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The quadratic variations of local martingales and the first-passage times of stochastic integrals

By

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Abstract

We obtain the tail estimation of the quadratic variation of a local martingale with no assumption with respect to positive jumps. Moreover, applying it, we also discuss a tail property of the first-passage times of stochastic integrals.

1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P)$ be a filtered probability space with usual conditions and $M = \{M_t\}_{t\in[0,\infty)}$ is a càdlàg local martingale with $M_0 = 0$ defined on it. There have been several works on the tail distribution of the predictable quadratic variation $\langle M\rangle$. In Azéma, Gundy, and Yor [1], the uniform integrability of a continuous martingale M was first characterized in terms of the tails of $\langle M\rangle$. For a continuous local martingale M, the tail distribution of $\langle M\rangle$ was studied by Elworthy, Li, and Yor [2], [3], Galtchouk and Novikov [4], Novikov [9], and Takaoka [11] etc. Recently, the above works were extended by Kaji [5], [6] and Liptser and Novikov [7] for a càdlàg local martingale. To state the results we introduce necessary notions. Set $\Delta M_t = M_t - M_{t-}, t > 0$. We define the counting measure μ of jumps of M on $[0,\infty)\times(\mathbf{R}\backslash\{0\})$ by

$$\mu((0,t]\times U) = \sum_{0<s\le t} 1_U(\Delta M_s),$$

for $t\in[0,\infty)$ and Borel subsets U of $\mathbf{R}\backslash\{0\}$. We denote by $\widehat{\mu}$ its predictable compensator. For any predictable function $\alpha(t,x)$ we denote the integral process $\alpha*\xi$ based on $\xi = \widehat{\mu}$ or $\widehat{\mu}^c$ by

$$(\alpha*\xi)_t = \int_{(0,t]\times(\mathbf{R}\backslash\{0\})} \alpha(s,x)\xi(dsdx)$$

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if $\alpha(s,x)$ is integrable on $(0,t]\times(\mathbf{R}\backslash\{0\})$. In Kaji [5], [6] we assumed either
(i) There exists $\lambda_0>0$ such that

$$E[\exp\{\lambda_0 M_\infty^- + (|\phi_{\lambda_0}|1_{\{|x|>K\}} * \widehat{\mu})_\infty\}] < \infty$$

or
(ii) M is quasi left continuous and there exists $\lambda_0>0$ such that

$$E[(|\phi_{\lambda_0}|1_{\{|x|>K\}} * \widehat{\mu})_\infty] < \infty$$

for some $K>0$, where $\phi_\lambda(x)=e^{-\lambda x}-1+\lambda x-\frac{\lambda^2}{2}x^2$ and $x^-=\max\{-x,0\}$. Then the asymptotic behaviour below was proved in Kaji [5], [6]

$$\lim_{\lambda\to\infty}\lambda P\left(\sqrt{\langle M\rangle_\infty}>\lambda\right) = -\sqrt{\frac{2}{\pi}}\,E[M_\infty].$$

In this note we try to relax the condition (i). Actually in Theorem1 we do not assume the integrability of the predictable compensator with respect to positive jumps.

In Section 3 we consider a Lévy process $X=\{X_t\}_{t\in[0,\infty)}$ represented by

$$X_t = \sigma W_t + \int_{(0,t]\times(\mathbf{R}\backslash\{0\})} x\{N(dsdx)-ds\nu(dx)\}, \tag{1}$$

where $W=\{W_t\}_{t\in[0,\infty)}$ is a standard Brownian motion starting from 0, $N(dsdx)$ is a Poisson random measure on $[0,\infty)\times(\mathbf{R}\backslash\{0\})$ with compensator $ds\nu(dx)$, and σ is a nonnegative constant. Here, we assume the measure $\nu(dx)$ on $\mathbf{R}\backslash\{0\}$ satisfies

$$\int_{\mathbf{R}\backslash\{0\}} x^2\nu(dx)<\infty.$$

The characteristic functions of the first-passage time of Lévy processes without positive jumps are known (see Theorem 46.3 of Sato [10]), from which asymptotic behaviours of their distributions are possible to obtain. However the author does not know that there exists any estimate of the first-passage time of the Lévy processes allowing positive jumps. In Theorem6, applying the theorem1, we obtain the asymptotics for the tail distributions of the first-passage times of a Lévy process X with $\nu((K,\infty))=0$ for some $K\geq 0$ as well as of a stochastic integral based on $N(dsdx)$.

2. The predictable quadratic variations of local martingales

Assume that M is a locally square integrable martingale such that

$$\langle M\rangle_\infty = \lim_{t\to\infty}\langle M\rangle_t<\infty \ \ a.s., \text{ and } \{M_\tau^-\}_{\tau\in\mathcal{T}} \text{ is uniformly integrable},$$

where $\mathcal{T}$ is the set of all stopping times and $x^-=\max\{-x,0\}$. Recall that $\langle M\rangle_\infty<\infty$ a.s. implies $M_\infty<\infty$ a.s. (see Theorem 5, Liptser and Shiryaev [8], p. 136). Then we have

Theorem 1. *Assume there exists $\lambda_0 > 0$ such that*

$$E\left[e^{-\lambda_0 M_\infty} + \left(e^{-\lambda_0 x} I_{\{x<-K\}} * \widehat{\mu}\right)_\infty\right] < \infty \tag{2}$$

for some $K > 0$. Then it holds that

$$\lim_{\lambda\to\infty} \lambda P\left(\sqrt{\langle M\rangle_\infty} > \lambda\right) = -\sqrt{\frac{2}{\pi}}\, E[M_\infty].$$

Set

$$\phi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x - \frac{\lambda^2}{2}x^2.$$

Then it is trivial that (2) holds if and only if there exists $\lambda_0 > 0$ such that

$$E[\exp(-\lambda_0 M_\infty) + (\phi_{\lambda_0} 1_{\{x<-K\}} * \widehat{\mu})_\infty] < \infty \tag{3}$$

for some $K > 0$. Noticing the inequality

$$|\phi_\lambda(x)| \le \frac{\lambda^2}{2}x^2 \text{ for } x \ge K$$

if $\lambda > 0$, $K > 0$, and the fact

$$(x^2 * \widehat{\mu})_\infty \le \langle M\rangle_\infty,$$

we easily see

$$(|\phi_\lambda| 1_{\{x>K\}} * \widehat{\mu})_\infty \le \frac{\lambda^2}{2}(x^2 1_{\{x>K\}} * \widehat{\mu})_\infty \le \frac{\lambda^2}{2}\langle M\rangle_\infty. \tag{4}$$

Let

$$\psi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x$$

and

$$\mathcal{E}(\lambda) = \exp\left\{-\lambda M - \frac{\lambda^2}{2}\langle M^c\rangle - (\psi_\lambda * \widehat{\mu}^c) - \sum\nolimits_{0<s\le \cdot} \log(1 + \Delta(\psi_\lambda * \widehat{\mu})_s)\right\}.$$

Then the inequality (4) and $\langle M\rangle_\infty < \infty$ a.s. imply that for all $\lambda \in (0, \lambda_0]$

$$(|\phi_\lambda| * \widehat{\mu})_\infty < \infty\ ,\ (|\psi_\lambda| * \widehat{\mu})_\infty < \infty\ ,\ \mathcal{E}(\lambda)_\infty < \infty \quad a.s.,$$

holds. The detail can be found in section 6.1 of Kaji [5].

Let

$$\eta_t^\lambda = \sum\nolimits_{0<s\le t} \left\{\frac{\lambda^2}{2}\Delta(x^2 * \widehat{\mu})_s - \log(1 + \Delta(\psi_\lambda * \widehat{\mu})_s)\right\};$$
$$\eta_\infty^\lambda = \lim\nolimits_{t\to\infty} \eta_t^\lambda.$$

We note $\eta_\infty^\lambda < \infty$ a.s..

Lemma 2. *For all $\lambda \in (0, \lambda_0]$ it holds that*

$$E\left[\mathcal{E}(\lambda)_\infty\right] = 1.$$

Proof. Lemma 6.2 of Kaji [5] shows the condition $E[e^{\lambda_0 M_\infty^-}] < \infty$ implies the result. In fact, we see

$$E[e^{\lambda_0 M_\infty^-}] \leq E[e^{-\lambda_0 M_\infty}] + 1,$$

and the right hand side is finite by (3). □

Lemma 3.

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left\{ e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty} - \mathcal{E}(\lambda)_\infty \right\} = -M_\infty \quad a.s.$$

is valid.

Proof. Fix $\lambda \in (0, \lambda_0]$ and observe an equality

$$\begin{aligned} \frac{1}{\lambda}\{e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty} - \mathcal{E}(\lambda)_\infty\} &= e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty} \cdot \frac{1}{\lambda}\{1 - e^{-\lambda M_\infty}\} \\ &\quad + e^{-\lambda M_\infty - \frac{\lambda^2}{2}\langle M \rangle_\infty} \cdot \frac{1}{\lambda}\{1 - e^{-(\phi_\lambda * \hat{\mu}^c)_\infty}\} \\ &\quad + e^{-\lambda M_\infty - \frac{\lambda^2}{2}\langle M \rangle_\infty - (\phi_\lambda * \hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda}\{1 - e^{\eta_\infty^\lambda}\} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

$I_1 \to -M_\infty$ a.s. as $\lambda \downarrow 0$. On the other hand, in Lemma 6.3 and 6.4 of Kaji [5] it is proved that from $(|\phi_{\lambda_0}| * \hat{\mu})_\infty < \infty$

$$\lim_{\lambda \downarrow 0} I_2 = \lim_{\lambda \downarrow 0} I_3 = 0 \quad a.s.$$

hold, which concludes the proof. □

Lemma 4. *For all $\lambda \in (0, \lambda_0 \wedge \frac{1}{2c_0 K})$*

$$\begin{aligned} &e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty} \Delta(x^2 * \hat{\mu})_s} \cdot \frac{1}{\lambda}|1 - e^{\eta_\infty^\lambda}| \\ &\quad \leq 1 + 2c_0 K e^{-1} + \sum_{0<s<\infty} \Delta\left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x < -K\}} * \hat{\mu}\right)_s \end{aligned}$$

is valid.

Proof. Fix $\lambda \in (0, \lambda_0 \wedge \frac{1}{2c_0 K})$. First we have

$$\begin{aligned} &e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty} \Delta(x^2 * \hat{\mu})_s} \cdot \frac{1}{\lambda}|1 - e^{\eta_\infty^\lambda}| \\ &\leq e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty} \Delta(x^2 * \hat{\mu})_s} \left(e^{\eta_\infty^\lambda} 1_{\{\eta_\infty^\lambda \geq 0\}} + \frac{1}{\lambda}(\eta_\infty^\lambda)^- 1_{\{\eta_\infty^\lambda < 0\}} \right) \\ &\leq e^{\eta_\infty^\lambda - \frac{\lambda^2}{2}\sum_{0<s<\infty} \Delta(x^2 * \hat{\mu})_s} 1_{\{\eta_\infty^\lambda \geq 0\}} + e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty} \Delta(x^2 * \hat{\mu})_s} \cdot \frac{1}{\lambda}(-\eta_\infty^\lambda) 1_{\{\eta_\infty^\lambda < 0\}}. \end{aligned}$$

Moreover the inequality $\log(1+x) \le x$ for $x \ge 0$ implies the right-hand side is dominated by

$$1 + e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \left\{ \frac{1}{\lambda} \sum\nolimits_{0<s<\infty} \Delta(\phi_\lambda * \hat{\mu})_s \right\} 1_{\{\eta_\infty^\lambda < 0\}}.$$

Therefore we have

$$\begin{aligned} & e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \cdot \frac{1}{\lambda}|1 - e^{\eta_\infty^\lambda}| \\ & \le 1 + e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \left\{ \frac{1}{\lambda} \sum\nolimits_{0<s<\infty} \Delta(\phi_\lambda * \hat{\mu})_s \right\} 1_{\{\eta_\infty^\lambda < 0\}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \left\{ \frac{1}{\lambda} \sum\nolimits_{0<s<\infty} \Delta(\phi_\lambda * \hat{\mu})_s \right\} 1_{\{\eta_\infty^\lambda < 0\}} \\ & \le e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \cdot \frac{1}{\lambda} \sum\nolimits_{0<s<\infty} \Delta(\phi_\lambda 1_{\{x<0\}} * \hat{\mu})_s \\ & \le e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \sum\nolimits_{0<s<\infty} \left(\Delta\left(\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu} \right)_s \right. \\ & \quad \left. + \Delta\left(\frac{\phi_\lambda}{\lambda} 1_{\{-K \le x < 0\}} * \hat{\mu} \right)_s \right), \end{aligned}$$

and moreover, since

$$\left| e^{-x} - 1 + x - \frac{x^2}{2} \right| \le c_0 \, |x|^3$$

for all $|x| \le \lambda_0 K$ with some $c_0 > 0$, the right-hand side is dominated by

$$\begin{aligned} & \le \sum\nolimits_{0<s<\infty} \Delta\left(\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu} \right)_s \\ & \quad + e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \cdot c_0 K \lambda^2 \sum\nolimits_{0<s<\infty} \Delta(x^2 1_{\{-K \le x < 0\}} * \hat{\mu})_s. \end{aligned}$$

By using the inequality $xe^{-x} \le e^{-1}$, $x \ge 0$ and Lemma 4.1 of Kaji [5], the last quantity is estimated as

$$\le \sum\nolimits_{0<s<\infty} \Delta\left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x<-K\}} * \hat{\mu} \right)_s + 2c_0 K e^{-1}.$$

Therefore we have

$$\begin{aligned} & e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \left\{ \frac{1}{\lambda} \sum\nolimits_{0<s<\infty} \Delta(\phi_\lambda * \hat{\mu})_s \right\} 1_{\{\eta_\infty^\lambda < 0\}} \\ & \le \quad \sum\nolimits_{0<s<\infty} \Delta\left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x<-K\}} * \hat{\mu} \right)_s + 2c_0 K e^{-1}, \end{aligned}$$

which completes the proof. □

Lemma 5. *For all $\lambda \in (0, \lambda_0 \wedge \frac{1}{2c_0K})$*

$$e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \cdot \left| \frac{1}{\lambda}\{e^{-\lambda M_\infty - (\phi_\lambda * \hat{\mu}^c)_\infty} - 1\} \right| \leq e^{-\lambda_0 M_\infty} + M_\infty^+ + \left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x<-K\}} * \hat{\mu}^c \right)_\infty + 2 + 2c_0 K e^{-1}$$

holds.

Proof. Fix $\lambda \in (0, \lambda_0 \wedge \frac{1}{2c_0K})$. Observe inequalities

$$\begin{aligned}
&e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \cdot \left| \frac{1}{\lambda}\{e^{-\lambda M_\infty - (\phi_\lambda * \hat{\mu}^c)_\infty} - 1\} \right| \\
&\quad \leq e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda} |1 - e^{-(\phi_\lambda 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty}| \\
&\qquad + e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty - (\phi_\lambda 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda} |1 - e^{-\lambda M_\infty - (\phi_\lambda 1_{\{|x|>K\}} * \hat{\mu}^c)_\infty}| \\
&\quad \leq e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda} |1 - e^{-(\phi_\lambda 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty}| \\
&\qquad + e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty - (\phi_\lambda 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty} \\
&\qquad \times e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|>K\}} * \hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda} |1 - e^{-\lambda M_\infty - (\phi_\lambda 1_{\{|x|>K\}} * \hat{\mu}^c)_\infty}| \\
&\quad = J_1 + J_2 \times J_3.
\end{aligned}$$

We will estimate J_1. We have

$$\begin{aligned}
J_1 &\leq e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \Bigg(e^{-(\phi_\lambda 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty} 1_{\{(\frac{\phi_\lambda}{\lambda} 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty \leq 0\}} \\
&\qquad + \left(- \left(\frac{\phi_\lambda}{\lambda} 1_{\{|x|\leq K\}} * \hat{\mu}^c \right)_\infty \right)^{-} 1_{\{(\frac{\phi_\lambda}{\lambda} 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty > 0\}} \Bigg) \\
&\leq e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \left(e^{(|\phi_\lambda| 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty} + \left(\left| \frac{\phi_\lambda}{\lambda} \right| 1_{\{|x|\leq K\}} * \hat{\mu}^c \right)_\infty \right).
\end{aligned}$$

Moreover, the right-hand side is dominated by

$$\begin{aligned}
&\leq e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \left(e^{c_0 K \lambda^3 (x^2 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty} + c_0 K \lambda^2 (x^2 1_{\{|x|\leq K\}} * \hat{\mu}^c)_\infty \right) \\
&\leq e^{\frac{\lambda^2}{2}(-1 + 2c_0 K \lambda)(x^2*\hat{\mu}^c)_\infty} + 2c_0 K \cdot \frac{\lambda^2}{2} (x^2 * \hat{\mu}^c)_\infty e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \\
&\leq 1 + 2c_0 K e^{-1}.
\end{aligned}$$

Therefore we have

$$J_1 \leq 1 + 2c_0 K e^{-1}.$$

J_2 can be estimated as follows:

$$\begin{aligned}(0 \le) J_2 &\le e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|\le K\}} * \hat{\mu}^c)_\infty + (|\phi_\lambda| 1_{\{|x|\le K\}} * \hat{\mu}^c)_\infty} \\ &\le e^{(c_0 K \lambda^3 - \frac{\lambda^2}{2})(x^2 1_{\{|x|\le K\}} * \hat{\mu}^c)_\infty} \\ &\le 1.\end{aligned}$$

Finally, we will estimate J_3. Observe inequalities

$$\begin{aligned}(0 \le) J_3 &\le e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|>K\}} * \hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda} |e^{-\lambda M_\infty - (\phi_\lambda 1_{\{x>K\}} * \hat{\mu}^c)_\infty - (\phi_\lambda 1_{\{x<-K\}} * \hat{\mu}^c)_\infty} - 1| \\ &\le e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|>K\}} * \hat{\mu}^c)_\infty - (\phi_\lambda 1_{\{x>K\}} * \hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda} |e^{-\lambda M_\infty - (\phi_\lambda 1_{\{x<-K\}} * \hat{\mu}^c)_\infty} - 1| \\ &+ e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|>K\}} * \hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda} |e^{-(\phi_\lambda 1_{\{x>K\}} * \hat{\mu}^c)_\infty} - 1| \\ &= J_{3,1} + J_{3,2}.\end{aligned}$$

Then we have

$$\begin{aligned}J_{3,1} &\le e^{-(\psi_\lambda 1_{\{x>K\}} * \hat{\mu}^c)_\infty - \frac{\lambda^2}{2}(x^2 1_{\{x<-K\}} * \hat{\mu}^c)_\infty} \\ &\times \Bigg(e^{-\lambda M_\infty - (\phi_\lambda 1_{\{x<-K\}} * \hat{\mu}^c)_\infty} 1_{\{M_\infty + (\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c)_\infty \le 0\}} \\ &+ \left(-M_\infty - \left(\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c \right)_\infty \right)^{-} 1_{\{M_\infty + (\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c)_\infty > 0\}} \Bigg).\end{aligned}$$

Moreover the right-hand side is dominated by

$$\begin{aligned}&\le 1 \times 1 \times \Bigg(e^{\lambda_0 \{-M_\infty - (\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c)_\infty\}} 1_{\{M_\infty + (\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c)_\infty \le 0\}} \\ &+ \left(M_\infty + \left(\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c \right)_\infty \right) 1_{\{M_\infty + (\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c)_\infty > 0\}} \Bigg), \\ &\le e^{-\lambda_0 M_\infty} + M_\infty^+ + \left(\frac{\phi_\lambda}{\lambda} 1_{\{x<-K\}} * \hat{\mu}^c \right)_\infty \\ &\le e^{-\lambda_0 M_\infty} + M_\infty^+ + \left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x<-K\}} * \hat{\mu}^c \right)_\infty,\end{aligned}$$

since the last line is valid by Lemma 4.1 of Kaji [5]. Therefore we see

$$J_{3,1} \le e^{-\lambda_0 M_\infty} + M_\infty^+ + \left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x<-K\}} * \hat{\mu}^c \right)_\infty.$$

On the other hand, we have

$$\begin{aligned}J_{3,2} &\le e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|>K\}} * \hat{\mu}^c)_\infty} \times \Bigg(e^{-(\phi_\lambda 1_{\{x>K\}} * \hat{\mu}^c)_\infty} 1_{\{-(\frac{\phi_\lambda}{\lambda} 1_{\{x>K\}} * \hat{\mu}^c)_\infty \ge 0\}} \\ &+ \left(-\left(\frac{\phi_\lambda}{\lambda} 1_{\{x>K\}} * \hat{\mu}^c \right)_\infty \right)^{-} 1_{\{-(\frac{\phi_\lambda}{\lambda} 1_{\{x>K\}} * \hat{\mu}^c)_\infty < 0\}} \Bigg).\end{aligned}$$

The right-hand side is easily estimated as

$$\leq e^{-\frac{\lambda^2}{2}(x^2 1_{\{|x|>K\}}*\hat{\mu}^c)_\infty} \times \left(e^{-(\phi_\lambda 1_{\{x>K\}}*\hat{\mu}^c)_\infty} 1_{\{(\frac{\phi_\lambda}{\lambda} 1_{\{x>K\}}*\hat{\mu}^c)_\infty \leq 0\}} + 0\right),$$
$$\leq 1,$$

since the last line holds by an inequality

$$-\frac{\lambda^2}{2}x^2 - \phi_\lambda(x) = -\psi_\lambda(x) \leq 0 \text{ on } (-\infty, \infty).$$

Therefore we have

$$J_{3,2} \leq 1.$$

Hence we see

$$(0 \leq) J_3 \leq e^{-\lambda_0 M_\infty} + M_\infty^+ + \left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x<-K\}}*\hat{\mu}^c\right)_\infty + 1,$$

and the proof is complete. □

Finally, we will prove Theorem 1. According to Lemma 2 and the Tauberian theorem (see Liptser and Novikov [7] or Kaji [5], [6]), it is sufficient to show

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda}(E[e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty}] - E[\mathcal{E}(\lambda)_\infty]) = -E[M_\infty].$$

First, the fact

$$\langle M \rangle_\infty = \langle M^c \rangle_\infty + (x^2*\hat{\mu}^c)_\infty + \sum_{0<s<\infty} \Delta(x^2*\hat{\mu})_s$$

implies

$$\begin{aligned}
\frac{1}{\lambda}|e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty} - \mathcal{E}(\lambda)_\infty| &= e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty} \cdot \frac{1}{\lambda}|1 - e^{-\lambda M_\infty - (\phi_\lambda * \hat{\mu}^c)_\infty + \eta_\infty^\lambda}| \\
&\leq e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty} \cdot \frac{1}{\lambda}|1 - e^{\eta_\infty^\lambda}| \\
&\quad + e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty + \eta_\infty^\lambda} \cdot \frac{1}{\lambda}|1 - e^{-\lambda M_\infty - (\phi_\lambda * \hat{\mu}^c)_\infty}| \\
&\leq e^{-\frac{\lambda^2}{2}\sum_{0<s<\infty}\Delta(x^2*\hat{\mu})_s} \cdot \frac{1}{\lambda}|1 - e^{\eta_\infty^\lambda}| \\
&\quad + e^{-\frac{\lambda^2}{2}(x^2*\hat{\mu}^c)_\infty} \cdot \frac{1}{\lambda}|1 - e^{-\lambda M_\infty - (\phi_\lambda * \hat{\mu}^c)_\infty}|.
\end{aligned}$$

for all $\lambda > 0$. Then, Lemmas 4 and 5 together with the last inequality imply

$$\frac{1}{\lambda}|e^{-\frac{\lambda^2}{2}\langle M \rangle_\infty} - \mathcal{E}(\lambda)_\infty| \leq e^{-\lambda_0 M_\infty} + M_\infty^+ + \left(\frac{\phi_{\lambda_0}}{\lambda_0} 1_{\{x<-K\}}*\hat{\mu}\right)_\infty + 3 + 4c_0 K e^{-1}$$

for all $\lambda \in (0, \lambda_0 \wedge \frac{1}{2c_0 K})$. The conclusion follows from this estimate and Lemma 3.

3. The first-passage times of stochastic integrals

For $a(s)$ be a measurable positive function on $[0,\infty)$ set

$$A(t) = \int_0^t a(s)^2 ds.$$

Assume

$$A(t) < \infty \quad \text{for any } t \in (0,\infty). \tag{5}$$

We consider the process in (1). Recall we are supposing the conditions

$$\int_{\mathbf{R}\setminus\{0\}} x^2 \nu(dx) < \infty, \quad \nu((K,\infty)) = 0 \quad \text{for some} \quad 0 \le K < \infty. \tag{6}$$

From (5), (6) we can define a stochastic integral

$$M_t = \int_0^t a(s) dX_s, \; t \in [0,\infty)$$

and set

$$\rho^2 = \sigma^2 + \int_{\mathbf{R}\setminus\{0\}} x^2 \nu(dx) < \infty.$$

For $b > 0$ we introduce the first-passage time of $M = \{M_t\}_{t\in[0,\infty)}$:

$$\tau_b = \begin{cases} \inf\{t > 0 | M_t > b\} & \text{if} \quad \{ \;\} \ne \phi, \\ \infty & \text{if} \quad \{ \;\} = \phi. \end{cases}$$

Theorem 6. *Assume*

$$\sup_{s\in[0,\infty)} a(s) < \infty \quad \text{and} \quad \int_0^\infty a(s)^2 ds = \infty. \tag{7}$$

Then, for any $b > 0$, $\tau_b < \infty$ a.s. is valid, and

$$\begin{cases} \lim_{t\to\infty} \sqrt{A(t)} P(\tau_b > t) = \sqrt{\frac{2}{\pi\rho^2}} E M_{\tau_b} \\ 0 \le E M_{\tau_b} < \infty \end{cases}$$

holds. In particular, if $K = 0$, that is, there is no positive jump, then $M_{\tau_b} = b$, and so $E M_{\tau_b} = b$.

Proof. First, set a new process

$$\widetilde{M}_t = -M_{t\wedge\tau_b}.$$

To prove this theorem, we take the following four steps.
[a] According to the corollary in Liptser and Shiryaev [8](p. 148), (6) and (7) imply $P(\tau_b < \infty) = 1$. Then it is clear that $\langle \widetilde{M} \rangle_\infty = \langle M \rangle_{\tau_b} = \rho^2 A(\tau_b) < \infty$ a.s. holds.

[b] We observe that

$$\widetilde{M}_\tau \geq -b \ \text{ if } \ \tau < \tau_b,$$
$$\widetilde{M}_\tau \geq -b - \Delta M_{\tau_b} \geq -b - K \cdot \sup_{s\in[0,\infty)} a(s) \ \text{ if } \ \tau \geq \tau_b$$

holds for any $\tau \in \mathcal{T}$, where $\mathcal{T}$ is the set of all stopping times with respect to X. Therefore $\{(\widetilde{M}_\tau)^-\}_{\tau\in\mathcal{T}}$ is uniformly integrable.
[c] We consider a random measure $\mu(dsdx)$ on $[0,\infty) \times (\mathbf{R}\backslash\{0\})$ defined by

$$\mu((0,t]\times U) = \sum_{0<s\leq t} 1_U(\Delta\widetilde{M}_s)$$

for all $t\in[0,\infty)$ and $U\in\mathcal{B}(\mathbf{R}\backslash\{0\})$. Then the right-hand side is

$$= \sum_{0<s\leq t\wedge\tau_b} 1_U(-a(s)\Delta X_s)$$
$$= \int_{(0,t\wedge\tau_b]\times(\mathbf{R}\backslash\{0\})} 1_U(-a(s)x)N(dsdx),$$

and so we have

$$\hat{\mu}((0,t]\times U) = \int_0^{t\wedge\tau_b} \int_{(\mathbf{R}\backslash\{0\})} 1_U(-a(s)x)ds\nu(dx),$$

where $\hat{\mu}(dsdx)$ is the predictable compensator of $\mu(dsdx)$. Therefore we have

$$\int_{(0,\infty)\times\{x<-L\}} e^{-x}\hat{\mu}(dsdx) = \int_0^{\tau_b} \int_{\{x>\frac{L}{a(s)}\}} e^{a(s)x}ds\nu(dx),$$

where $L > 0$. Then we see

$$\begin{aligned}\int_{(0,\infty)\times\{x<-L\}} e^{-x}\hat{\mu}(dsdx) &= \int_0^{\tau_b} \int_{\{x>\frac{L}{a(s)}\}} e^{a(s)x}ds\nu(dx) \\ &\leq \int_0^{\tau_b} \int_{\{x>K\}} e^{a(s)x}ds\nu(dx) = 0,\end{aligned}$$

where $L = K \cdot \sup_{s\in[0,\infty)} a(s) > 0$.
[d] Finally, according to Theorem 1, [a], [b] and [c] imply

$$\lim_{t\to\infty} \sqrt{t}P(\langle\widetilde{M}\rangle_\infty > t) = -\sqrt{\frac{2}{\pi}}E[\widetilde{M}_\infty];$$

which completes the proof. □

Remark 7. In particular, we have the asymptotics for the tail of the first passage time of the Lévy process (1) allowing positive jumps if we set $a(s) = 1$.

Remark 8. The condition

$$\int_0^\infty a(s)^2 ds = \infty$$

does not necessarily imply $\tau_b < \infty$ a.s. without an extra condition

$$\sup_{s\in[0,\infty)} a(s) < \infty.$$

A counterexample is given as follows. Let $a(s)$ be a measurable positive function on $[0,\infty)$ with

$$\int_0^\infty a(s)ds < \infty \quad \text{and} \quad \int_0^\infty a(s)^2 ds = \infty.$$

Note that $a(s)$ does not satisfy $\sup_{s\in[0,\infty)} a(s) < \infty$ in this case. Set

$$X_t = -N_t + \lambda t,$$

where $N = \{N_t\}_{t\in[0,\infty)}$ is a Poisson process with parameter $\lambda > 0$ and

$$M_t = \int_0^t a(s)dX_s.$$

It follows from $\int_0^\infty a(s)ds < \infty$ that

$$\int_0^\infty a(s)dN_s = \lim_{t\to\infty} \int_0^t a(s)dN_s$$

exists a.s., because

$$E\left[\int_0^\infty a(s)dN_s\right] = \lambda \int_0^\infty a(s)ds < \infty$$

holds. Hence we have

$$\sup_{t\geq 0} M_t \leq \int_0^\infty a(s)dN_s + \lambda \int_0^\infty a(s)ds < \infty \quad \text{a.s.},$$

which implies that for some $b > 0$, $\tau_b = \infty$ with positive probability.

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