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On a result of H. Fujimoto

By

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Abstract

Let $P(\omega)$ be a uniqueness polynomial of degree q without multiple zeros whose derivative has mutually distinct k zeros d_l with multiplicities q_l for $l = 1, 2, \ldots, k$ respectively, and let $S := \{a_1, a_2, \cdots, a_q\}$ be the zero set of $P(\omega)$. Under the assumption that $P(d_{l_s}) \neq P(d_{l_t})$ $(1 \leq l_s < l_t \leq k)$, we give some sufficient conditions for the set S to be a unique range set with some weak value-sharing hypothesis, namely, to satisfy the condition that $\sum_{j=1}^{q} \nu_{f,m_0)}^{a_j} \equiv \sum_{j=1}^{q} \nu_{g,m_0)}^{a_j}$ $\big(m_0 \in \mathbb{Z}^+ \cup \{\infty\}\big)$ implies $f \equiv g$ for any two nonconstant meromorphic or entire functions f and g on $\mathbb{C}$, which improve a result of H. Fujimoto. Also, we discuss some other related topics.

1. Introduction and main results

In this paper, by a meromorphic function we mean a meromorphic function in the complex plane $\mathbb{C}$, and by a divisor we mean a map $\nu : \mathbb{C} \to \mathbb{Z}$ whose support $\overline{\{z|\nu(z) \neq 0\}}$ is discrete. For a divisor ν, we set $\bar{\nu}(z) := \min\{\nu(z), 1\}$. For a nonconstant meromorphic function f and a value $a \in \mathbb{C}$, we define the divisor $\nu_f^a : \mathbb{C} \to \mathbb{Z}$ by

$$\nu_f^a(z) := \begin{cases} 0 & \text{if } f(z) \neq a, \\ m & \text{if } f(z) - a = 0 \text{ with multiplicity } m; \end{cases}$$

and set $\nu_f^\infty := \nu_{1/f}^0$. We call a discrete subset $S \subset \mathbb{C}$ a unique range set for meromorphic (resp. entire) functions counting multiplicities, i.e., URSM (resp. URSE) if for any two nonconstant meromorphic (resp. entire) functions f and g, the condition that $\sum_{a\in S} \nu_f^a \equiv \sum_{a\in S} \nu_g^a$ implies $f \equiv g$; and a unique range set for meromorphic (resp. entire) functions ignoring multiplicities, i.e., URSM-IM (resp. URSE-IM) if it has the analogous property for which the condition that $\sum_{a\in S} \nu_f^a \equiv \sum_{a\in S} \nu_g^a$ is replaced by $\sum_{a\in S} \bar{\nu}_f^a \equiv \sum_{a\in S} \bar{\nu}_g^a$. Particularly, if S

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consists of only one element, these definitions coincide with those of CM or IM shared values.

In 1982, F. Gross and C. C. Yang showed that the set $S := \{\omega \in \mathbb{C} | \omega + e^{\omega} = 0\}$ is a URSE. Afterwards, many efforts were made to seek unique range sets that their cardinality are as small as possible (*cf.* S. Bartels [1], G. Frank-M. Reinders [2], H. Fujimoto [3, 4], P. Li-C. C. Yang [6, 7], E. Mues-M. Reinders [8], Y. Xu [9] and H. X. Yi [11–13]).

Now, we turn to some necessary terminologies and notations. It is assumed that the readers are familiar with those standard ones such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting function $N(r, f)$ of poles for a meromorphic function f, and the fundamental results in Nevanlinna's value distribution theory of meromorphic functions (*cf.* W. K. Hayman [5], L. Yang [10]). Also, we use $S(r, f)$ to denote a term satisfying $S(r, f) = o(T(r, f))$ as r tends to infinity outside a subset $E \subset \mathbb{R}^+$ with $\int_E dr < +\infty$.

The counting function of a divisor $\nu : \mathbb{C} \to \mathbb{Z}$ is defined as

$$N(r, \nu) := \int_0^r \left(\sum_{0<|z|\leq t} \nu(z) \right) \frac{dt}{t} + \nu(0) \log r.$$

Under this notation, $N(r, f)$ could be rewritten as $N(r, \nu_f^{\infty})$.

H. Fujimoto [3, 4] gave his definition of uniqueness polynomials in a broad sense and uniqueness polynomials as follows

Definition 1.1. Let $P(\omega)$ be a nonconstant monic polynomial. We call $P(\omega)$ a uniqueness polynomial in a broad sense if $P(f) \equiv P(g)$ implies $f \equiv g$ for any two nonconstant meromorphic functions f and g; while a uniqueness polynomial if $P(f) \equiv cP(g)$ implies $f \equiv g$ for any two nonconstant meromorphic functions f and g, and any nonzero constant $c \in \mathbb{C}$.

For a discrete subset $S := \{a_1, a_2, \cdots, a_q\} \subset \mathbb{C}$, we consider its generated polynomial with the following form

$$P(\omega) := (\omega - a_1)(\omega - a_2) \cdots (\omega - a_q). \tag{1.1}$$

Assume that the derivative of $P(\omega)$ has mutually distinct k zeros $d_1, d_2, \cdots, d_k$ with multiplicities $q_1, q_2, \cdots, q_k$, respectively. Under the assumption that

$$P(d_{l_s}) \neq P(d_{l_t}) \quad (1 \leq l_s < l_t \leq k), \tag{1.2}$$

H. Fujimoto [3, 4] proved the following two theorems that enable us to find out many uniqueness polynomials in a broad sense and uniqueness polynomials.

Theorem 1.2 ([4, p. 35])**.** *Let $P(\omega)$ be a polynomial of the form* (1.1) *satisfying the condition* (1.2). *Then, $P(\omega)$ is a uniqueness polynomial in a broad sense if and only if*

$$\sum_{1 \leq l_s < l_t \leq k} q_{l_s} q_{l_t} > \sum_{l_s=1}^{k} q_{l_s}. \tag{1.3}$$

In particular, (1.3) *is always satisfied whenever* $k \geq 4$. *Also,* (1.3) *holds whenever* $\max\{q_1,\ q_2,\ q_3\} \geq 2$ *for the case* $k = 3$, *and whenever* $\min\{q_1,\ q_2\} \geq 2$ *and* $q_1 + q_2 \geq 5$ *for the case* $k = 2$.

Moreover, he gave an example to show that any polynomial can not be a uniqueness polynomial in a broad sense for the case $k = 1$ (*cf.* [3, p. 1183]) and hence, the conclusions of Theorem 1.2 is beat possible.

Theorem 1.3 ([3, p. 1192] or [4, pp. 34, 35 and 44])**.** *Under the same situation as in Theorem* 1.2, *we assume furthermore that* $P(d_1)+P(d_2)+\cdots+P(d_k) \neq 0$ *for the cases* $k \geq 4$; *that* $P(d_{l_s}) \pm P(d_{l_t}) \neq 0$ *with* $1 \leq l_s < l_t \leq 3$, *and* $P^2(d_{l_r}) \neq P(d_{l_s})P(d_{l_t})$ *with* $\{l_r, l_s, l_t\} = \{1, 2, 3\}$ *for the case* $k = 3$; *and that* $q_1 \leq q_2$ *with* $q_1 \geq 2$ *and* $q_2 \geq q_1 + 3$ *or with* $q_1 \geq 3$ *and* $P(d_1) + P(d_2) \neq 0$ *for the case* $k = 2$. *Then,* $P(\omega)$ *is a uniqueness polynomial.*

Moreover, he investigated the relationship between a uniqueness polynomial and its zero set as a unique range set, and obtained the following uniqueness theorem that generalized the main results of S. Bartels [1] and G. Frank-M. Reinders [2].

Theorem 1.4 ([3, p. 1177])**.** *Suppose that* $P(\omega)$ *is a uniqueness polynomial of form* (1.1) *satisfying condition* (1.2), *and that* $k \geq 3$, *or* $k = 2$ *and* $\min\{q_1,\ q_2\} \geq 2$. *Then,* S *is a URSM* (*resp. URSE*) *whenever* $q > 2k + 6$ (*resp.* $q > 2k+2$), *while a URSM-IM* (*resp. URSE-IM*) *whenever* $q > 2k+12$ (*resp.* $q > 2k + 5$).

For the sake of convenience, we introduce the following notation. For a divisor ν, we define the m_0-truncated divisor $\nu_{m_0)}$ generated by ν as

$$\nu_{m_0)}(z) := \begin{cases} \nu(z), & \text{if } \nu(z) \leq m_0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

The purpose of this paper is to give some sufficient conditions for the set S to be a unique range set, which improve and extend Theorem 1.4 through weaker value-sharing hypothesis or smaller lower bounds for q.

Theorem 1.5. *Under the same situation as in Theorem* 1.4, *we assume furthermore that* $m_0 \geq 3$ *is a positive integer or infinity, and that* $q > 2k + 6$ (*resp.* $q > 2k + 2$). *Then, to satisfy the condition that* $\sum_{j=1}^{q} \nu_{f,m_0)}^{a_j} \equiv \sum_{j=1}^{q} \nu_{g,m_0)}^{a_j}$ *implies* $f \equiv g$ *for any two nonconstant meromorphic* (*resp. entire*) *functions* f *and* g.

Remark. The condition that $\sum_{j=1}^{q} \min\{\nu_f^{a_j}(z), m_0\} \equiv \sum_{j=1}^{q} \min\{\nu_g^{a_j}(z),\ m_0\}$ in [3, p. 1195] implies that $\sum_{j=1}^{q} \bar{\nu}_f^{a_j} \equiv \sum_{j=1}^{q} \bar{\nu}_g^{a_j}$, $\sum_{j=1}^{q} \nu_{f,m_0-1)}^{a_j} \equiv \sum_{j=1}^{q} \nu_{g,m_0-1)}^{a_j}$ and $\sum_{j=1}^{q} \nu_f^{a_j} \equiv \sum_{j=1}^{q} \nu_g^{a_j} \equiv m_0$ whenever $\min\{\nu_f^{a_i}(z),\ \nu_g^{a_j}(z)\} \geq m_0$ for some $i,\ j \in \{1,\ 2,\ \ldots,\ q\}$ and a positive integer m_0; while $\sum_{j=1}^{q} \nu_f^{a_j} \equiv \sum_{j=1}^{q} \nu_g^{a_j}$ implies $\sum_{j=1}^{q} \nu_{f,\infty)}^{a_j} \equiv \sum_{j=1}^{q} \nu_{g,\infty)}^{a_j}$ for $m_0 = \infty$. So, the value-sharing assumption in Theorem 1.5 is weaker than that in Theorem 1.4 for cases of URSM (resp. URSE).

Theorem 1.6. *Under the same situation as in Theorem* 1.4, *we assume furthermore that the union of the sets of the double a_j-points of f for $j = 1$, 2, ..., q is the same as that of g. Then, S is a URSM-IM (resp. URSE-IM) whenever $q > 2k + 9$ (resp. $q > 2k + \frac{7}{2}$).*

2. Preliminary lemmas

The following the first lemma plays quite an important role in studying the relationship between a uniqueness polynomial and its zero set as a unique range set, while the second one is used for controlling the growth of zeros of the derivative of a meromorphic function by that of its own zeros and poles.

Lemma 2.1 ([3, p. 1200]). *Under the same situation as in Theorem* 1.2, *we assume furthermore that $q \geq 5$ and there are two meromorphic function f and g such that, and that*

$$\frac{1}{P(f)} \equiv \frac{c_0}{P(g)} + c_1 \tag{2.1}$$

for any two constants $c_0 (\neq 0)$ and c_1. If $k \geq 3$ or if $k = 2$ and $\min\{q_1, q_2\} \geq 2$, then, $c_1 = 0$.

Lemma 2.2 ([13, p. 379]). *For any nonconstant meromorphic function f,*

$$N(r, \nu^0_{f'}) \leq N(r, \nu^0_f) + N(r, \bar{\nu}^\infty_f) + S(r, f). \tag{2.2}$$

3. Proof of Theorem 1.5

Proof. Set $F := P(f)$ and $G := P(g)$; then the condition $\sum_{j=1}^q \nu^{a_j}_{f,m_0)} \equiv \sum_{j=1}^q \nu^{a_j}_{g,m_0)}$ implies $\nu^0_{F,m_0)} \equiv \nu^0_{G,m_0)}$. By Lemma 1 in [9, p. 1490], $T(r, F) = qT(r, f) + O(1)$ and $T(r, G) = qT(r, g) + O(1)$, and hence $S(r, F) = S(r, f)$ and $S(r, G) = S(r, g)$, since F and f, and G and g have the same growth estimates, respectively.

Define the function H as

$$H := \left(\frac{F''}{F'} - 2\frac{F'}{F}\right) - \left(\frac{G''}{G'} - 2\frac{G'}{G}\right).$$

Suppose that $H \not\equiv 0$. By *the lemma on logarithmic derivative* (*cf.* [5, p. 36–42]),

$$m(r, H) = S(r)\big(:= S(r, f) + S(r, g)\big).$$

For any common simple zero z_0 of F and G, by the local Laurent expansions of F and G around z_0, we have $H(z_0) = 0$, and hence

$$\begin{aligned} N(r, \bar{\nu}^0_{F,1)}) &= N(r, \bar{\nu}^0_{G,1)}) \leq N(r, \nu^0_H) + O(1) \\ &\leq T(r, H) + O(1) \leq N(r, \bar{\nu}^\infty_H) + S(r), \end{aligned} \tag{3.1}$$

since all the poles of H are simple.

By our assumption, we know that $F' = q(f-d_1)^{q_1}(f-d_2)^{q_2}\cdots(f-d_k)^{q_k}f'$ and $G' = q(g-d_1)^{q_1}(g-d_2)^{q_2}\cdots(g-d_k)^{q_k}g'$. From the expression of H and the fact that $\nu^0_{F,m_0)} \equiv \nu^0_{G,m_0)}$, we have

(3.2)
$$N(r,\bar\nu_H^\infty) \le \sum_{l=1}^{k}\big(N(r,\bar\nu_f^{d_l}) + N(r,\bar\nu_g^{d_l})\big) + N(r,\bar\nu^0_{F,(m_0+1}) + N(r,\bar\nu^0_{G,(m_0+1}) + N(r,\bar\nu_f^\infty) + N(r,\bar\nu_g^\infty) + N(r,\bar{\hat\nu}^0_{f'}) + N(r,\bar{\hat\nu}^0_{g'}) + O(1),$$

where $\nu^0_{F,(m_0+1}$ denotes the subtraction of the two divisors $\nu^0_F - \nu^0_{F,m_0)}$ and $\hat\nu^0_{f'} := \nu^0_{f'}\big|_{\prod_{j=1}^q(f-a_j)\cdot\prod_{l=1}^k(f-d_l)\neq 0}$, and $\nu^0_{G,(m_0+1}$ and $\hat\nu^0_{g'}$ are similarly defined.

Obviously, $a_j \neq d_l$ ($j = 1, 2, \ldots, q$; $l = 1, 2, \ldots, k$) since all the a_j's are distinct for $j = 1, 2, \ldots, q$. Then, applying the second main theorem to the functions f, g and the values $a_1, a_2, \cdots, a_q, d_1, d_2, \cdots, d_k, \infty$, respectively to conclude

(3.3)
$$\begin{aligned}(q+k-1)T(r)\big(:= T(r,f)+T(r,g)\big) &\le N(r,\bar\nu_f^\infty) + N(r,\bar\nu_g^\infty) + N(r,\bar\nu^0_F) \\ &\quad + N(r,\bar\nu^0_G) + \sum_{l=1}^{k}\big(N(r,\bar\nu_f^{d_l}) + N(r,\bar\nu_g^{d_l})\big) - N(r,\hat\nu^0_{f'}) - N(r,\hat\nu^0_{g'}) + S(r) \\ &= N(r,\bar\nu_f^\infty) + N(r,\bar\nu_g^\infty) + \sum_{l=1}^{k}\big(N(r,\bar\nu_f^{d_l}) + N(r,\bar\nu_g^{d_l})\big) + N(r,\bar\nu^0_{F,1)}) \\ &\quad + N(r,\bar\nu^0_F) + N(r,\bar\nu^0_G) - N(r,\bar\nu^0_{F,1)}) - N(r,\hat\nu^0_{f'}) - N(r,\hat\nu^0_{g'}) + S(r).\end{aligned}$$

Noting that $\nu^0_{F,m_0)} \equiv \nu^0_{G,m_0)}$, we have the following

(3.4)
$$\begin{aligned}&N(r,\bar\nu^0_F) + N(r,\bar\nu^0_G) + N(r,\bar\nu^0_{F,(m_0+1}) + N(r,\bar\nu^0_{G,(m_0+1}) - N(r,\bar\nu^0_{F,1)}) \\ &\le N(r,\bar\nu^0_F) + N(r,\bar\nu^0_{F,(m_0+1}) - \frac{1}{2}N(r,\bar\nu^0_{F,1)}) + N(r,\bar\nu^0_G) + N(r,\bar\nu^0_{G,(m_0+1}) \\ &\quad - \frac{1}{2}N(r,\bar\nu^0_{G,1)}) \le \frac{1}{2}N(r,\nu^0_F) + \frac{1}{2}N(r,\nu^0_G) + O(1) \le \frac{q}{2}T(r) + O(1)\end{aligned}$$

if $m_0 \ge 3$ is a positive integer or infinity.

Substituting (3.1), (3.2) into (3.3) and noting (3.4), we have

$$\begin{aligned}(q+k-1)T(r) &\le 2\big(N(r,\bar\nu_f^\infty) + N(r,\bar\nu_g^\infty)\big) \\ &\quad + 2\sum_{l=1}^{k}\big(N(r,\bar\nu_f^{d_l}) + N(r,\bar\nu_g^{d_l})\big) + \frac{q}{2}T(r) + S(r),\end{aligned} \tag{3.5}$$

which implies that $(q-2k-6)T(r) \le S(r)$,since $2\sum_{l=1}^{k}\big(N(r,\bar\nu_f^{d_l}) + N(r,\bar\nu_g^{d_l})\big) \le$

$2kT(r)$ and $2\big(N(r,\bar\nu_f^\infty)+N(r,\bar\nu_g^\infty)\big)\le 2T(r)$. This is a contradiction against the assumption that $q>2k+6$, and hence $H\equiv 0$.

Integrating the equation $H\equiv 0$ twice, we have

$$\frac{1}{F}\equiv c_0\frac{1}{G}+c_1 \quad (c_0\ne 0, c_1\in\mathbb{C}),$$

that is,

$$\frac{1}{P(f)}\equiv c_0\frac{1}{P(g)}+c_1.$$

Now, applying Lemma 2.1 to the above equality and noting the assumption that $P(\omega)$ is a uniqueness polynomial, we have $f\equiv g$.

In particular, in the case where f and g are entire, if $H\not\equiv 0$, (3.5) becomes

$$(3.6)\qquad (q+k-1)T(r)\le 2\sum_{l=1}^{k}\big(N(r,\bar\nu_f^{d_l})+N(r,\bar\nu_g^{d_l})\big)+\frac{q}{2}T(r)+S(r),$$

which yields $(q-2k-2)T(r)\le S(r)$. This contradicts the assumption $q>2k+2$. □

4. Some further results concerning Theorem 1.5

Just as the discussions of H. Fujimoto in [3, p. 1196–1202], we could compute the relationship between q and k more precisely if we assume $m_0=2$ or $m_0=1$ with the weaker value-sharing hypothesis that $\sum_{j=1}^q \nu_{f,m_0)}^{a_j}\equiv\sum_{j=1}^q \nu_{g,m_0)}^{a_j}$.

Theorem 4.1. *Under the same situation as in Theorem* 1.4*, we assume furthermore that* $m_0=2$*, and that* $q>2k+7$ (*resp.* $q>2k+2$)*. Then, the condition that* $\sum_{j=1}^q \nu_{f,2)}^{a_j}\equiv\sum_{j=1}^q \nu_{g,2)}^{a_j}$ *implies* $f\equiv g$ *for any two nonconstant meromorphic* (*resp. entire*) *functions* f *and* g.

Proof. Similar to the proof of Theorem 1.5, if we assume $H\not\equiv 0$, then (3.1), (3.2) and (3.3) hold.

By Lemma 2.2 and noting the fact that $\nu_{F,2)}^0\equiv\nu_{G,2)}^0$, we have

$$\begin{aligned}N(r,\bar\nu_{F,(3}^0)&\le\sum_{j=1}^q N(r,\bar\nu_{f,(3}^{a_j})\le N(r,\bar\nu_{f',(2}^0)+O(1)\\&\le\frac12 N(r,\nu_{f'}^0)+O(1)\le\frac12 T(r,f)+\frac12 N(r,\bar\nu_f^\infty)+S(r,f),\end{aligned}$$

and $\quad N(r,\bar\nu_{G,(3}^0)\le\frac12 T(r,g)+\frac12 N(r,\bar\nu_g^\infty)+S(r,g);$

therefore,

(4.1)
$$\begin{aligned}
&N(r,\bar\nu^0_F)+N(r,\bar\nu^0_G)+N(r,\bar\nu^0_{F,(3})+N(r,\bar\nu^0_{G,(3})-N(r,\bar\nu^0_{F,1)})\\
&\le N(r,\bar\nu^0_F)+\frac12 N(r,\bar\nu^0_{F,(3})-\frac12 N(r,\bar\nu^0_{F,1)})+\frac12 N(r,\bar\nu^0_{F,(3})\\
&\quad+N(r,\bar\nu^0_G)+\frac12 N(r,\bar\nu^0_{G,(3})-\frac12 N(r,\bar\nu^0_{G,1)})+\frac12 N(r,\bar\nu^0_{G,(3})\\
&\le \frac12 N(r,\nu^0_F)+\frac14 T(r,f)+\frac14 N(r,\bar\nu^\infty_f)\\
&\quad+\frac12 N(r,\nu^0_G)+\frac14 T(r,g)+\frac14 N(r,\bar\nu^\infty_g)+S(r)\\
&\le \frac12\big(N(r,\nu^0_F)+N(r,\nu^0_G)\big)+\frac14\big(N(r,\bar\nu^\infty_f)+N(r,\bar\nu^\infty_g)\big)+\frac14 T(r)+S(r)\\
&\le \frac14\big(N(r,\bar\nu^\infty_f)+N(r,\bar\nu^\infty_g)\big)+\frac{2q+1}{4}T(r)+S(r).
\end{aligned}$$

Combining (3.1), (3.2) and (3.3) with (4.1) yields

$$\begin{aligned}
(q+k-1)T(r)\le{}&\frac94\big(N(r,\bar\nu^\infty_f)+N(r,\bar\nu^\infty_g)\big)\\
&+2\sum_{l=1}^{k}\big(N(r,\bar\nu^{d_l}_f)+N(r,\bar\nu^{d_l}_g)\big)+\frac{2q+1}{4}T(r)+S(r),
\end{aligned}\tag{4.2}$$

which implies $(q-2k-7)T(r)\le S(r)$. This contradicts the assumption $q>2k+7$.

If f and g are entire, (4.2) becomes

$$(q+k-1)T(r)\le 2\sum_{l=1}^{k}\big(N(r,\bar\nu^{d_l}_f)+N(r,\bar\nu^{d_l}_g)\big)+\frac{2q+1}{4}T(r)+S(r),\tag{4.3}$$

which gives $(q-2k-\frac52)T(r)\le S(r)$. Noting q is a natural number, This is a contradiction to the assumption $q>2k+2$.

Both cases imply $H\equiv 0$, and hence $f\equiv g$ by Lemma 2.1. □

Theorem 4.2. *Under the same situation as in Theorem* 1.4*, we assume furthermore that* $m_0=1$*, and that* $q>2k+10$ (*resp.* $q>2k+4$)*. Then, the condition that* $\sum_{j=1}^q \nu^{a_j}_{f,1)}\equiv\sum_{j=1}^q \nu^{a_j}_{g,1)}$ *implies* $f\equiv g$ *for any two nonconstant meromorphic* (*resp. entire*) *functions* f *and* g.

Proof. Similar to the proof of Theorem 4.1, for the situation $H\not\equiv 0$, noting that $\nu^0_{F,1)}\equiv\nu^0_{G,1)}$ and by Lemma 2.2, we have

$$N(r,\bar\nu^0_{F,(2})\le\sum_{j=1}^{q}N(r,\bar\nu^{a_j}_{f,(2})\le N(r,\bar\nu^0_{f'})\le T(r,f)+N(r,\bar\nu^\infty_f)+S(r,f),$$

and
$$N(r,\bar\nu^0_{G,(2})\le T(r,g)+N(r,\bar\nu^\infty_g)+S(r,g);$$

hence,

$$
\begin{aligned}
&N(r,\bar{\nu}_F^0)+N(r,\bar{\nu}_G^0)+N(r,\bar{\nu}_{F,(2}^0)+N(r,\bar{\nu}_{G,(2}^0)-N(r,\bar{\nu}_{F,1)}^0)\\
&\leq N(r,\bar{\nu}_F^0)-\frac{1}{2}N(r,\bar{\nu}_{F,1)}^0)+N(r,\bar{\nu}_{F,(2}^0)+N(r,\bar{\nu}_G^0)-\frac{1}{2}N(r,\bar{\nu}_{G,1)}^0)+N(r,\bar{\nu}_{G,(2}^0)\\
&\leq \frac{1}{2}N(r,\nu_F^0)+T(r,f)+N(r,\bar{\nu}_f^\infty)+\frac{1}{2}N(r,\nu_G^0)+T(r,g)+N(r,\bar{\nu}_g^\infty)+S(r)\\
&\leq \frac{1}{2}\big(N(r,\nu_F^0)+N(r,\nu_G^0)\big)+N(r,\bar{\nu}_f^\infty)+N(r,\bar{\nu}_g^\infty)+T(r)+S(r)\\
&\leq N(r,\bar{\nu}_f^\infty)+N(r,\bar{\nu}_g^\infty)+\frac{q+2}{2}T(r)+S(r).
\end{aligned}
\tag{4.4}
$$

Combining (3.1), (3.2) and (3.3) with (4.4) yields

$$
\begin{aligned}
(q+k-1)T(r) \leq{}& 3\big(N(r,\bar{\nu}_f^\infty)+N(r,\bar{\nu}_g^\infty)\big)\\
&+2\sum_{l=1}^{k}\big(N(r,\bar{\nu}_f^{d_l})+N(r,\bar{\nu}_g^{d_l})\big)+\frac{q+2}{2}T(r)+S(r),
\end{aligned}
\tag{4.5}
$$

which shows $(q-2k-10)T(r)\leq S(r)$. This contradicts the assumption $q>2k+10$.

If f and g are entire, (4.5) becomes

$$
(q+k-1)T(r)\leq 2\sum_{l=1}^{k}\big(N(r,\bar{\nu}_f^{d_l})+N(r,\bar{\nu}_g^{d_l})\big)+\frac{q+2}{2}T(r)+S(r),
\tag{4.6}
$$

which implies $(q-2k-4)T(r)\leq S(r)$. This contradicts the assumption $q>2k+4$.

Both cases imply $H\equiv 0$, and hence $f\equiv g$ by Lemma 2.1. □

5. Proof of Theorem 1.6

Set $F:=P(f)$ and $G:=P(g)$. Then the condition $\sum_{j=1}^q\bar{\nu}_f^{a_j}\equiv\sum_{j=1}^q\bar{\nu}_g^{a_j}$ implies $\bar{\nu}_F^0\equiv\bar{\nu}_G^0$. Similarly, we know that any common simple zero of F and G is a zero of H if $H\not\equiv 0$, where H is the same one as that defined in Section 3.

For two divisors ν_1 and ν_2 satisfying $\bar{\nu}_1\equiv\bar{\nu}_2$, define

$$
\nu_{i,\mathcal{D}}(z):=\begin{cases}\nu_i(z), & \text{if } \nu_i(z)>\nu_j(z) \text{ with } \{i,j\}=\{1,2\};\\ 0, & \text{otherwise.}\end{cases}
$$

for $i=1$, 2 and
$$
\nu_{1\&2,\mathcal{C}}(z):=\begin{cases}\nu_1(z), & \text{if } \nu_1(z)=\nu_2(z);\\ 0, & \text{otherwise.}\end{cases}
$$

Then, if $H\not\equiv 0$, we have

$$
\begin{aligned}
N(r,\bar{\nu}_H^\infty)\leq{}&\sum_{l=1}^{k}\big(N(r,\bar{\nu}_f^{d_l})+N(r,\bar{\nu}_g^{d_l})\big)+N(r,\bar{\nu}_{F,\mathcal{D}}^0)+N(r,\bar{\nu}_{G,\mathcal{D}}^0)\\
&+N(r,\bar{\nu}_f^\infty)+N(r,\bar{\nu}_g^\infty)+N(r,\bar{\hat{\nu}}_{f'}^0)+N(r,\bar{\hat{\nu}}_{g'}^0)+O(1).
\end{aligned}
\tag{5.1}
$$

Applying the second main theorem to the functions f, g and the values $a_1, a_2, \cdots, a_q, d_1, d_2, \cdots, d_k, \infty$, respectively to conclude that

(5.2)

$$(q+k-1)T(r) \le N(r,\bar{\nu}_f^{\infty}) + N(r,\bar{\nu}_g^{\infty}) + N(r,\bar{\nu}_F^0) + N(r,\bar{\nu}_G^0) + \sum_{l=1}^{k}\big(N(r,\bar{\nu}_f^{d_l}) + N(r,\bar{\nu}_g^{d_l})\big) - N(r,\hat{\nu}_{f'}^0) - N(r,\hat{\nu}_{g'}^0) + S(r).$$

Noting that the intersection of the support of the divisors $\bar{\nu}^0_{F\&G,\mathcal{C},(2}$ and $\bar{\nu}^0_{G,\mathcal{D}}$ is empty. A simple computation leads to

$$N(r,\bar{\nu}^0_{F\&G,\mathcal{C},(2}) + N(r,\bar{\nu}^0_{G,\mathcal{D}}) + N(r,\bar{\nu}^0_G) \le N(r,\nu^0_G) \le T(r,G).$$

Since $\bar{\nu}_F^0 \equiv \bar{\nu}_G^0$, we could derive that

$$\bar{\nu}_F^0 = \bar{\nu}^0_{F\&G,\mathcal{C},1)} + \bar{\nu}^0_{F\&G,\mathcal{C},(2} + \bar{\nu}^0_{F,\mathcal{D}} + \bar{\nu}^0_{G,\mathcal{D}},$$

and we could further obtain

$$\begin{aligned} &N(r,\bar{\nu}_F^0) + N(r,\bar{\nu}_G^0) = N(r,\bar{\nu}^0_{F\&G,\mathcal{C},1)}) + N(r,\bar{\nu}^0_{F\&G,\mathcal{C},(2}) \\ &\quad + N(r,\bar{\nu}^0_{F,\mathcal{D}}) + N(r,\bar{\nu}^0_{G,\mathcal{D}}) + N(r,\bar{\nu}^0_G) = N(r,\bar{\nu}^0_{F\&G,\mathcal{C},1)}) \\ &\quad + N(r,\bar{\nu}^0_{F,\mathcal{D}}) + T(r,G) \le N(r,\bar{\nu}^0_{F\&G,\mathcal{C},1)}) + N(r,\bar{\nu}^0_{F,\mathcal{D}}) + qT(r,g). \end{aligned} \tag{5.3}$$

Substituting (5.1) and (5.3) into (5.2), noting the fact that $N(r,\bar{\nu}^0_{F\&G,\mathcal{C},1)}) \le N(r,\bar{\nu}_H^{\infty})$ by similar reasonings as those to (3.1), we could derive

$$(q+k-1)T(r) \le 2\big(N(r,\bar{\nu}_f^{\infty}) + N(r,\bar{\nu}_g^{\infty})\big) + 2\sum_{l=1}^{k}\big(N(r,\bar{\nu}_f^{d_l}) + N(r,\bar{\nu}_g^{d_l})\big) + 2N(r,\bar{\nu}^0_{F,\mathcal{D}}) + N(r,\bar{\nu}^0_{G,\mathcal{D}}) + qT(r,g) + S(r).$$

Interchanging the positions between f and g, we obtain

$$(q+k-1)T(r) \le 2\big(N(r,\bar{\nu}_f^{\infty}) + N(r,\bar{\nu}_g^{\infty})\big) + 2\sum_{l=1}^{k}\big(N(r,\bar{\nu}_f^{d_l}) + N(r,\bar{\nu}_g^{d_l})\big) + N(r,\bar{\nu}^0_{F,\mathcal{D}}) + 2N(r,\bar{\nu}^0_{G,\mathcal{D}}) + qT(r,f) + S(r).$$

Summing up the above two inequalities, we get

(5.4)

$$(q+2k-2)T(r) \le 4\big(N(r,\bar{\nu}_f^{\infty}) + N(r,\bar{\nu}_g^{\infty})\big) + 4\sum_{l=1}^{k}\big(N(r,\bar{\nu}_f^{d_l}) + N(r,\bar{\nu}_g^{d_l})\big) + 3\big(N(r,\bar{\nu}^0_{F,\mathcal{D}}) + N(r,\bar{\nu}^0_{G,\mathcal{D}})\big) + S(r).$$

Since we assume that the union of the sets of the double a_j-points of f for $j = 1, 2, \ldots, q$ is the same as that of g, then $\nu^0_{F,\mathcal{D}}(z) \ge 3$ if it does not vanish at point z, and analogous property holds for $\nu^0_{G,\mathcal{D}}$. Similar to the methods employed in the proof of Theorem 4.1, we can obtain

$$N(r,\bar\nu^0_{F,\mathcal{D}}) \le N(r,\bar\nu^0_{F,(3}) \le \frac{1}{2}T(r,f) + \frac{1}{2}N(r,\bar\nu^\infty_f) + S(r,f),$$

and
$$N(r,\bar\nu^0_{G,\mathcal{D}}) \le N(r,\bar\nu^0_{G,(3}) \le \frac{1}{2}T(r,g) + \frac{1}{2}N(r,\bar\nu^\infty_g) + S(r,g).$$

Hence, (5.4) becomes

$$\begin{aligned}(q+2k-2)T(r) \le{}& \frac{11}{2}\big(N(r,\bar\nu^\infty_f) + N(r,\bar\nu^\infty_g)\big)\\ &+ 4\sum_{l=1}^{k}\big(N(r,\bar\nu^{d_l}_f) + N(r,\bar\nu^{d_l}_g)\big) + \frac{3}{2}T(r) + S(r),\end{aligned} \tag{5.5}$$

which means $(q-2k-9)T(r) \le S(r)$. This contradicts the assumption $q > 2k+9$.

In particular, in the case where f and g are entire, (5.5) becomes

$$(q+2k-2)T(r) \le 4\sum_{l=1}^{k}\big(N(r,\bar\nu^{d_l}_f) + N(r,\bar\nu^{d_l}_g)\big) + \frac{3}{2}T(r) + S(r), \tag{5.6}$$

which implies $(q-2k-\frac{7}{2})T(r) \le S(r)$. This contradicts the assumption $q > 2k+\frac{7}{2}$.

Both cases imply $H \equiv 0$, and hence $f \equiv g$ by Lemma 2.1. □

Remark. The polynomial given by G. Frank and M. Reinders in [2]

$$P^{FR}(\omega) := \frac{(q-1)(q-2)}{2}\omega^q - q(q-2)\omega^{q-1} + \frac{q(q-1)}{2}\omega^{q-2} - c_0 \quad (c_0 \ne 0,1)$$

is a uniqueness polynomial in a broad sense for the case $k=2$ by Theorem 1.2 whenever $q > 5$. Also, according to their original argument in [2, p. 191, Case 2], we know that $P^{FR}(f) \equiv cP^{FR}(g)$ implies $P^{FR}(f) \equiv P^{FR}(g)$ whenever $q > 7$ (resp. $q > 6$) for any two nonconstant meromorphic (resp. entire) functions f and g. Hence, if we denote its zero set by S^{FR}, there exists an unique range set. That is S^{FR} with the weaker value-sharing condition that $\sum_{j=1}^q \nu^{a_j}_{f,m_0)} \equiv \sum_{j=1}^q \nu^{a_j}_{g,m_0)}$ ($a_j \in S^{FR}$, $j = 1, 2, \ldots, q$) consists of 11, 12 and 15 (resp. 7, 7 *and* 9) elements for any two nonconstant meromorphic (resp. entire) functions for cases $m_0 \ge 3$ or $m_0 = \infty$, $m_0 = 2$ and $m_0 = 1$; and an URSM-IM (resp. URSE-IM), i.e., S^{FR}, together with the condition that the union of the sets of the double a_j-points of f for $j = 1, 2, \ldots, q$ is the same as that of g, consisting of 14 (resp. 8) elements.

6. Some applications

It is interesting to know that, for any two nonconstant meromorphic functions f and g with $N(r,\bar\nu^\infty_f) = S(r,f)$ and $N(r,\bar\nu^\infty_g) = S(r,g)$, the conclusions of our results for the cases of entire functions hold. Hence, the differences between the lower bounds of q's for cases of meromorphic functions and those for cases of entire functions derive from the two quantities $N(r, \bar{\nu}_f^{\infty})$ and $N(r, \bar{\nu}_g^{\infty})$. In the following, we shall make a research on the influences of these two terms on the lower bounds of q's for cases of meromorphic functions more accurately, and the natural instrument for our research is the quantity $\Theta_f(\infty)$ defined as

$$\Theta_f(\infty) := 1 - \limsup_{r\to+\infty, r\in\mathbb{R}^+} \frac{N(r, \bar{\nu}_f^{\infty})}{T(r, f)}.$$

Obviously, $0 \le \Theta_f(\infty) \le 1$. Moreover, Nevanlinna's second main theorem with counting functions of reduced form says that $\sum_{a\in\mathbb{C}\cup\{\infty\}} \Theta_f(a) \le 2$ for any nonconstant meromorphic function f, and the upper bound 2 is attained by the exponent function e^z since $\Theta_{e^z}(0) = \Theta_{e^z}(\infty) = 1$.

Theorem 6.1. *Under the same situation as in Theorem* 1.4*, we assume furthermore that* $\Theta_f(\infty) + \Theta_g(\infty) > 3 + k - \frac{q}{2}$ *for the cases* $m_0 \ge 3$ *or* $m_0 = \infty$; *that* $\Theta_f(\infty) + \Theta_g(\infty) > \frac{28+8k-4q}{9}$ *for the case* $m_0 = 2$; *and that* $\Theta_f(\infty) + \Theta_g(\infty) > \frac{20+4k-2q}{6}$ *for the case* $m_0 = 1$. *Then, the condition that* $\sum_{j=1}^q \nu_{f,m_0)}^{a_j} \equiv \sum_{j=1}^q \nu_{g,m_0)}^{a_j}$ *implies* $f \equiv g$ *for any two nonconstant meromorphic functions* f *and* g.

Proof. Similar to the proof of Theorem 1.5, if we assume that $H \not\equiv 0$, then (3.5) holds for the cases $m_0 \ge 3$ or $m_0 = \infty$. For any given $\varepsilon > 0$, noting that $2\sum_{l=1}^{k} \big(N(r, \bar{\nu}_f^{d_l}) + N(r, \bar{\nu}_g^{d_l})\big) \le 2kT(r)$, we have

(6.1)
$$(2k-q+2)T(r)+4(1-\Theta_f(\infty)+\varepsilon)T(r,f)+4(1-\Theta_g(\infty)+\varepsilon)T(r,g)+S(r) \ge 0.$$

Set $T_0(r) := \max\{T(r,f),\ T(r,g)\}$. Then, $T(r) \le 2T_0(r)$. With this notation and noting that $0 \le \Theta_f(\infty) \le 1$, (6.1) becomes

$$\big(4k - 2q + 12 - 4(\Theta_f(\infty) + \Theta_g(\infty) - 2\varepsilon)\big)T_0(r) + S(r) \ge 0,$$

which derives a contradiction since we may take ε as small as we like.

For the cases $m_0 = 2$ or $m_0 = 1$, analogous analysis as above with (4.2) or (4.5) respectively would still yields contradictions if we assume $H \not\equiv 0$. Thus $H \equiv 0$ and by Lemma 2.1, we have $f \equiv g$. □

Theorem 6.2. *In the same situation as in Theorem* 1.4*, we assume furthermore that* $\Theta_f(\infty) + \Theta_g(\infty) > \frac{24+4k-2q}{7}$. *Then, S is a URSM-IM.*

Proof. Similarly as in the proof of Theorem 1.6, if $H \not\equiv 0$, (5.4) holds.

Generally speaking, $\nu_{F,\mathcal{D}}^0(z) \ge 2$ if $\nu_{F,\mathcal{D}}^0(z) \ne 0$, and $\nu_{G,\mathcal{D}}^0(z) \ge 2$ if $\nu_{G,\mathcal{D}}^0(z) \ne 0$. Similar to the methods employed in the proof of Theorem 4.2, we can get

$$N(r, \bar{\nu}_{F,\mathcal{D}}^0) \le N(r, \bar{\nu}_{F,(2}^0) \le T(r,f) + N(r, \bar{\nu}_f^{\infty}) + S(r,f),$$

and
$$N(r, \bar{\nu}_{G,\mathcal{D}}^0) \le N(r, \bar{\nu}_{G,(2}^0) \le T(r,g) + N(r, \bar{\nu}_g^{\infty}) + S(r,g).$$

Hence, (5.4) becomes

$$(q+2k-2)T(r) \le 7\big(N(r,\bar{\nu}_f^{\infty})+N(r,\bar{\nu}_g^{\infty})\big)+(4k+3)T(r)+S(r),$$

and then we have

$$\big(4k-2q+24-7(\Theta_f(\infty)+\Theta_g(\infty)-2\varepsilon)\big)T_0(r)+S(r)\ge 0. \tag{6.2}$$

This is a contradiction against our assumption follows immediately. □

Concluding remark. Now, we turn back to $P^{FR}(\omega)$ for illustration again. From the argument in [2, p. 191, Case 2], we know that $P^{FR}(f)\equiv cP^{FR}(g)$ implies $P^{FR}(f)\equiv P^{FR}(g)$ whenever $q>6$ for any two nonconstant meromorphic functions f and g if either $\Theta_f(\infty)>0$ or $\Theta_g(\infty)>0$. Hence, we know that there exists an unique range set, say, S^{FR}, it consists 7 elements for any two nonconstant meromorphic functions f and g, with the assumptions that $\sum_{j=1}^{q}\nu_{f,m_0)}^{a_j}\equiv\sum_{j=1}^{q}\nu_{g,m_0)}^{a_j}$ and $\Theta_f(\infty)+\Theta_g(\infty)>\frac{3}{2}$ (resp. $\Theta_f(\infty)+\Theta_g(\infty)>\frac{16}{9}$) for the cases $m_0\ge 3$ or $m_0=\infty$ (resp. for the case $m_0=2$). While an URSM-IM, say, S^{FR} consists 10 elements with the assumption that $\Theta_f(\infty)+\Theta_g(\infty)>\frac{12}{7}$, where 7 and 10 are the best-known lower bounds of the numbers of elements of any URSE and URSE-IM, respectively. A special case of these results with $m_0=\infty$ was obtained by Y. Xu in [9] as below

Corollary 6.3 ([9, p. 1490]). *Let f and g be any two nonconstant meromorphic functions such that $\Theta_f(\infty)+\Theta_g(\infty)>\frac{3}{2}$. Then, there exists an URSM, i.e., S^{FR}, consisting of 7 elements. In particular, there exists an URSM consisting of 7 elements whenever $\Theta_f(\infty)>\frac{3}{4}$ and $\Theta_g(\infty)>\frac{3}{4}$.*

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References

[1] S. Bartels, *Meromorphic functions sharing a set with* 17 *elements ignoring multiplicities*, Complex Variables Theory Appl. **39** (1999), 85–90.

[2] G. Frank and M. Reinders, *A unique range set for meromorphic functions with* 11 *elements*, Complex Variables Theory Appl. **37** (1998), 185–193.

[3] H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets*, Amer. J. Math. **122** (2000), 1175–1203.

[4] ———, *On uniqueness polynomials for meromorphic functions*, Nagoya Math. J. **170** (2003), 33–46.

[5] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[6] P. Li and C. C. Yang, *On the unique range sets for meromorphic functions*, Proc. Amer. Math. Soc. **124** (1996), 177–185.

[7] ———, *Some further results on the unique range sets for meromorphic functions*, Kodai Math. J. **18** (1995), 437–450.

[8] E. Mues and M. Reinders, *Meromorphic functions sharing one value and unique range sets*, Kodai Math. J. **18** (1995), 515–522.

[9] Y. Xu, *Meromorphic functions sharing one finite set*, Comput. Math. Appl. **45** (2003), 1489–1495.

[10] L. Yang, Value Distribution Theory, Springer-Verlag, Berlin, 1993.

[11] H. X. Yi, *On a question of Gross*, Sci. China Ser. A **38** (1995), 8–16.

[12] ———, *A question of Gross and the uniqueness of entire functions*, Nagoya Math. J. **138** (1995), 169–177.

[13] ———, *Some further results on uniqueness of meromorphic functions*, Complex Variables Theory Appl. **38** (1999), 375–385.