J. Math. Kyoto Univ. (JMKYAZ)
49-2 (2009), 389–417

A construction of processes with one dimensional martingale marginals, based upon path-space Ornstein-Uhlenbeck processes and the Brownian sheet

By

Francis Hirsch and Marc Yor

Abstract

Using a variation from the construction of the Ornstein-Uhlenbeck process on canonical path-space $C([0,1];\mathbb{R})$ in terms of the Brownian sheet, we obtain a large class of processes, adapted to the Brownian filtration, which admit the one dimensional marginals of a martingale.

1. Introduction

1.1.

Recently, Carr et al. [5] showed that the arithmetic average A of geometric Brownian motion $\mathcal{E}^{(\lambda)}$:

$$A_t = \frac{1}{t} \int_0^t \mathcal{E}_s^{(\lambda)} \, \mathrm{d}s \,, \quad t \geq 0$$

where

$$\mathcal{E}_s^{(\lambda)} = \exp\left(\lambda \, B_s - \frac{\lambda^2}{2} \, s\right) \,, \quad s \geq 0$$

with (B_s) a standard Brownian motion and $\lambda \in \mathbb{R}$, is increasing in the convex order, meaning that:
for every convex function $f : \mathbb{R} \longrightarrow \mathbb{R}$, the function

$$t \longrightarrow \mathbb{E}[f(A_t)] \quad (\in (-\infty, +\infty])$$

increases.

2000 *Mathematics Subject Classification*(*s*). 60Gxx, 60Hxx.
Received January 14, 2009
Revised May 13, 2009

1.2.

Later, Baker-Yor [1] showed that $(A_t\ ,\ t \geq 0)$ is a 1-martingale, meaning that it has the one dimensional marginals of a martingale $(M_t\ ,\ t \geq 0)$. Thus, the previous result in 1.1 follows from Jensen's inequality.
Precisely, in [1], it is shown that one can take

$$M_t = \int_0^1 \exp\left(\lambda\, W_{u,t} - \frac{\lambda^2}{2}\, ut\right)\, \mathrm{d}u\ , \quad t \geq 0$$

which is a $(\mathcal{W}_t := \sigma\{W_{u,s}\ ;\ u \leq 1\ ,\ s \leq t\})$-martingale, where $\{W_{u,s}\}$ denotes the standard Brownian sheet (see, e.g., Cairoli-Walsh [4] for a deep study of the Brownian sheet).

1.3.

Our aim in this paper is to develop systematically the above approach in 1.2, by exhibiting many processes $(\Phi^\sharp_t\ ,\ t \leq 1)$, which are adapted to the Brownian filtration $(\mathcal{B}_t := \sigma\{B_s\ ;\ s \leq t\}\ ,\ t \leq 1)$ and are 1-martingales. Basically, our findings rest on the simple, but powerful, observation that, for any fixed t,

$$(B_{t\,u}\ ,\ u \geq 0) \overset{\text{(law)}}{=} (W_{u,t}\ ,\ u \geq 0)$$

1.4.

Our paper is organised as follows:

- In Section 2, we define, via Brownian chaos expansions, Markov operators $(R_t\ ,\ t \leq 1)$ on $L^2(\mathcal{B}_1)$. These operators can also be described from the Ornstein-Uhlenbeck semigroup (T_h) and scaling operators on canonical path-space.

- In Section 3, we show that for any $\Phi \in L^2(\mathcal{B}_1)$, the process $\Phi^\sharp$ defined by

  $$\Phi^\sharp_t(B) = R_t\Phi(B) = T_{-\log t}\Phi(t^{-1/2}\, B_{t\,\bullet})\ , \quad t \leq 1$$

  has the same one dimensional marginals as the process Φ^m defined by

  $$\Phi^m_t(W) = T_{-\log t}\Phi(t^{-1/2}\, W_{\bullet,t})\ , \quad t \leq 1$$

  which is a $(\mathcal{W}_t)$-martingale. We illustrate this result with many examples, notably the example presented in 1.2 above. Moreover, we show that there exist Φ's such that the <u>processes</u> $(\Phi^\sharp_t(B)\ ,\ t \leq 1)$ and $(\Phi^m_t(W)\ ,\ t \leq 1)$ are identical in law; some such examples are closely connected with the construction of non-canonical Brownian motions (see, Jeulin-Yor [8], Hitsuda [6], Chiu [3], Hibino-Hitsuda-Muraoka [7], $\cdots$). However, this identity in law between the two processes is the exception, rather than the rule (see notably Proposition 5.2 below).

- In Section 4, we exhibit a Markov process $(Y^h\ ,\ h \geq 0)$ with semigroup $Q_h = R_{\mathrm{e}^{-h}}\ , h \geq 0$, and we study the infinitesimal generator of (Q_h).

- In Section 5, we study the vector spaces $\mathcal{M}, \mathcal{V}, \mathcal{S}$, which consist of Φ's in $L^2(\mathcal{B}_1)$ such that $\Phi^\sharp$ is respectively a $(\mathcal{B}_t)$-martingale, a continuous process of bounded variation on $[0,1]$, a $(\mathcal{B}_t)$-semimartingale.

2. The operators $(R_t\ ,\ 0 \le t \le 1)$

2.1. Notation

We first introduce some basic notation.

• We denote by E the standard Wiener space $C([0,1];\mathbb{R})$ equipped with the sup-norm:

$$\| B \| = \sup_{0 \le t \le 1} |B_t|$$

The generic element of E shall often be denoted by B or $B_\bullet$, the Wiener measure on E by $\mathbb{P}_{(B)}$, and the corresponding expectation by $\mathbb{E}_{(B)}$. If no confusion is possible, we omit (B) in the notation.
We will use the notation L^2 to denote the L^2-space with respect to $\mathbb{P}_{(B)}$. The corresponding norm will be denoted by $\| \cdot \|_{L^2}$.

• We denote by $(\mathcal{B}_t)_{0 \le t \le 1}$ the usual Brownian filtration on E.

• If $0 < t \le 1$, we denote by σ_t the scaling operator on E defined by:

$$\sigma_t(B) = \frac{1}{\sqrt{t}}\, B_{t\,\bullet}$$

We also denote $\sigma_t(B)$ by $B^{(t)}$, that is:

$$\forall u \in [0,1] \quad B^{(t)}_u = \frac{1}{\sqrt{t}}\, B_{t\,u}$$

• If $0 \le t \le 1$ and $n \ge 1$, we set

$$\Delta_n(t) = \{u \in \mathbb{R}^n\ ;\ t \ge u_1 \cdots \ge u_n \ge 0\}$$

and $\Delta_n = \Delta_n(1)$.
If φ_n is defined on $\Delta_n(t)$, $\| \varphi_n \|_{\Delta_n(t)}$ denotes the L^2-norm of φ_n with respect to the Lebesgue measure on $\Delta_n(t)$.
If φ_n is defined on Δ_n and $0 < t \le 1$, we set

$$\forall u \in \Delta_n(t) \quad \varphi_n^t(u) = \varphi_n\left(\frac{1}{t}\, u\right)$$

Thus,

$$\| \varphi_n^t \|_{\Delta_n(t)} = t^{n/2}\, \| \varphi_n \|_{\Delta_n}$$

• Let $(B_t)_{0 \le t \le 1}$ be a standard linear Brownian motion on the time interval $[0,1]$. We set, for $0 \le t \le 1$, $n \ge 1$ and $\varphi_n \in L^2(\Delta_n(t))$,

$$\begin{aligned} I_n^t(\varphi_n)(B) &= \int_{\Delta_n(t)} \varphi_n(u)\, \mathrm{d}^{(n)} B_u \\ &= \int_0^t \mathrm{d}B_{u_1} \int_0^{u_1} \mathrm{d}B_{u_2} \cdots \int_0^{u_{n-1}} \mathrm{d}B_{u_n}\, \varphi_n(u_1, \cdots, u_n) \end{aligned}$$

We omit t in the notation if $t = 1$; thus, $I_n(\varphi_n) = I_n^1(\varphi_n)$.

2.2. Definition of R_t

Let $\Phi \in L^2$ given by its chaos expansion:

$$\Phi = \sum_{n\geq 0} I_n(\varphi_n) \tag{2.1}$$

where, by convention, $I_0(\varphi_0)$ is set for $\mathbb{E}(\Phi)$. We define $R_t\Phi$ by

$$R_0\Phi = \mathbb{E}(\Phi) \quad \text{ and, for } 0 < t \leq 1,$$

$$R_t\Phi(B) = \mathbb{E}(\Phi) + \sum_{n\geq 1} \int_{\Delta_n} \varphi_n(u)\, \mathrm{d}^{(n)} B_{t\,u} \tag{2.2}$$

Equivalently, for $0 < t \leq 1$,

$$R_t\Phi(B) = \mathbb{E}(\Phi) + \sum_{n\geq 1} t^{n/2} I_n(\varphi_n)(B^{(t)}) \tag{2.3}$$

We also have, for $0 < t \leq 1$,

$$R_t\Phi(B) = \mathbb{E}(\Phi) + \sum_{n\geq 1} \int_{\Delta_n(t)} \varphi_n\left(\frac{1}{t}\,u\right)\, \mathrm{d}^{(n)} B_u \tag{2.4}$$

which can be written:

$$R_t\Phi(B) = \mathbb{E}(\Phi) + \sum_{n\geq 1} I_n^t(\varphi_n^t)(B) \tag{2.5}$$

2.3. Some properties of $(R_t\ ,\ t \leq 1)$

Proposition 2.1.

i) *$(R_t)_{0\leq t\leq 1}$ is a family of linear contractions of L^2.*
More precisely, if Φ is given by its chaos expansion (2.1)*,*

$$\| R_t\Phi \|_{L^2}^2 = (\mathbb{E}(\Phi))^2 + \sum_{n\geq 1} t^n \| I_n(\varphi_n) \|_{\Delta_n}^2 \leq \| \Phi \|_{L^2}^2 \tag{2.6}$$

In particular, if $\mathbb{E}(\Phi) = 0$, $\| R_t\Phi \|_{L^2} \leq \sqrt{t}\, \| \Phi \|_{L^2}$.

ii) *R_1 is the identity operator on L^2 and, for any $t, s \in [0,1]$,*

$$R_t\, R_s = R_{t\,s}$$

iii) *For any $\Phi \in L^2$, the map*

$$t \in [0,1] \longrightarrow R_t\Phi \in L^2$$

is continuous.

Proof. Property i) is a direct consequence of formula (2.3) and of the fact that σ_t preserves the Wiener measure. The expression (2.5) could also be used.

Property ii) is clear.

The continuity of the map $t \in [0,1] \longrightarrow R_t\Phi \in L^2$ at 0 follows from i). By the expression (2.5), the continuity on $(0,1]$ is easy if, for any $n \geq 1$, $\varphi_n \in C(\Delta_n)$. The general case follows by density, according to i). □

2.4. Relation with the Ornstein-Uhlenbeck semigroup

We first recall the definition of the Ornstein-Uhlenbeck semigroup: $T = (T_h)_{h\geq 0}$, on L^2 (see, e.g., Bouleau-Hirsch [2, Chapter II, Section 2] or Nualart [9, p. 49, Definition 1.4.1]). If $\Phi \in L^2$, $\Phi = \sum_{n\geq 0} I_n(\varphi_n)$, and $h \geq 0$,

$$T_h\Phi = \sum_{n\geq 0} \mathrm{e}^{-nh/2} I_n(\varphi_n) \tag{2.7}$$

Then, the so-called Mehler's formula holds:

$$T_h\Phi(B) = \mathbb{E}_{(\widetilde{B})}[\Phi(\mathrm{e}^{-h/2}\,B + \sqrt{1-\mathrm{e}^{-h}}\,\widetilde{B})] \tag{2.8}$$

where $\widetilde{B}$ denotes a Brownian motion independent of B.

We also set, for $0 < t \leq 1$,

$$\Sigma_t\Phi(B) = \Phi(B^{(t)}) = \Phi \circ \sigma_t(B)$$

As σ_t preserves the Wiener measure, Σ_t is an isometry of L^2.

Lemma 2.1. *For $h \geq 0$ and $0 < t \leq 1$*

$$T_h\,\Sigma_t = \Sigma_t\,T_h$$

Proof. By Mehler's formula (2.8),

$$T_h\Sigma_t\Phi(B) = \mathbb{E}_{(\widetilde{B})}[\Phi(\mathrm{e}^{-h/2}\,B^{(t)} + \sqrt{1-\mathrm{e}^{-h}}\,\widetilde{B}^{(t)})]$$

and

$$\Sigma_tT_h\Phi(B) = \mathbb{E}_{(\widetilde{B})}[\Phi(\mathrm{e}^{-h/2}\,B^{(t)} + \sqrt{1-\mathrm{e}^{-h}}\,\widetilde{B})]$$

Therefore the equality follows from the fact that $\widetilde{B}^{(t)}$ and $\widetilde{B}$ have the same law. □

We now note the following useful expression of R_t.

Proposition 2.2. *One has, for any $t \in (0,1]$,*

$$R_t = T_{-\log t}\,\Sigma_t = \Sigma_t\,T_{-\log t}$$

Proof. By formulas (2.3) and (2.7), we have $R_t = \Sigma_t\,T_{-\log t}$, and we may then apply the previous lemma. □

Corollary 2.1. *For $0 \le t \le 1$, R_t is a Markovian operator, and*

$$\forall \Phi \in L^2 \quad R_t\Phi(B) = \mathbb{E}_{(\widetilde{B})}[\Phi(B_{t\bullet} + \sqrt{1-t}\,\widetilde{B})] \tag{2.9}$$

2.5. Extension

We can more generally define, for $\alpha > 0$ and $\beta \ge 0$, a family of Markovian operators $(R_t^{\alpha,\beta})_{0 \le t \le 1}$ by

$$R_t^{\alpha,\beta} = T_{-\alpha \log t}\,\Sigma_{t^\beta} = \Sigma_{t^\beta}\,T_{-\alpha \log t}$$

In particular, we have, for $\Phi \in L^2$, $\Phi = \sum_{n \ge 0} I_n(\varphi_n)$,

$$R_t^{\alpha,\beta}\Phi(B) = \mathbb{E}(\Phi) + \sum_{n \ge 1} t^{\alpha\, n/2} I_n(\varphi_n)(B^{(t^\beta)})$$

or

$$R_t^{\alpha,\beta}\Phi(B) = \mathbb{E}_{(\widetilde{B})}[\Phi(t^{(\alpha-\beta)/2}\, B_{t^\beta\bullet} + \sqrt{1-t^\alpha}\,\widetilde{B})$$

The previous results given for R_t are easily extended to $R_t^{\alpha,\beta}$.

3. Definitions and some properties of the processes $\Phi^\sharp$ and Φ^m

3.1. Notation

We denote by E_2 the space $C([0,1]^2, \mathbb{R})$ equipped with the law $P_{(W)}$ of a Brownian sheet $(W_{s,t})_{0 \le s,t \le 1}$. The generic element of E_2 will be denoted by W.

We also define the filtration $(\mathcal{W}_t)_{0 \le t \le 1}$ on E_2 by

$$\mathcal{W}_t = \sigma\{W_{u,v}\ ;\ 0 \le u \le 1, 0 \le v \le t\}$$

3.2. Definitions of $\Phi^\sharp$ and Φ^m

Let $\Phi \in L^2$. We associate with Φ two processes denoted by $\Phi^\sharp$ and Φ^m, and defined as follows:

$\Phi^\sharp$ is the process defined on the filtered probability space $(E, P_{(B)}, (\mathcal{B}_t)_{0 \le t \le 1})$ by

$$\Phi_t^\sharp(B) = R_t\Phi(B) = \mathbb{E}_{(\widetilde{B})}[\Phi(B_{t\bullet} + \sqrt{1-t}\,\widetilde{B})] \tag{3.1}$$

Φ^m is the process defined on the filtered probability space $(E_2, P_{(W)}, (\mathcal{W}_t)_{0 \le t \le 1})$ by

$$\Phi_t^m(W) = T_{-\log t}\Phi\left(\frac{1}{\sqrt{t}}\, W_{\bullet,t}\right) = \mathbb{E}_{(\widetilde{B})}[\Phi(W_{\bullet,t} + \sqrt{1-t}\,\widetilde{B})] \tag{3.2}$$

In formula (3.1), $\widetilde{B}$ denotes a Brownian motion independent of B, and in formula (3.2), $\widetilde{B}$ denotes a Brownian motion independent of W.

3.3. Main properties of $\Phi^\sharp$ and Φ^m

The following theorem summarizes our main objective in this paper.

Theorem 3.1. *Let $\Phi \in L^2$.*

i) *The process $\Phi^\sharp$ is $(\mathcal{B}_t)$-adapted and L^2-continuous.*

ii) *The process Φ^m is a $(\mathcal{W}_t)$-martingale.*

iii) *For any $t \in [0,1]$, $\Phi^\sharp_t$ and Φ^m_t have the same law.*

Proof. The property i) is clear by the definition and Proposition 2.1, iii). Let $0 \le s \le t$. We have by (3.2):

$$\Phi^m_t(W) = \mathbb{E}_{(\widetilde{B})}[\Phi(W_{\bullet,s} + (W_{\bullet,t} - W_{\bullet,s}) + \sqrt{1-t}\,\widetilde{B})]$$

Therefore, from the properties of the Brownian sheet,

$$\mathbb{E}(\Phi^m_t \mid \mathcal{W}_s)(W) = \mathbb{E}_{(\widehat{B},\widetilde{B})}[\Phi(W_{\bullet,s} + \sqrt{t-s}\,\widehat{B} + \sqrt{1-t}\,\widetilde{B})]$$

where $(\widehat{B}, \widetilde{B})$ denotes a two-dimensional Brownian motion, independent of W. We can thus write:

$$\mathbb{E}(\Phi^m_t \mid \mathcal{W}_s)(W) = \mathbb{E}_{(\widetilde{B})}[\Phi(W_{\bullet,s} + \sqrt{1-s}\,\widetilde{B})] = \Phi^m_s(W)$$

The property iii) follows from the formulas (3.1) and (3.2), since the processes $B_{t\bullet}$ and $W_{\bullet,t}$ have the same law. □

In particular, the processes $\Phi^\sharp$ are 1-*martingales*, meaning that they have the same one dimensional marginals as a martingale. This result is remarkable because, as we will see in the next subsection, there exist many such processes which are continuous and of finite variation. Now, continuous processes with square-integrable variation cannot be 2-*martingales*, that is they cannot have the same two dimensional marginals as a martingale, unless they are constant. A more general result is shown below.

Lemma 3.1. *Let $V = (V_t)_{0 \le t \le 1}$ be a continuous process of finite variation such that*

i) $\mathbb{E}\left[\left(\int_0^1 |dV_s|\right)^2\right] < \infty$

ii) *V has orthogonal increments*

Then, $V_t = V_0$ for $0 \le t \le 1$.

Proof. Let (σ_n) be a sequence of subdivisions of $[0,t]$ whose meshes tend to 0 when n tends to infinity. By hypothesis ii),

$$\mathbb{E}\left[\sum_{\sigma_n}(V_{t_{i+1}} - V_{t_i})^2\right] = \mathbb{E}\left[(V_t - V_0)^2\right]$$

On the other hand,

$$\left[\sum_{\sigma_n}(V_{t_{i+1}} - V_{t_i})^2\right] \le \left(\int_0^1 |dV_s|\right)^2$$

and

$$\left[\sum_{\sigma_n}(V_{t_{i+1}} - V_{t_i})^2\right] \le \sup_{\sigma_n} |V_{t_{i+1}} - V_{t_i}| \left(\int_0^1 |dV_s|\right)$$

Therefore the result follows from i), by using the dominated convergence theorem. □

An interesting consequence of Theorem 3.1 is the following result, which actually is a general result valid for any 1-martingale.

Proposition 3.1. *The process $\Phi^\sharp$ is increasing for the convex order, which means: For any convex function f on $\mathbb{R}$, the map*

$$t \in [0,1] \longrightarrow \mathbb{E}[f(\Phi^\sharp_t)] \in (-\infty, +\infty]$$

is increasing.

Proof. This follows directly from the fact that $\Phi^\sharp$ is a 1-martingale, by Jensen's inequality for conditional expectations. □

Concerning the continuity of the process $\Phi^\sharp$, there is the following partial result.

Proposition 3.2. *Suppose that Φ is continuous on E and that there exist $\lambda \ge 0$ and $c < 1/2$ such that*

$$\forall B \in E \quad |\Phi(B)| \le \lambda \exp(c \,\| B \|^2)$$

Then $\Phi^\sharp$ admits a continuous version which is given by

$$\forall B \in E,\ \forall t \in [0,1], \quad \Phi^\sharp_t(B) = \mathbb{E}_{(\widetilde{B})}[\Phi(B_{t\,\bullet} + \sqrt{1-t}\,\widetilde{B})]$$

Proof. The result follows from the dominated convergence theorem thanks to the following lemma.

Lemma 3.2. *One has, for $0 \le c < 1/2$,*

$$(1-2c)^{-1/2} \le \mathbb{E}[\exp(c \,\| B \|^2)] \le 2(1-2c)^{-1/2}$$

Proof. The first inequality is obvious since $\| B \|^2 \ge (B_1)^2$.

For the second inequality, set $S(B) = \sup_{0 \le s \le 1} B_s$. We have $\| B \| = \sup(S(B), S(-B))$, and, $S(B)$ and $S(-B)$ have the same law as $|B_1|$. Therefore,

$$\mathbb{E}[\exp(c \,\| B \|^2)] = \mathbb{E}[\sup(\exp(c\, S(B)^2), \exp(c\, S(-B)^2))] \le 2\,\mathbb{E}[\exp(c\,(B_1)^2)]$$

which yields the result. □

□

Concerning the continuity of the process Φ^m, we have:

Proposition 3.3. *The process Φ^m admits a continuous version on* $[0,1]$.

Proof. Let, for $0 \leq s, t \leq 1$,

$$\mathcal{W}_{s,t} = \sigma\{W_{u,v} \ ; \ 0 \leq u \leq s, \ 0 \leq v \leq t\}$$

and

$$\Phi_{s,t} = \mathbb{E}(\Phi(W_{\bullet,1}) \mid \mathcal{W}_{s,t})$$

Then, from Cairoli-Walsh [4], the two-parameter martingale $\Phi_{s,t}$ admits a continuous version. Now, since Φ^m is a $(\mathcal{W}_{1,t})$-martingale, then $\Phi^m = \Phi_{1,\bullet}$ and, therefore, Φ^m also admits a continuous version. □

Remark 1. Here is a more direct proof of the fact that any square-integrable $(\mathcal{W}_t)$-martingale admits a continuous version. We now sketch this proof in three steps.

1- Let $(h_n)_{n\geq 0}$ be an orthonormal basis of $L^2([0,1])$ and set, for $n \geq 0$, $t \in [0,1]$,

$$W_t^{(n)} = \int_0^1 h_n(u) \, \mathrm{d}_u W_{u,t}$$

Then $(W^{(n)} \ , \ n \geq 0)$ is a sequence of independent Brownian motions and

$$\mathcal{W}_t = \sigma\{W_s^{(n)} \ ; \ n \geq 0, \ 0 \leq s \leq t\}$$

2- Any $X \in L^2(\mathcal{W}_1)$ admits the following representation:

$$X = c + \sum_{n=0}^{\infty} \int_0^1 H_n(s) \, \mathrm{d}W_s^{(n)}$$

where (H_n) is a sequence of $(\mathcal{W}_t)$-predictable processes such that

$$\mathbb{E}\left[\sum_{n=0}^{\infty} \int_0^1 H_n^2(s) \, \mathrm{d}s\right] < \infty$$

To prove this property, we first consider $X = \mathcal{E}_n^{\varphi}$ with

$$\mathcal{E}_n^{\varphi} = \exp\left[\sum_{k=0}^{n}\left(\int_0^1 \varphi_k(s) \, \mathrm{d}W_s^{(k)} - \frac{1}{2}\int_0^1 \varphi_k^2(s) \, \mathrm{d}s\right)\right]$$

and $\varphi = (\varphi_k)_{0\leq k\leq n} \in (L^2([0,1]))^{n+1}$. We then reason by density.

3- If X admits the above representation, then, for $0 \leq t \leq 1$,

$$X_t := \mathbb{E}(X \mid \mathcal{W}_t) = c + \sum_{n=0}^{\infty} \int_0^t H_n(s) \, \mathrm{d}W_s^{(n)}$$

It is then clear, using Doob's maximal inequality, that $(X_t \ , \ t \in [0,1])$ admits a continuous version.

3.4. Examples

In what follows, $P = (P_t)_{t\geq 0}$ denotes the Gaussian (or Heat) semigroup, and, for $0 < r \leq 1$, we denote by γ_r the normal law with variance r.

$\mathbf{\Phi(B) = f(B_r)}$

Proposition 3.4. *Let $r \in (0,1]$, $f \in L^2(\gamma_r)$ and $\Phi(B) = f(B_r)$. Then, for any $t \in [0,1]$, we have*:

$$\Phi_t^\sharp(B) = P_{(1-t)r} f(B_{t\,r}) = \mathbb{E}(\Phi(B) \mid \mathcal{B}_{t\,r}) \tag{3.3}$$

As a consequence, $(\Phi_t^\sharp(B))$ is a $(\mathcal{B}_{t\,r})$-martingale.

We also have:

$$\Phi_t^m(W) = P_{(1-t)r} f(W_{r,t}) \tag{3.4}$$

Consequently, the processes $\Phi^\sharp$ and Φ^m have the same law.

Proof. We have

$$\Phi_t^\sharp(B) = \mathbb{E}_{(\widetilde{B})}[f(B_{t\,r} + \sqrt{1-t}\,\widetilde{B}_r)] = \mathbb{E}[f(B_{t\,r} + \sqrt{(1-t)r}\,N)]$$

where N denotes a standard normal variable, independent of B. This yields formula (3.3).

Concerning

$$\Phi_t^m(W) = \mathbb{E}_{(\widetilde{B})}[f(W_{r,t} + \sqrt{1-t}\,\widetilde{B}_r)]$$

we get:

$$\Phi_t^m(W) = P_{(1-t)r} f(W_{r,t})$$

Finally, we use that $(B_{t\,r}\ ,\ t \le 1)$ and $(W_{r,t}\ ,\ t \le 1)$ have the same law to conclude. □

Corollary 3.1. *Let φ ($\varphi = \varphi(t,x)$) belong to $C^{1,2}([0,1],\mathbb{R})$. We assume that φ is a time-space harmonic function, that is*

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} = 0 \quad \text{on } [0,1] \times \mathbb{R}$$

Let $r \in [0,1]$. We assume that $\Phi(B) := \varphi(r, B_r) \in L^2$. Then, for any $t \in [0,1]$,

$$\Phi_t^\sharp(B) = \varphi(t\,r, B_{t\,r})$$

We also have:

$$\Phi_t^m(W) = \varphi(t\,r, W_{r,t})$$

Thus, again, the processes $\Phi^\sharp$ and Φ^m have the same law.

Proof. By Itô's formula, $(\varphi(t, B_t)\ ,\ t \le 1)$ is a $(\mathcal{B}_t)$-martingale. So, applying the above proposition to

$$f(x) = \varphi(r,x)\ , \quad x \in \mathbb{R}$$

we get the result. □

Particular cases

(1) $\Phi(B) = \exp(\lambda\, B_r - \frac{\lambda^2 r}{2})$ $(0 \le r \le 1$ and $\lambda \in \mathbb{R})$. Then

$$\Phi^\sharp_t(B) = \exp\left(\lambda\, B_{t\,r} - \frac{\lambda^2\, t\, r}{2}\right)$$

In this case, $\Phi^\sharp$ is thus a geometric Brownian motion.

(2) $\Phi(B) = H_n(r, B_r)$ with $r \in [0,1]$, $n \in \mathbb{N}$ and

$$H_n(t,x) = t^{n/2}\, h_n\left(\frac{x}{\sqrt{t}}\right)$$

where h_n denotes the n-th Hermite polynomial. Then

$$\Phi^\sharp_t(B) = H_n(t\,r, B_{t\,r})$$

This case (2) can also be obtained from the case (1), according to the formula:

$$\exp\left(\lambda\, x - \frac{\lambda^2\, t}{2}\right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, H_n(t,x)$$

The set $\mathcal{I}$ The results in the previous paragraph motivated us to introduce the set

$$\mathcal{I} = \{\Phi \in L^2 \;;\; (\Phi^\sharp_t(B)\;,\; t \le 1) \;\text{ and }\; (\Phi^m_t(W)\;,\; t \le 1) \;\text{ are identical in law }\}$$

Indeed, we have just seen in the previous subsection that $\Phi(B) = f(B_r)$, or $\Phi(B) = \varphi(r, B_r)$ for φ a time-space harmonic function, belong to $\mathcal{I}$. We now characterize the elements in the first Wiener chaos which belong to $\mathcal{I}$.

Proposition 3.5. *Let $h \in L^2([0,1])$ and let ν_h be the norm, in $L^2([0,1])$, of h. We assume $\nu_h \neq 0$ and we set*

$$\Phi(B) = \Phi_h(B) = \int_0^1 h(u)\, \mathrm{d}B_u$$

The following properties are equivalent.

(i) $\Phi \in \mathcal{I}$

(ii) $(\Phi^\sharp_t(B)\;,\; t \le 1)$ *is identical in law to* $(\nu_h\, B_t\;,\; t \le 1)$

(iii) *For every $z \in (0,1]$,*

$$\int_0^1 h(u)\, h(zu)\, \mathrm{d}u = \nu_h^2$$

Condition (iii) is equivalent to the fact that the function $L : [1, \infty] \longrightarrow \mathbb{R}$ defined by $L(v) = (\nu_h)^{-1} h(v^{-1})$ is a *Brownian motion preserving* function, which means that

$$B_t^L := \int_0^t L\left(\frac{t}{s}\right) \, \mathrm{d}B_s \; , \quad t \leq 1$$

is still a Brownian motion.
This topic was introduced by P. Lévy and has been for example discussed in Jeulin-Yor [8]. It is a particular case of the more general family of *non-canonical Brownian motions* (see, e.g., [6, 3, 7]).

Proof. We have:

$$\Phi_t^m(W) = \int_0^1 h(u) \, \mathrm{d}_u W_{u,t}$$

Therefore, the process $(\Phi_t^m(W) \; , \; t \leq 1)$ is clearly distributed as $(\nu_h \, B_t \; , \; t \leq 1)$. This yields the equivalence between properties (i) and (ii).

On the other hand,

$$\Phi_t^\sharp(B) = \int_0^t h\left(\frac{u}{t}\right) \, \mathrm{d}B_u$$

from which, the equivalence between properties (ii) and (iii) follows easily. □

In the following, we denote by $\mathcal{Y}$ the set of functions h satisfying the equivalent conditions of Proposition 3.5. The following corollary is an extension of Proposition 3.4, which corresponds to the case where h is the indicator function of $[0, r]$.

Corollary 3.2. *Let $h \in \mathcal{Y}$. If $f \in L^2(\gamma_{\nu_h^2})$, then $f(\Phi_h) \in \mathcal{I}$.*
In particular, for any $n \geq 1$, $I_n(h^{(n)}) \in \mathcal{I}$, where $h^{(n)}$ is defined on Δ_n by

$$h^{(n)}(u_1, u_2, \cdots, u_n) = h(u_1) \, h(u_2) \, \cdots \, h(u_n)$$

Proof. Let $\Phi = f(\Phi_h)$. Then,

$$\Phi_t^\sharp(B) = \mathbb{E}_{(\widetilde{B})} \left[f\left((\Phi_h)_t^\sharp(B) + \sqrt{1-t} \int_0^1 h(u) \, \mathrm{d}\widetilde{B}_u \right) \right] \quad \text{and}$$
$$\Phi_t^m(W) = \mathbb{E}_{(\widetilde{B})} \left[f\left((\Phi_h)_t^m(W) + \sqrt{1-t} \int_0^1 h(u) \, \mathrm{d}\widetilde{B}_u \right) \right]$$

Then the result follows from Proposition 3.5.

In particular, we can take for f the n-th Hermite polynomial, which yields the second part of the corollary. □

$\mathbf{\Phi = \int_0^1 F_u^\sharp\, h(u)\,du}$ We consider a Borel function h on $[0,1]$ and $F \in L^2$. We assume

$$\int_0^1 \| F_u^\sharp \|_{L^2} \, |h(u)| \, \mathrm{d}u < \infty \tag{3.5}$$

and we set

$$\Phi = \int_0^1 F_u^\sharp \, h(u) \, \mathrm{d}u$$

Proposition 3.6. *For $t \in (0,1]$,*

$$\Phi_t^\sharp = \frac{1}{t} \int_0^t F_u^\sharp \, h\left(\frac{u}{t}\right) \, \mathrm{d}u$$

Proof. By Proposition 2.1, ii),

$$(F_u^\sharp)_t^\sharp = F_{u\,t}^\sharp$$

Therefore

$$\Phi_t^\sharp = \int_0^1 F_{u\,t}^\sharp \, h(u) \, \mathrm{d}u = \frac{1}{t} \int_0^t F_u^\sharp \, h\left(\frac{u}{t}\right) \, \mathrm{d}u$$

□

Proposition 3.7. *Suppose (without loss of generality) that $\mathbb{E}(F) = 0$. We assume that h is an absolutely continuous function on $(0,1]$ satisfying*

$$\int_0^1 u^{3/2} \, |h'(u)| \, \mathrm{d}u < \infty$$

Then the condition (3.5) is satisfied and the process $\Phi^\sharp$ is continuous and of finite variation on $[0,1]$. Moreover, the variation on $[0,1]$ is square-integrable.

Proof. We have

$$\int_0^1 \sqrt{u} \, |h(u)| \, \mathrm{d}u \leq |h(1)| + \int_0^1 s^{3/2} \, |h'(s)| \, \mathrm{d}s < \infty$$

On the other hand, as $\mathbb{E}(F) = 0$, we have by Proposition 2.1, i),

$$\| F_u^\sharp \|_{L^2} \leq \sqrt{u} \, \| F \|_{L^2}$$

Consequently, the condition (3.5) is satisfied.

We have, for $0 < u \leq t \leq 1$,

$$h\left(\frac{u}{t}\right) = h(1) - \int_u^t \frac{u}{s^2} \, h'\left(\frac{u}{s}\right) \, \mathrm{d}s$$

Then, by the previous proposition,

$$\Phi_t^\sharp = \frac{1}{t}\,\left(h(1)\,\int_0^t F_u^\sharp\,\mathrm{d}u - \int_0^t \frac{1}{s^2}\,\left(\int_0^s F_u^\sharp\,u\,h'\left(\frac{u}{s}\right)\,\mathrm{d}u\right)\,\mathrm{d}s\right)$$
$$= \frac{1}{t}\,\left(h(1)\,\int_0^t F_u^\sharp\,\mathrm{d}u - \int_0^t \left(\int_0^1 F_{u\,s}^\sharp\,u\,h'(u)\,\mathrm{d}u\right)\,\mathrm{d}s\right)$$

Therefore, $\Phi_t^\sharp$ is absolutely continuous on $(0,1]$ and

$$\frac{d}{dt}\,\Phi_t^\sharp = \frac{1}{t}\,\left(h(1)\,F_t^\sharp - \int_0^1 F_{u\,t}^\sharp\,u\,h'(u)\,\mathrm{d}u - \Phi_t^\sharp\right)$$

Hence,

$$\left\|\frac{d}{dt}\,\Phi_t^\sharp\right\|_{L^2} \leq \frac{2}{\sqrt{t}}\,\|\,F\,\|_{L^2}\,\left(|h(1)| + \int_0^1 u^{3/2}\,|h'(u)|\,\mathrm{d}u\right)$$

which yields the result. □

As a consequence of the previous proposition and of Lemma 3.1, under the above hypotheses, $\Phi^\sharp$ is a 1-martingale, but it is not a 2-martingale (unless $\Phi = 0$).

Particular case

A particular case is the case $h = 1$. Let $F \in L^2$. We set $\Phi = \int_0^1 F_u^\sharp\,\mathrm{d}u$. Then, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t}\,\int_0^t F_u^\sharp\,\mathrm{d}u$$

and $\Phi_0^\sharp = \mathbb{E}(F)$. By Proposition 3.7, $\Phi^\sharp$ is a continuous process with finite variation on $[0,1]$.

If $F(B) = \exp(\lambda\,B_1 - \frac{\lambda^2}{2})$, then by what we saw in the first paragraph of this subsection, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t}\,\int_0^t \exp\left(\lambda\,B_u - \frac{\lambda^2\,u}{2}\right)\,\mathrm{d}u$$

and $\Phi_0^\sharp = 1$. Proposition 3.1 for this particular case was shown by P. Carr et al. [5] by a completely different method. The proof given later in Baker-Yor [1] is at the origin of our present generalization.

We now give a density result.

Proposition 3.8. *We set*

$$\mathcal{U} = \left\{\Phi = \int_0^1 F_u^\sharp\,h(u)\,\mathrm{d}u\;;\;F \in L^2, h \in C^1([0,1])\right\}$$

Then, for any $\Phi \in \mathcal{U}$, $\Phi^\sharp$ is continuous and of finite variation on $[0,1]$, and $\mathcal{U}$ is dense in L^2.

Proof. By Proposition 3.7, we only need to prove the density. For this purpose, we consider:

$$\Phi_n = \int_0^1 F_u^\sharp \, h_n(u) \, \mathrm{d}u$$

with $h_n \in C^1([0,1])$, $h_n = 0$ on $[0, 1 - 1/n]$, $h_n \geq 0$ and $\int_0^1 h_n(x) \, \mathrm{d}x = 1$. We then have by Proposition 2.1, iii),

$$\lim_{n \to \infty} \Phi_n = F$$

in L^2. □

$\mathbf{\Phi = \int_0^1 F_u^\sharp \, h(u) \, dB_u}$ We begin with a general result.

Proposition 3.9. *Let $\Phi \in L^2$ given by its predictable representation*

$$\Phi = a + \int_0^1 H_u \, \mathrm{d}B_u$$

Then, for any $t \in (0,1]$,

$$\Phi_t^\sharp = a + \int_0^t (H_{u/t})_t^\sharp \, \mathrm{d}B_u$$

Proof. We have easily, according to formula (3.1),

$$\begin{aligned}
\Phi_t^\sharp(B) &= a + \int_0^1 \mathbb{E}_{(\widetilde{B})}[H_u(B_{t\,\bullet} + \sqrt{1-t}\,\widetilde{B})] \, \mathrm{d}B_{t\,u} \\
&= a + \int_0^1 (H_u)_t^\sharp \, \mathrm{d}B_{t\,u} = a + \int_0^t (H_{u/t})_t^\sharp \, \mathrm{d}B_u
\end{aligned}$$

□

Corollary 3.3. *Let $F \in L^2$ and let h be a Borel function on $[0,1]$ such that*

$$\int_0^1 \| F_u^\sharp \|_{L^2}^2 \, |h(u)|^2 \, \mathrm{d}u < \infty$$

We consider

$$\Phi = \int_0^1 F_u^\sharp \, h(u) \, \mathrm{d}B_u$$

Then, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \int_0^t F_u^\sharp \, h\left(\frac{u}{t}\right) \, \mathrm{d}B_u$$

Proof. This is a direct consequence of the previous proposition and of Proposition 2.1, ii). □

Corollary 3.4. *Let $F \in L^2$, $r \in [0,1]$ and $\Phi = \int_0^r F_u^\sharp \, \mathrm{d}B_u$. Then*

$$\Phi_t^\sharp = \int_0^{t\,r} F_u^\sharp \, \mathrm{d}B_u$$

As a consequence, $\Phi^\sharp$ is a $(\mathcal{B}_{t\,r})$-martingale.

We now show that the functions Φ of the previous form generate the whole space L^2.

Proposition 3.10. *Let*

$$\mathcal{H} = \left\{ a + \int_0^r F_u^\sharp \, \mathrm{d}B_u \ ; \ a \in \mathbb{R}, r \in [0,1], F \in L^2 \right\}$$

Then the vector space spanned by $\mathcal{H}$ is dense in L^2.

Proof. Let $G = b + \int_0^1 H_u \, \mathrm{d}B_u$ be orthogonal to $\mathcal{H}$ in L^2. Then

$$\forall a \in \mathbb{R},\ \forall r \in [0,1],\ \forall F \in L^2,\ \ a\,b + \int_0^r \mathbb{E}(F_u^\sharp \, H_u) \, \mathrm{d}u = 0$$

Therefore, $b = 0$ and, for every $F \in L^2$, $\mathbb{E}(F_u^\sharp \, H_u) = 0$ for almost every $u \in [0,1]$. As L^2 is separable, for almost every $u \in [0,1]$, $\mathbb{E}(F_u^\sharp \, H_u) = 0$ for every $F \in L^2$.
Now, by the expression of $F_u^\sharp$ on the Brownian chaoses, we have, for u fixed in $(0,1]$,

$$\{F_u^\sharp \ ; \ F \in L^2\} = \left\{ \sum_{n \geq 0} I_n^u(\varphi_n) \ ; \ \sum_{n \geq 0} u^{-n} \, \| \, \varphi_n \, \|^2_{\Delta_n(u)} < \infty \right\}$$

In particular, $\{F_u^\sharp \ ; \ F \in L^2\}$ is dense in $L^2(\mathcal{B}_u)$. Hence, for almost every $u \in [0,1]$, $H_u = 0$ and, finally, $G = 0$. □

$\mathbf{\Phi = L_r^a}$ In this last example, we take as Φ the local time of B at $a \in \mathbb{R}$ and at time $r \in (0,1]$: $\Phi = L_r^a$.

Proposition 3.11. *For $0 \leq t < 1$,*

$$\Phi_t^\sharp = \frac{1}{\sqrt{2\,\pi\,(1-t)}} \int_0^r \exp\left(-\frac{(B_{t\,s} - a)^2}{2\,s\,(1-t)} \right) \frac{1}{\sqrt{s}} \, \mathrm{d}s \tag{3.6}$$

The process $\Phi^\sharp$ is continuous and of finite variation on $[0,1)$.

Proof. By the occupation times formula, for any $f \in C_c(\mathbb{R})$[*1]

$$\int_0^r f(B_s) \, \mathrm{d}s = \int_{-\infty}^{+\infty} f(a) \, L_r^a \, \mathrm{d}a$$

[*1] Here and in what follows, the subscript: c, means: with compact support.

By Proposition 3.4, we get then, for $t \in [0,1]$,

$$\int_0^r P_{(1-t)\,s} f(B_{t\,s})\,\mathrm{d}s = \int_{-\infty}^{+\infty} f(a)\,(L_r^a)_t^\sharp\,\mathrm{d}a$$

and the formula (3.6) follows by identification.

The continuity of $\Phi^\sharp$ on $[0,1)$ is clear on the formula (3.6).

By change of variable, for $t \in (0,1)$,

$$\Phi_t^\sharp = \frac{1}{\sqrt{2\,\pi\,t\,(1-t)}}\int_0^{t\,r} \exp\left(-\frac{(B_u-a)^2\,t}{2\,u\,(1-t)}\right)\frac{1}{\sqrt{u}}\,\mathrm{d}u$$

Therefore, $\Phi^\sharp$ is of class C^1 on $(0,1)$ and we can write its derivative as a sum of four terms ℓ_t^i, $1 \le i \le 4$ defined below:

$$\ell_t^1 = \frac{2t-1}{2t(1-t)}(\Phi_t^\sharp - \Phi_0^\sharp)$$

$$\ell_t^2 = \frac{\sqrt{r}}{t\sqrt{2\pi(1-t)}}\left[\exp\left(-\frac{(B_{t\,r}-a)^2}{2r(1-t)}\right) - \exp\left(-\frac{a^2}{2r(1-t)}\right)\right]$$

$$\ell_t^3 = -\frac{1}{2t(1-t)^2\sqrt{2\pi(1-t)}}\times$$

$$\int_0^r \left[(B_{tu}-a)^2\exp\left(-\frac{(B_{tu}-a)^2}{2u(1-t)}\right) - a^2\exp\left(-\frac{a^2}{2u(1-t)}\right)\right]u^{-3/2}\mathrm{d}u$$

$$\sqrt{2\pi}\,\ell_t^4 = \frac{2t-1}{2t(1-t)}\int_0^r \exp\left(-\frac{a^2}{2u}\right)u^{-1/2}\mathrm{d}u + \frac{\sqrt{r}}{t\sqrt{(1-t)}}\exp\left(-\frac{a^2}{2r(1-t)}\right)$$
$$- \frac{1}{2t(1-t)^2\sqrt{(1-t)}}\int_0^r a^2\exp\left(-\frac{a^2}{2u(1-t)}\right)u^{-3/2}\mathrm{d}u$$

By Proposition 2.1, i), $\|\,\ell_t^1\,\|_{L^2} = \mathrm{O}(t^{-1/2})$ when t tends to 0. On the other hand, it is not difficult to see that $\|\,\ell_t^2\,\|_{L^2}$ and $\|\,\ell_t^3\,\|_{L^2}$ are $\mathrm{O}(t^{-1/2})$ if $a \neq 0$ and $\mathrm{O}(1)$ if $a = 0$, when t tends to 0. By integration by parts in the last integral, we can write $\sqrt{2\pi}\,\ell_t^4$ as the difference of the two following terms:

$$\frac{1}{2t(1-t)}\int_0^r \left[(1-t)^{-1/2}\exp\left(-\frac{a^2}{2u(1-t)}\right) - (1-2t)\exp\left(-\frac{a^2}{2u}\right)\right]u^{-1/2}\mathrm{d}u$$

and

$$\frac{\sqrt{r}}{(1-t)\sqrt{(1-t)}}\exp\left(-\frac{a^2}{2r(1-t)}\right)$$

Hence,

$$\sqrt{2\pi}\lim_{t\to 0}\ell_t^4 = \frac{1}{4}\int_0^r u^{-3/2}(5u-a^2)\exp\left(-\frac{a^2}{2u}\right)\mathrm{d}u - \sqrt{r}\exp\left(-\frac{a^2}{2r}\right)$$

Finally, $\| \frac{d}{dt}\Phi^\sharp_t \|_{L^2}$ is $\mathrm{O}(t^{-1/2})$ when t tends to 0, which entails that $\Phi^\sharp$ is of finite variation on any interval $[0, s]$ with $0 < s < 1$, the variation being square-integrable on any such interval. □

4. Semigroup Q

4.1. Definition of Q

In this section, we are interested in the family $Q = (Q_h)_{h\geq 0}$ of operators in L^2, defined by

$$\forall h \geq 0 \quad Q_h = R_{\mathrm{e}^{-h}}$$

where the operators R_t were defined in Section 2. As a direct consequence of Proposition 2.1 and Corollary 2.1, we have:

Proposition 4.1. *$Q = (Q_h)_{h\geq 0}$ is a strongly continuous semigroup of Markovian operators in L^2.*

4.2. Infinitesimal generator of Q

In this subsection, we look for a description of the infinitesimal generator $\mathcal{A}$, of the semigroup Q. The domain of $\mathcal{A}$ will be denoted by $\mathrm{dom}(\mathcal{A})$.

We first introduce another notation: If φ_n is a C^1 function on the interior of Δ_n (denoted by $\mathrm{int}(\Delta_n)$), we denote by $\widehat{\varphi}_n$ the function defined by

$$\forall u \in \mathrm{int}(\Delta_n) \quad \widehat{\varphi}_n(u) = \sum_{j=1}^{n} \frac{\partial \varphi_n}{\partial u_j}(u)\, u_j = \varphi_n'(u) \cdot u$$

We also denote by $\Delta_n' = \{u \in \Delta_n \; ; \; u_1 \neq 0\}$

Theorem 4.1. *We denote by $D_{\mathcal{A}}$ the space of functions $\Phi = \sum_{n\geq 0} I_n(\varphi_n) \in L^2$ such that*

1- *for any $n \geq 1$, φ_n is continuous on Δ_n' and of class C^1 on int(Δ_n),*

2- *$\forall n \geq 1$, $\forall v \in \Delta_{n-1}$, $\varphi_n(1, v) = 0$,*

3- *$\sum_{n\geq 1} \| \widehat{\varphi}_n \|^2_{\Delta_n} < \infty$.*

Then $D_{\mathcal{A}} \subset$ dom$(\mathcal{A})$, and,

$$\forall \Phi \in D_{\mathcal{A}} \quad \mathcal{A}\Phi = \sum_{n\geq 1} I_n(\widehat{\varphi}_n)$$

Proof. Let $\Phi \in D_{\mathcal{A}}$. We have, by formula (2.4) in Section 2,

$$Q_h\Phi(B) = \mathbb{E}(\Phi) + \sum_{n\geq 1} \int_{\Delta_n(\mathrm{e}^{-h})} \varphi_n(\mathrm{e}^h\, u)\, \mathrm{d}^{(n)} B_u$$

Now, for $n \geq 1$ and $u \in \Delta_n(\mathrm{e}^{-h})$ with $u_1 \neq 0$, we have by hypotheses 1 and 2:

$$\varphi_n(\mathrm{e}^h\, u) = -\int_h^{-\log u_1} \widehat{\varphi}_n(\mathrm{e}^k\, u)\; \mathrm{d}k = -\int_{u_1}^{\mathrm{e}^{-h}} \frac{1}{s}\,\widehat{\varphi}_n\left(\frac{1}{s}\,u\right)\; \mathrm{d}s$$

Therefore,

$$\int_{\Delta_n(\mathrm{e}^{-h})} \varphi_n(\mathrm{e}^h\, u)\; \mathrm{d}^{(n)}B_u = -\int_0^{\mathrm{e}^{-h}} \frac{1}{s}\left[\int_{\Delta_n(s)} \widehat{\varphi}_n\left(\frac{1}{s}\,u\right)\; \mathrm{d}^{(n)}B_u\right]\; \mathrm{d}s$$

The result then follows by taking the derivative with respect to h at $h = 0$. □

We now can complete the description of the infinitesimal generator $\mathcal{A}$.

Theorem 4.2. *The infinitesimal generator $\mathcal{A}$ is the closure of its restriction to $D_{\mathcal{A}}$.*

Proof. We proceed in three steps.

1) Let

$$\Psi = \sum_{n\geq 0} I_n(\psi_n) \in L^2$$

such that, for every $n \geq 1$, $\psi_n \in C_c^1(\mathrm{int}(\Delta_n))$ and let $\ell \in C_c^1((0,\infty))$. We set

$$\Phi = \int_0^\infty Q_h \Psi\; \ell(h)\; \mathrm{d}h$$

We have

$$\begin{aligned}\Phi(B) &= \sum_{n\geq 0} \int \ell(h) \left(\int_{\Delta_n(\mathrm{e}^{-h})} \psi_n(\mathrm{e}^h\, u)\; \mathrm{d}^{(n)}B_u\right)\; \mathrm{d}h \\ &= \sum_{n\geq 0} \int_{\Delta_n} \left(\int_0^{-\log u_1} \ell(h)\, \psi_n(\mathrm{e}^h\, u)\; \mathrm{d}h\right)\; \mathrm{d}^{(n)}B_u\end{aligned}$$

Therefore, for $n \geq 1$,

$$\varphi_n(u) = \int_0^{-\log u_1} \ell(h)\, \psi_n(\mathrm{e}^h\, u)\; \mathrm{d}h$$

Consequently, Φ satisfies the conditions 1 and 2 of the statement of Theorem 4.1 and, for $n \geq 1$,

$$\begin{aligned}\widehat{\varphi}_n(u) &= \int_0^{-\log u_1} \ell(h)\, \widehat{\psi}_n(\mathrm{e}^h\, u)\; \mathrm{d}h = \int_0^{-\log u_1} \ell(h)\, \frac{\partial}{\partial h}[\psi_n(\mathrm{e}^h\, u)]\; \mathrm{d}h \\ &= -\int_0^{-\log u_1} \ell'(h)\, \psi_n(\mathrm{e}^h\, u)\; \mathrm{d}h\end{aligned}$$

Then, setting $C = \int_0^\infty (\ell'(h))^2 \, \mathrm{d}h$, we have:

$$\begin{aligned} \| \widehat{\varphi}_n \|^2_{\Delta_n} &\le C \int_{\Delta_n} \left[\int_0^{-\log u_1} \psi_n^2(\mathrm{e}^h\, u) \, \mathrm{d}h \right] \mathrm{d}u \\ &= C \int_0^\infty \left[\int_{\Delta_n(\mathrm{e}^{-h})} \psi_n^2(\mathrm{e}^h\, u) \, \mathrm{d}u \right] \mathrm{d}h = \frac{C}{n} \, \| \psi_n \|^2_{\Delta_n} \end{aligned}$$

Therefore, Φ also satisfies the condition 3 of Theorem 4.1. Thus, $\Phi \in D_{\mathcal{A}}$. Moreover,

$$I_n(\widehat{\varphi}_n) = - \int_0^\infty \ell'(h) \left(\int_{\Delta_n(\mathrm{e}^{-h})} \psi_n(\mathrm{e}^h\, u) \, \mathrm{d}^{(n)} B_u \right) \mathrm{d}h$$

and consequently

$$\mathcal{A}\Phi = - \int_0^\infty Q_h \Psi \, \ell'(h) \, \mathrm{d}h$$

2) Assume now that $\Psi \in L^2$ and $\ell \in C_c^1((0,\infty))$, and set as before

$$\Phi = \int_0^\infty Q_h \Psi \, \ell(h) \, \mathrm{d}h$$

It is easy to see directly that

$$\Phi \in \mathrm{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\Phi = - \int_0^\infty Q_h \Psi \, \ell'(h) \, \mathrm{d}h$$

We can approximate Ψ in L^2 by a sequence $\Psi_p = \sum_{n \ge 0} I_n(\psi_{n,p})$ such that $\psi_{n,p} \in C_c^1(\mathrm{int}(\Delta_n))$. We set

$$\Phi_p = \int_0^\infty Q_h \Psi_p \, \ell(h) \, \mathrm{d}h$$

Then, by the first step, $\Phi_p \in D_{\mathcal{A}}$, $\mathcal{A}\Phi_p = - \displaystyle\int_0^\infty Q_h \Psi_p \, \ell'(h) \, \mathrm{d}h$, and therefore

$$\lim_{p \to \infty} (\Phi_p, \mathcal{A}\Phi_p) = (\Phi, \mathcal{A}\Phi)$$

in $L^2 \times L^2$.

3) Finally, let $\Phi \in \mathrm{dom}(\mathcal{A})$. We consider a sequence (ℓ_p) in $C_c^1((0,\infty))$ such that, for every p,

$$\ell_p \ge 0, \quad \ell_p(h) = 0 \text{ on } [1/p, \infty), \quad \int_0^\infty \ell_p(h) \, \mathrm{d}h = 1$$

We set

$$\Phi_p = \int_0^\infty Q_h \Phi \, \ell_p(h) \, \mathrm{d}h$$

We have clearly

$$\Phi_p \in \mathrm{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\Phi_p = \int_0^\infty Q_h \mathcal{A}\Phi \, \ell_p(h) \, \mathrm{d}h$$

and therefore

$$\lim_{p\to\infty} (\Phi_p, \mathcal{A}\Phi_p) = (\Phi, \mathcal{A}\Phi)$$

in $L^2 \times L^2$.

The result follows from both previous approximations. □

We now give a large subset of $\mathrm{dom}(\mathcal{A})$; it consists in the functions appearing in Proposition 3.7.

Proposition 4.2. *We assume that ℓ is an absolutely continuous function on $(0,1]$ satisfying*

$$\int_0^1 u^{3/2} \, |\ell'(u)| \, \mathrm{d}u < \infty$$

Let $F \in L^2$ such that $\mathbb{E}(F) = 0$ and

$$\Phi = \int_0^1 F_u^\sharp \, \ell(u) \, \mathrm{d}u$$

Then

$$\Phi \in \mathit{dom}(\mathcal{A}) \quad \mathit{and} \quad \mathcal{A}\Phi = \int_0^1 F_u^\sharp \, [\ell(u) + u\,\ell'(u)] \, \mathrm{d}u - \ell(1)F$$

Proof. We saw, in the proof of Proposition 3.7, that

$$\| F_u^\sharp \|_{L^2} \le \sqrt{u} \, \| F \|_{L^2} \quad \text{and} \quad \int_0^1 u^{1/2} \, |\ell(u)| \, \mathrm{d}u < \infty$$

In particular, $\Phi \in L^2$. We have, for $h > 0$,

$$\frac{1}{h} (Q_h\Phi - \Phi) = \frac{1}{h} \int_0^1 (F^\sharp_{\mathrm{e}^{-h}u} - F_u^\sharp) \, \ell(u) \, \mathrm{d}u$$

Therefore,

$$\frac{1}{h} (Q_h\Phi - \Phi) = \frac{1}{h} \left(\int_0^{\mathrm{e}^{-h}} F_u^\sharp \, [\mathrm{e}^h \ell(\mathrm{e}^h u) - \ell(u)] \, \mathrm{d}u - \int_{\mathrm{e}^{-h}}^1 F_u^\sharp \, \ell(u) \, \mathrm{d}u \right)$$

Therefore, letting h tend to 0, we get the result. □

Remark 2. The previous result still holds if $\mathbb{E}(F) \neq 0$, provided we assume that ℓ satisfies the additional condition:

$$\int_0^1 |\ell(u)| \, \mathrm{d}u < \infty$$

4.3. A Markovian process with semigroup (Q_h)

In this subsection, we shall associate a Markov process with the semigroup Q. We adopt, for this subsection, the following notation.

We denote by $(W(s,t)\ ,\ s \geq 0, t \geq 0)$ a standard Brownian sheet, and we set, for $h \geq 0$,

$$\widehat{\mathcal{W}}_h = \sigma\{W(u,v)\ ;\ 0 \leq u \leq 1,\ 1 \leq v \leq \mathrm{e}^h\}$$

We define a process $(Y^h\ ,\ h \geq 0)$ taking values in the Wiener space E by:

$$\forall u \in [0,1] \quad Y_u^h = W(\mathrm{e}^{-h}u, \mathrm{e}^h)$$

Proposition 4.3. *For $h \geq 0$, $k \geq 0$, and for any $\Phi \in L^2$, one has*:

$$\mathbb{E}[\Phi(Y^{h+k}) \mid \widehat{\mathcal{W}}_k] = Q_h \Phi(Y^k)$$

Proof. We have

$$\Phi(Y^{h+k}) = \Phi(W(\mathrm{e}^{-(h+k)}\bullet, \mathrm{e}^k) + (W(\mathrm{e}^{-(h+k)}\bullet, \mathrm{e}^{h+k}) - W(\mathrm{e}^{-(h+k)}\bullet, \mathrm{e}^k)))$$

Therefore,

$$\mathbb{E}[\Phi(Y^{h+k}) \mid \widehat{\mathcal{W}}_k] = \mathbb{E}_{(\widetilde{B})}[\Phi(Y^k_{\mathrm{e}^{-h}\bullet} + \sqrt{1-\mathrm{e}^{-h}}\, \widetilde{B})]$$

where $\widetilde{B}$ denotes a Brownian motion independent of W. Hence the result follows by the definition of Q_h and formula (2.9). □

We remark that the process Y^h is nothing else but $\sigma_{\mathrm{e}^{-h}}(X^h)$, where (X^h) denotes the classical Ornstein-Uhlenbeck process in the Wiener space, and σ_t denotes as previously the scaling operator with parameter t.

5. Spaces $\mathcal{M}$, $\mathcal{V}$ and $\mathcal{S}$

In this section, we are interested in description and properties of the following spaces:

$$\mathcal{M} = \{\Phi \in L^2\ ;\ \Phi^\sharp \text{ is a } (\mathcal{B}_t)\text{-martingale}\}$$
$$\mathcal{V} = \{\Phi \in L^2\ ;\ \Phi^\sharp \text{ is a continuous process with finite variation on } [0,1]\}$$
$$\mathcal{S} = \{\Phi \in L^2\ ;\ \Phi^\sharp \text{ is a } (\mathcal{B}_t)\text{-semi-martingale}\}$$

5.1. Space $\mathcal{M}$

On Itô's integrand for $\boldsymbol{\Phi} \in \mathcal{M}$

Theorem 5.1. *Let $\Phi \in L^2$ given by its predictable representation*

$$\Phi = a + \int_0^1 H_u \, \mathrm{d}B_u$$

Then $\Phi \in \mathcal{M}$ if and only if the following condition is fulfilled:

There exists a version of H which is L^2-continuous on $[0,1)$ and satisfies

$$\forall u \in [0,1),\ \forall t \in [0,1], \quad (H_u)^\sharp_t = H_{u\,t}$$

Proof. By Proposition 3.9, $\Phi \in \mathcal{M}$ if and only if

$$\forall t \in (0,1] \quad (H_{u/t})^\sharp_t = H_u \ \text{ for almost every } u \in [0,t]$$

or

$$\forall t \in (0,1] \quad (H_u)^\sharp_t = H_{u\,t} \ \text{ for almost every } u \in [0,1]$$

The condition of the theorem is therefore sufficient.

Conversely, suppose that $\Phi \in \mathcal{M}$. Then, by Fubini's theorem, for almost every $u \in (0,1]$,

$$(H_u)^\sharp_t = H_{u\,t} \ \text{ for almost every } t \in [0,1]$$

or

$$(H_u)^\sharp_{t/u} = H_t \ \text{ for almost every } t \in [0,u]$$

Then, considering a sequence (u_n) tending to 1 and such that the above property holds for each u_n, we see, according to Theorem 3.1, i), that there exists a version of H which is L^2- continuous on each interval $[0,u_n]$ and hence on $[0,1)$. For such a version,

$$\forall t \in (0,1] \quad (H_u)^\sharp_t = H_{u\,t} \ \text{ for every } u \in [0,1)$$

Letting t tend to 0, we also have

$$(H_u)^\sharp_0 = H_0 \ \text{ for every } u \in [0,1)$$

Thus, the condition of the theorem is necessary. □

Definition of the space $\mathcal{N}$ We now introduce

$$\mathcal{N} = \left\{ \Phi = a + \int_0^1 F^\sharp_u \, \mathrm{d}B_u \ ; \ F \in L^2 \text{ and } a \in \mathbb{R} \right\}$$

As a consequence of Theorem 5.1 or of Corollary 3.4, we have: $\mathcal{N} \subset \mathcal{M}$.

The following proposition clarifies the situation in the framework of the first example of Subsection 3.4.

Proposition 5.1. *Let $f \in L^2(\gamma_1)$ and $\Phi(B) = f(B_1)$. Then $\Phi \in \mathcal{N}$ if and only if the function f is absolutely continuous on $\mathbb{R}$ and its derivative f' belongs to $L^2(\gamma_1)$. In this case,*

$$\Phi(B) = \mathbb{E}_{\gamma_1}(f) + \int_0^1 F_u^\sharp \, \mathrm{d}B_u$$

with $F(B) = f'(B_1)$.

Proof. Let $(h_n)_{n\geq 0}$ be the sequence of Hermite polynomials. As $f \in L^2(\gamma_1)$, f admits the following expansion in $L^2(\gamma_1)$:

$$f = \sum_{n\geq 0} a_n \, h_n \quad \text{with} \quad \sum_{n\geq 0} \frac{1}{n!} \, a_n^2 < \infty$$

Let $\Phi(B) = f(B_1)$. Then

$$\Phi = a_0 + \int_0^1 H_s \, \mathrm{d}B_s$$

with

$$H_s = \sum_{n\geq 1} a_n \int_{\Delta_{n-1}(s)} \mathrm{d}^{(n-1)} B_u$$

Therefore

$$\|H_s\|_{L^2}^2 = \sum_{n\geq 1} \frac{1}{(n-1)!} \, a_n^2 \, s^{n-1}$$

Consequently, $\Phi \in \mathcal{N}$ if and only if $\sum_{n\geq 1} \frac{1}{(n-1)!} \, a_n^2 < \infty$.
Equivalently, $\Phi \in \mathcal{N}$ if and only if f belongs to the domain of the canonical Dirichlet form on $L^2(\gamma_1)$, that is if and only if f is an absolutely continuous function on $\mathbb{R}$ such that f and f' belong to $L^2(\gamma_1)$ (see, e.g., Bouleau-Hirsch [2]).
In this case, we have $H_s = F_s^\sharp$ with

$$F(B) = \sum_{n\geq 1} a_n \int_{\Delta_{n-1}} \mathrm{d}^{(n-1)} B_u = \sum_{n\geq 1} a_n \, h_{n-1}(B_1) = f'(B_1)$$

□

Comparison of $\mathcal{M}$ and $\mathcal{N}$

Theorem 5.2. *The following properties hold:*

i) $\mathcal{N} \subset \mathcal{M}$

ii) $\mathcal{N} \neq \mathcal{M}$

iii) $\mathcal{M}$ *is the closure of* $\mathcal{N}$ *in* L^2

Proof. As it was already mentioned, the property i) is contained in Corollary 3.4. It also is a consequence of Theorem 5.1.

The property ii) is a direct consequence of the previous proposition 5.1 (consider for example $\Phi(B) = f(B_1)$ with f the indicator function of $\mathbb{R}_+$).

As $\mathcal{M}$ is closed, the property i) entails that $\mathcal{M}$ contains the closure of $\mathcal{N}$. Suppose then that $\Phi \in \mathcal{M}$ and consider its predictable representation fulfilling the condition of Theorem 5.1. We set, for $0 < v < 1$,

$$\Phi^{(v)} = a + \int_0^1 H_{u\,v}\ \mathrm{d}B_u$$

By definition, $\Phi^{(v)} \in \mathcal{N}$ and

$$\| \Phi^{(v)} - \Phi \|_{L^2}^2 = \int_0^1 \| H_{u\,v} - H_u \|_{L^2}^2\ \mathrm{d}u$$

Now, for $u \in [0,1)$,

$$\lim_{v \to 1} \| H_{u\,v} - H_u \|_{L^2}^2 = 0 \quad \text{and}$$
$$\forall 0 < v < 1 \quad \| H_{u\,v} \|_{L^2}^2 = \| (H_u)_v^\sharp \|_{L^2}^2 \le \| H_u \|_{L^2}^2$$

Therefore

$$\lim_{v \to 1} \| \Phi^{(v)} - \Phi \|_{L^2}^2 = 0$$

□

Comparison of $\mathcal{M}$ and $\mathcal{I}$ In Subsection 3.4, we introduced the set

$$\mathcal{I} = \{\Phi \in L^2\ ;\ (\Phi_t^\sharp(B)\ ,\ t \le 1)\ \text{ and }\ (\Phi_t^m(W)\ ,\ t \le 1)\ \text{ are identical in law}\}$$

and we gave examples of functions in $\mathcal{I}$. It is immediate, from the discussion following Proposition 3.5, that if h is assumed regular, then Φ_h in that proposition belongs to $\mathcal{M}$ if and only if it is constant. Thus, $\mathcal{I}$ is not contained in $\mathcal{M}$. Conversely, a natural question is to decide whether the set $\mathcal{M}$ is contained in $\mathcal{I}$. First, we note the following simple identity.

Lemma 5.1. *For any bounded Borel function* $h : [0,1] \longrightarrow \mathbb{R}$ *and* $\Phi \in \mathcal{M}$*, there is the identity*

$$\mathbb{E}\left[\left(\int_0^1 h(t)\ \mathrm{d}\Phi_t^m\right)^2\right] = \mathbb{E}\left[\left(\int_0^1 h(t)\ \mathrm{d}\Phi_t^\sharp\right)^2\right] \tag{5.1}$$

Proof. As (Φ_t^m) and $(\Phi_t^\sharp)$ are square-integrable continuous martingales, we have:

$$\mathbb{E}\left[\left(\int_0^1 h(t)\,\mathrm{d}\Phi_t^m\right)^2\right] = \mathbb{E}\left[\int_0^1 h^2(t)\,\mathrm{d}\langle\Phi^m\rangle_t\right] = \int_0^1 h^2(t)\,\mathrm{d}_t\mathbb{E}[(\Phi_t^m)^2]$$

and similarly

$$\mathbb{E}\left[\left(\int_0^1 h(t)\,\mathrm{d}\Phi_t^\sharp\right)^2\right] = \mathbb{E}\left[\int_0^1 h^2(t)\,\mathrm{d}\langle\Phi^\sharp\rangle_t\right] = \int_0^1 h^2(t)\,\mathrm{d}_t\mathbb{E}[(\Phi_t^\sharp)^2]$$

Consequently the identity (5.1) holds, since Φ^m and $\Phi^\sharp$ have the same one dimensional marginals. □

Despite Lemma 5.1, the following proposition shows that, in general, the elements of $\mathcal{N}$ do not belong to $\mathcal{I}$.

Proposition 5.2. *Let* $0 < a < 1$. *We set*

$$\Phi(B) = \int_0^1 B_{a\,u}\,\mathrm{d}B_u$$

Then, $\Phi \in \mathcal{N}$ *and* $\Phi \notin \mathcal{I}$. *More precisely, if* $0 < t < 1$,

$$\mathbb{E}[(\Phi)^2\,\Phi_t^\sharp] \neq \mathbb{E}[(\Phi_1^m)^2\,\Phi_t^m]$$

Proof. Obviously, since $B_{a\,u} = F_u^\sharp(B)$ with $F(B) = B_a$, $\Phi \in \mathcal{N}$.
We have:

$$\Phi_t^\sharp(B) = \int_0^t B_{a\,u}\,\mathrm{d}B_u \quad \text{and} \quad \Phi_t^m(W) = \int_0^1 W_{au,t}\,\mathrm{d}_u W_{u,t}$$

Now, a tedious computation yields:

$$\mathbb{E}[(\Phi)^2\,\Phi_t^\sharp)] = \frac{1}{3}\,[2a^3t^3 + (t\wedge a)^3 + 3t^2(a - a\wedge t)]$$

whereas:

$$\mathbb{E}[(\Phi_1^m)^2\,\Phi_t^m] = a^3t^2$$

which entails

$$\mathbb{E}[(\Phi)^2\,\Phi_t^\sharp)] > \mathbb{E}[(\Phi_1^m)^2\,\Phi_t^m]$$

□

5.2. Space $\mathcal{V}$

Proposition 5.3. *We have*

$$dom(\mathcal{A}) \subset \mathcal{V}$$

Moreover, if $\Phi \in dom(\mathcal{A})$, *the variation of* $\Phi^\sharp$ *on* $[0,1]$ *is square-integrable.*

Proof. Let $\Phi \in \mathrm{dom}(\mathcal{A})$. Then, for $h \geq 0$,

$$Q_h \Phi = \Phi + \int_0^h Q_k \mathcal{A}\Phi \, \mathrm{d}k$$

So, after change of variable, we get for $t > 0$

$$\Phi_t^\sharp = \Phi + \int_t^1 (\mathcal{A}\Phi)_s^\sharp \, \frac{1}{s} \, \mathrm{d}s$$

As $\mathbb{E}(\mathcal{A}\Phi) = 0$, then $\| (\mathcal{A}\Phi)_s^\sharp \|_{L^2} \leq \sqrt{s} \, \| \mathcal{A}\Phi \|_{L^2}$ and the result follows. More precisely, we obtain that the L^2-norm of the variation of $\Phi^\sharp$ on $[0, 1]$ is smaller than $(2 \, \| \mathcal{A}\Phi \|_{L^2})$. □

As a consequence of the previous proposition and of Proposition 4.2, we obtain again Proposition 3.7.

5.3. Space $\mathcal{S}$

In this subsection, we give a sufficient condition, based on the chaos expansion, entailing that a function $\Phi \in L^2$ belongs to $\mathcal{S}$.

We keep the notation of Section 4 and, for $n \geq 1$ and for φ_n a function on Δ_n, we denote by $\widetilde{\varphi}_n$ the function defined on Δ'_n by

$$\widetilde{\varphi}_n(u) = \varphi_n \left(\frac{1}{u_1} \, u \right)$$

We remark that

$$\| \, \widetilde{\varphi}_n \, \|_{\Delta_n}^2 = \frac{1}{n} \, \| \, \varphi_n(1, \bullet) \, \|_{\Delta_{n-1}}^2$$

Theorem 5.3. *Let $\Phi = \sum_{n \geq 0} I_n(\varphi_n) \in L^2$ such that*

1- *for any $n \geq 1$, φ_n is continuous on Δ'_n, φ_n is of class C^1 on int(Δ_n) and $\varphi_n(1, \bullet)$ is of class C^1 on int(Δ_{n-1}),*

2- $\sum_{n \geq 1} \| \, \widetilde{\varphi}_n \, \|_{\Delta_n}^2 < \infty$,

3- $\sum_{n \geq 1} \| \, \widehat{\varphi}_n \, \|_{\Delta_n}^2 < \infty$.

Then $\Phi \in \mathcal{S}$. More precisely, $\Phi = \Phi_1 + \Phi_2$ with

$$\Phi_1 = \mathbb{E}(\Phi) + \sum_{n \geq 1} I_n(\widetilde{\varphi}_n) \in \mathcal{M} \quad \textit{and}$$

$$\Phi_2 = \sum_{n \geq 1} I_n(\varphi_n - \widetilde{\varphi}_n) \in D_{\mathcal{A}} \;\; (\subset \mathit{dom}(\mathcal{A}) \subset \mathcal{V})$$

Proof. We can write

$$\Phi_1 = \mathbb{E}(\Phi) + \int_0^1 H_s \, \mathrm{d}B_s$$

with, for $s \in (0,1)$,

$$H_s = \sum_{n \geq 1} \int_{\Delta_{n-1}(s)} \varphi_n \left(1, \frac{1}{s}\, v\right) \, \mathrm{d}^{(n)} B_v$$

Hence, we see that Φ_1 satisfies the condition of Theorem 5.1.

On the other hand, it is clear, by the definition of $D_{\mathcal{A}}$ given in Theorem 4.1, that $\Phi_2 \in D_{\mathcal{A}}$. We can then apply the results of this section to conclude that $\Phi \in \mathcal{S}$. □

Remark 3.

If we replace, in the statement of the previous theorem, the condition 2 by the stronger condition:

$$\sum_{n \geq 1} n \parallel \widetilde{\varphi}_n \parallel^2_{\Delta_n} < \infty$$

then $\Phi_1 \in \mathcal{N}$.

Acknowledgements. The second author (M. Y) would like to thank RIMS, Kyoto, and in particular Professor Takahashi for hospitality in November 2008, when this work was started. He also thanks Professor Shigekawa for a discussion about Malliavin Calculus.

Laboratoire d'Analyse et Probabilités
Université d'Évry - Val d'Essonne, Boulevard F. Mitterrand
F-91025 Évry Cedex
France
e-mail: francis.hirsch@univ-evry.fr

Laboratoire de Probabilités et Modèles Aléatoires
Université Paris VI et VII, 4 Place Jussieu - Case 188
F-75252 Paris Cedex 05
France
e-mail: deaproba@proba.jussieu.fr
and Institut Universitaire de France

References

[1] D. Baker and M. Yor, *A Brownian sheet martingale with the same marginals as the arithmetic average of geometric Brownian motion*, Prepublications LPMA, n^o 1265, Dec. 2008.

[2] N. Bouleau and F. Hirsch, Dirichlet Forms and Analysis on Wiener Space, Walter de Gruyter, 1991.

[3] Y. Chiu, *From an example of Lévy's*, in Sém. Prob. XXIX, Lecture Notes in Math. **1613**, Springer, 1995, pp. 162–165.

[4] R. Cairoli and J. B. Walsh, *Stochastic integrals in the plane*, Acta Math. **134** (1977), 111–183.

[5] P. Carr, C.-O. Ewald and Y. Xiao, *On the qualitative effect of volatility and duration on prices of Asian options*, Finance Research Letters **5**-3 (2008), 162–171.

[6] M. Hitsuda, *Gaussian innovations and integral operators*, in Proceedings of the Sixth USSR-Japan Symposium, Kiev, Aug. 5-10 1991, World Scientific, 1992, pp. 125–131.

[7] Y. Hibino, M. Hitsuda and H. Muraoka, *Constructions of non-canonical representations of a Brownian motion*, Hiroshima Math. J. **27** (1997), 439–448.

[8] T. Jeulin and M. Yor, *Moyennes mobiles et semimartingales*, in Sém. Prob. XXVII, Lecture Notes in Math. **1557**, Springer, 1993, pp. 53–77.

[9] D. Nualart, The Malliavin Calculus and related topics, Springer, 1995.