcal{F}(A_0,A_1) := \{f : \text{bounded continuous function on } S, \text{analytic in } S_0,$$
$$f(j+i\cdot)\in C(\mathbb{R};A_j) \text{ and } f(j+it)\to 0 \text{ as } |t|\to\infty\ (j=0,1)\}.$$

Then $\Lambda(\theta) := (1+|\boldsymbol{\xi}|^2+|\lambda|)^{-s_0(1-\theta)-s_1\theta}$ is an isometry isomorphism from $\mathcal{F}(K^0,K^0)$ onto $\mathcal{F}(K^{s_0},K^{s_1})$. Hence we obtain

$$[K^{s_0},K^{s_1}]_\theta = \Lambda(\theta)[L^2(\mathbb{R}_+\times\mathbb{T}^2\times\mathbb{R}), L^2(\mathbb{R}_+\times\mathbb{T}^2\times\mathbb{R})]_\theta = K^s.$$

(ii) $\Lambda(\theta)$ in (i) is also an isometry isomorphism from $\mathcal{F}(\langle t\rangle^{r_0}K^0,\langle t\rangle^{r_1}K^0)$ onto $\mathcal{F}(\langle t\rangle^{r_0}K^{s_0},\langle t\rangle^{r_1}K^{s_1})$. Since $[\langle t\rangle^{r_0}K^0,\langle t\rangle^{r_1}K^0]_\theta = \langle t\rangle^{r}K^0$, we have the desired result. □

We next give two propositions concerning product estimates.

Proposition 6.2. *Let $s>3/2$.*
(i) *Let $\nabla\boldsymbol{v}\in H^s(\Omega)$. We set $\bar\eta(\boldsymbol{x}',x_3)=\sum_{\boldsymbol{\xi}'\in\mathbb{Z}^2} e^{ix'\cdot\xi'}e^{|\xi'|x_3}\hat\eta(\xi')$ for $\eta\in H^{1/2}(\mathbb{T}^2)$. Then*

$$\|(\nabla\overline{\eta})\nabla\boldsymbol{v}\|_{{}_0H^{-1}(\Omega)} \le c\|\overline{\eta}\|_{L^2(\Omega)}\|\nabla\boldsymbol{v}\|_{H^s(\Omega)}.$$

(ii) *For $\nabla\overline{\eta}\in H^s(\Omega)$ and $\boldsymbol{v}\in {}_0H^1(\Omega)=\{\boldsymbol{u}\in H^1(\Omega) : \boldsymbol{u}=\boldsymbol{0}$ on $S_B\}$, we have*

$$\|(\nabla\overline{\eta})\nabla\boldsymbol{v}\|_{{}_0H^{-1}(\Omega)} \le c\|\nabla\overline{\eta}\|_{H^s(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)}.$$

(iii)*For $\nabla\overline{\eta}\in H^{1/4}(\Omega)$ and $\nabla\boldsymbol{v}\in H^{1/4}(\Omega)$, we have*

$$\|(\nabla\overline{\eta})\nabla\boldsymbol{v}\|_{{}_0H^{-1}(\Omega)} \le c\|\nabla\overline{\eta}\|_{H^{1/4}(\Omega)}\|\nabla\boldsymbol{v}\|_{H^{1/4}(\Omega)}.$$

Proof. For (i), regarding multiplication by $\nabla\boldsymbol{v}$ on ${}_0H^{-1}(\Omega)$ as the adjoint multiplication on ${}^0H^1(\Omega)$, we are enough to show $\|\nabla\overline{\eta}\|_{{}_0H^{-1}(\Omega)}\le c\|\overline{\eta}\|_{L^2(\Omega)}$. Since $\nabla\overline{\eta}\in L^2(\Omega)$, we have for any vector function $\varphi\in {}^0H^1(\Omega)$,

$$\begin{aligned}\langle\nabla\overline{\eta},\varphi\rangle_{{}^0H^1(\Omega)} &= (\nabla\overline{\eta},\varphi)_{L^2(\Omega)}\\ &= (\widehat{\overline{\eta}}, {}^t(i\boldsymbol{\xi}',|\boldsymbol{\xi}'|)\cdot\hat\varphi)_{\ell^2(\mathbb{Z}^2\,;\,L^2(-1,0))}\\ &\le c\|\overline{\eta}\|_{L^2(\Omega)}\|\varphi\|_{{}^0H^1(\Omega)}.\end{aligned}$$

This gives (i). For (ii), if we regard multiplication by $\nabla\overline{\eta}$ on ${}_0H^{-1}(\Omega)$ as the adjoint multiplication on ${}^0H^1(\Omega)$, it can be verified by $\|(\nabla\overline{\eta})\varphi\|_{{}^0H^1(\Omega)}\le c\|\nabla\overline{\eta}\|_{H^s(\Omega)}\|\varphi\|_{{}^0H^1(\Omega)}$. The proof for (iii) is easy since the compact embeddings ${}^0H^1(\Omega)\subset L^6(\Omega)$ and $H^{1/4}(\Omega)\subset L^{12/5}(\Omega)$ hold. □

Proposition 6.3. *Let $s\in(0,1)$. Assume that X_i $(i=0,1,2)$ are B-spaces having a property*: $\|fg\|_{X_0}\le c\|f\|_{X_1}\|g\|_{X_2}$ *for all $f\in X_1(\Omega)$ and $g\in X_2(\Omega)$. Then, the inequality*

$$\|fg\|_{H^s(\mathbb{R}_+;X_0)} \le c(\|f\|_{H^s(\mathbb{R}_+;X_1)}\|g\|_{H^r(\mathbb{R}_+;X_2)} + \|f\|_{H^r(\mathbb{R}_+;X_1)}\|g\|_{H^s(\mathbb{R}_+;X_2)}).$$

holds for $r>1/2, f\in H^{s_0}(\mathbb{R}_+;X_1)$, and $g\in H^{s_0}(\mathbb{R}_+;X_2)$ with $s_0=\max(s,r)$.

Proof. After replacing $\mathbb{R}_+$ by $\mathbb{R}$ by standard extension-restriction arguments, we define a norm of $H^s(\mathbb{R}\,;X_0)$ by

$$\|fg\|^2_{H^s(\mathbb{R}\,;\,X_0)} = \|fg\|^2_{L^2(\mathbb{R}\,;\,X_0)} + \int_{\mathbb{R}^2}\frac{\|fg(\tau)-fg(t)\|^2_{X_0}}{|\tau-t|^{2s+1}}d\tau dt.$$

The second term is bounded by

$$\|f\|^2_{L^\infty(\mathbb{R}\,;\,X_1)} \int_{\mathbb{R}^2} \frac{\|g(\tau)-g(t)\|^2_{X_2}}{|\tau-t|^{2s+1}} d\tau dt + \int_{\mathbb{R}^2} \frac{\|f(\tau)-f(t)\|^2_{X_1}}{|\tau-t|^{2s+1}} d\tau dt\, \|g\|^2_{L^\infty(\mathbb{R}\,;\,X_2)}$$
$$\le \|f\|^2_{L^\infty(\mathbb{R}\,;\,X_1)} \|g\|^2_{H^s(\mathbb{R}\,;\,X_2)} + \|f\|^2_{H^s(\mathbb{R}\,;\,X_1)} \|g\|^2_{L^\infty(\mathbb{R}\,;\,X_2)},$$

which completes the proof. □

Remark 6.4. By the same arguments, we have

$$\|fg\|_{H^s(\mathbb{R}_+\,;\,X_0)} \le c(\|f\|_{H^s(\mathbb{R}_+;X_1)}\|g\|_{H^r(\mathbb{R}_+;X_2)} + \|f\|_{L^2(\mathbb{R}_+;X_1)}\|g\|_{H^{s+r}(\mathbb{R}_+;X_2)}),$$

for $r > 1/2$, because

$$\int_{\mathbb{R}^2} \|f(t)\|^2_{X_1} \frac{\|g(\tau)-g(t)\|^2_{X_2}}{|\tau-t|^{2s+1}} d\tau dt \le$$
$$\le c\Bigg(\sup_{\{(\tau,t)\in\mathbb{R}^2 \,|\, \tau\neq t\}} \frac{\|g(\tau)-g(t)\|^2_{X_2}}{|\tau-t|^{2(s+\varepsilon)}} \int_{\mathbb{R}} \|f(t)\|^2_{X_1} dt \int_{|\tau-t|<1} \frac{d\tau}{|\tau-t|^{1-2\varepsilon}} +$$
$$+ \sup_{\{(\tau,t)\in\mathbb{R}^2 \,|\, \tau\neq t\}} \frac{\|g(\tau)-g(t)\|^2_{X_2}}{|\tau-t|^{2(s-\varepsilon)}} \int_{\mathbb{R}} \|f(t)\|^2_{X_1} dt \int_{|\tau-t|>1} \frac{d\tau}{|\tau-t|^{1+2\varepsilon}} \Bigg)$$
$$\le c\|f\|^2_{L^2(\mathbb{R}\,;\,X_1)} \|g\|^2_{C^{s+\varepsilon}(\mathbb{R}\,;\,X_2)}.$$

□

Graduate School of Science and Technology
Yamaguchi University
Yamaguchi 753-8512, Japan
e-mail: hataya@yamaguchi-u.ac.jp

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