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The relation between stationary and periodic solutions of the Navier-Stokes equations in two or three dimensional channels

By

Teppei Kobayashi

Abstract

In this paper we will consider whether there exists a time periodic solution of the Navier-Stokes equations for infinite channels in $\mathbb{R}^n(n = 2, 3)$. H. Beirão da Veiga [4] treated such a problem. This paper is the special case of his paper and we argue the relation between the existence of stationary and time periodic solutions of the Navier-Stokes equations.

1. Introduction

In a bounded domain $D \subset \mathbb{R}^n(n = 2, 3)$ we consider the Navier-Stokes eqautions

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad (0,T)\times D, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad (0,T)\times D, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{\beta} \quad \text{on} \quad (0,T)\times\partial D. \tag{1.3}$$

Then the boundary condition $\boldsymbol{\beta}$ must satisfy

$$\int_{\partial D}\boldsymbol{\beta}(t)\cdot\boldsymbol{n}d\sigma = 0 \quad (\forall t\in(0,T)), \tag{1.4}$$

where $\boldsymbol{n}$ is the unit outward to ∂D. We call the condition (1.4) "*General Outflow Condition*", (*GOC*). Let $\gamma_j(1 \le j \le L)$ be boundary components of D. If $\boldsymbol{\beta}$ satisfies

$$\int_{\gamma_j}\boldsymbol{\beta}(t)\cdot\boldsymbol{n}d\sigma = 0 \quad (\forall t\in(0,T),\ 0\le j\le L), \tag{1.5}$$

the condition (1.5) is called "*Stringent Outflow Condition*", (*SOC*).

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H. Fujita [6] proved the existence of stationary solutions of the Navier-Stokes equations with the Dirichlet boundary condition satisfying (SOC). V. I. Yudovič [21] proved that there exists a time periodic solutions of the Navier-Stokes equations with the Dirichlet boundary condition satisfying (SOC). H. Fujita [7] ensured that in a certain symmetric bounded domain there exists symmetric stationary solutions of the Navier-Stokes equations with the symmetric Dirichlet boundary condition satisfying (GOC). H. Morimoto [15] and T-P. Kobayashi [11] proved that there exists symmetric time periodic solutions of the Navier-Stokes equations with the symmetric Dirichlet boundary condition satisfying (GOC). We appreciate that in some situations there exist stationary solutions and time periodic solutions of the Navier-Stokes equations under the same condition. We know that there exists statioary solutions of the Navier-Stokes equations with the nonhomogeneous boundary condition satisfying (GOC) besides H. Fujita [6], [7]. For example, see W. Borchers and K. Pileckas [5], G. P. Galdi [9], H. Fujita and H. Morimoto [8]. In this paper we will prove that for two and three dimensional unbounded channels there exists time periodic solutions of the Navier-Stokes equations under the same condition as the existence of stationary solutions of the Navier-Stokes equations with the Poiseuille velocities investigated by C. J. Amick [2]. The condition of the existence of stationary solutions of the Navier-Stokes equations is stated in the Section 4.

2. Problem

Let ω_1 be an infinite channel which is centered on the x_1-axis with the radius a_1, that is to say,

$$(n=2) \quad \omega_1 = \{x \in \mathbb{R}^2; -a_1 < x_2 < a_1\},$$
$$(n=3) \quad \omega_1 = \{x \in \mathbb{R}^3; (x_2^2 + x_3^2)^{\frac{1}{2}} < a_1\}.$$

Let ω_2 be an infinite channel which is centered on a certain line with the radius a_2. Without loss of generality, the center line of ω_2 is the x_1'-axis, that is to say, for a fixed $y \in \mathbb{R}^2$ or $\mathbb{R}^3$

$$(n=2) \quad \omega_2 = \{x' = Ox + y; x \in \mathbb{R}^2; -a_2 < x_2' < a_2\},$$
$$(n=3) \quad \omega_2 = \{x' = Ox + y; x \in \mathbb{R}^3; ((x_2')^2 + (x_3')^2)^{\frac{1}{2}} < a_2\},$$

where O is an appropriate matrix. The Poiseuille velocities

$$(n=2) \quad \boldsymbol{V}_1^{P,\alpha}(x) = \left(\frac{3\alpha}{4a_1^3}(a_1^2 - x_2^2), 0\right),$$
$$(n=2) \quad \boldsymbol{V}_2^{P,\alpha}(x') = \left(\frac{3\alpha}{4a_2^3}(a_2^2 - (x_2')^2), 0\right),$$
$$(n=3) \quad \boldsymbol{V}_1^{P,\alpha}(x) = \left(\frac{2\alpha}{a_1^4}(a_1^2 - x_2^2 - x_3^2), 0, 0\right),$$
$$(n=3) \quad \boldsymbol{V}_2^{P,\alpha}(x') = \left(\frac{2\alpha}{a_2^4}(a_2^2 - (x_2')^2 - (x_3')^2), 0, 0\right)$$

are the basic flows in ω_1 or ω_2 respectively with a flux α. We suppose that

$$\omega_{10} = \{x \in \omega_1; x_1 \leq 0\},$$
$$\omega_{20} = \{x' = Ox + y; x \in \omega_2, x_1' \geq 0\}.$$

Let Ω be a smooth and unbounded domain in $\mathbb{R}^n (n = 2, 3)$ and $\partial\Omega$ be a boundary of the domain Ω. We suppose $\Omega \cap \omega_{10} = \omega_{10}$ and $\Omega \cap \omega_{20} = \omega_{20}$. Let ω_0 be $\Omega \backslash (\omega_{10} \cup \omega_{20})$. The domain ω_0 is bounded but not smooth. The domain ω_0 has multiple boundary components.

The domain Ω is filled with an incompressible viscous fluid. Here let $\boldsymbol{u} = \boldsymbol{u}(t, x)$, $p = p(t, x)$ and be the unknown velocity and pressure of the fluid in Ω respectively. Let ν be the kinematic viscosity. Then in the domain Ω we consider the nonstationary Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad (0, T) \times \Omega, \tag{2.1}$$
$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad (0, T) \times \Omega \tag{2.2}$$

with the boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad (0, T) \times \partial\Omega, \tag{2.3}$$
$$\boldsymbol{u} \to \boldsymbol{V}_i^P \quad \text{as} \quad |x| \to \infty \quad \text{in} \quad \omega_{i0} \quad (i = 1, 2) \tag{2.4}$$

and the time periodic condition

$$\boldsymbol{u}(0) = \boldsymbol{u}(T) \quad \text{in} \quad \Omega, \tag{2.5}$$

where $\boldsymbol{f}$ is the prescribed external force. We know that there exists a smooth vector function $\boldsymbol{V}^P$ which satisfies

$$\operatorname{div} \boldsymbol{V}^P = 0 \quad \text{in} \quad \Omega, \tag{2.6}$$
$$\boldsymbol{V}^P = \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \tag{2.7}$$
$$\boldsymbol{V}^P = \boldsymbol{V}_i^P \quad \text{in} \quad \omega_{i0} \quad (i = 1, 2). \tag{2.8}$$

See C. J. Amick [2]. Let us call $\boldsymbol{V}^P$ "the extended Poiseuille velocity". Set $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{V}^P$, then we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} - \nu \Delta \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{V}^P + (\boldsymbol{V}^P \cdot \nabla)\boldsymbol{v} + \nabla p & \\ = \boldsymbol{f} - \nu \Delta \boldsymbol{V}^P - (\boldsymbol{V}^P \cdot \nabla)\boldsymbol{V}^P & \quad \text{in} \quad (0, T) \times \Omega, \end{aligned} \tag{2.9}$$
$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad (0, T) \times \Omega, \tag{2.10}$$
$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad (0, T) \times \partial\Omega, \tag{2.11}$$
$$\boldsymbol{v} \to \boldsymbol{0} \quad \text{as} \quad |x| \to \infty, \tag{2.12}$$
$$\boldsymbol{v}(0) = \boldsymbol{v}(T) \quad \text{in} \quad \Omega. \tag{2.13}$$

In more genaralized infinite channels (than treated in this paper) H. Beirão da Veiga [4] proved that there exists the time periodic solutions of the Navier-Stokes equations with a small time periodic fulx. S. Kaniel and M. Shinbrot

[10] studied the uniqueness of time periodic solutions of the Navier-Stokes equations in three dimensional bounded domains. J. L. Lions [13] considered time priodic problems for the Navier-Stokes equations with the homogeneous boundary condition. A. Takeshita [18] studied the existence and uniqness of time periodic solutions of the Navier-Stokes equations in two dimensional bounded domains. H. Morimoto [15] proved that in a two dimensional bounded symmetric domain there exists symmetric time periodic solutions of the Navier-Stokes equations with the time-independent symmetric Dirichlet boundary condition satisfying (GOC). T-P. Kobayashi [11] treated similar problems to H. Morimoto [15] with the time-dependent symmetric Dirichlet boundary condition satisfying (GOC).

3. Preliminary

Before we state our results, we introduce some function spaces.

$\mathbb{C}_0^\infty(\Omega)$ is the set of all real smooth vector functions with compact support in Ω. $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ is the set of all $\mathbb{C}_0^\infty(\Omega)$ functions $\boldsymbol{\varphi}$ with $\operatorname{div}\boldsymbol{\varphi} = 0$. $\boldsymbol{V}(\Omega)$ and $\boldsymbol{H}(\Omega)$ are the completion of $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ with respect to the usual $\mathbb{H}^1(\Omega)$ and $\mathbb{L}^2(\Omega)$ norm respectively. $\mathbb{H}_0^1(\Omega)$ is the completion of $\mathbb{C}_0^\infty(\Omega)$ with respect to the $\mathbb{H}^1(\Omega)$ norm. $\|\cdot\|_{2,\Omega}$ and $(\cdot,\cdot)_\Omega$ denotes the $\mathbb{L}^2$ norm and inner product on Ω respectively. We often omit Ω from $\|\cdot\|_{2,\Omega}$ and $(\cdot,\cdot)_\Omega$. $\mathbb{H}_0^1(\Omega)$ and $\boldsymbol{V}(\Omega)$ are the Hilbert spaces with respect to the inner product $((\boldsymbol{u},\boldsymbol{v}))_\Omega = (\nabla\boldsymbol{u},\nabla\boldsymbol{v})_\Omega$. We usually omit Ω. Let $(\boldsymbol{V}(\Omega))'$ and $(\boldsymbol{H}(\Omega))'$ be the dual spaces of $\boldsymbol{V}(\Omega)$ and $\boldsymbol{H}(\Omega)$ respectively. Then we obtain inclusions

$$\boldsymbol{V}(\Omega) \subset \boldsymbol{H}(\Omega) \equiv (\boldsymbol{H}(\Omega))' \subset (\boldsymbol{V}(\Omega))',$$

where the inclusions are dense and the injections are continuous.

Let X be a Banach space. $L^2((0,T);X)$ is the set of all the measurable functions $\boldsymbol{u}(t)$ with values in X satisfying $\int_0^T \|\boldsymbol{u}(t)\|_X^2 dt < \infty$. $L^\infty((0,T);X)$ is the set of all the measurable functions $\boldsymbol{u}(t)$ with value in X satisfying $\operatorname{esssup}_{t\in(0,T)} \|\boldsymbol{u}(t)\|_X < \infty$.

In this paper we use following Lemmas. These Lemmas are used in various situations except the last Lemma.

Lemma 3.1 (The Poincaré inequality).
The inequality

$$\|\boldsymbol{u}\|_2 \le C(\Omega)\|\nabla\boldsymbol{u}\|_2 \quad (\boldsymbol{u} \in \mathbb{H}_0^1(\Omega))$$

holds true, where the constant $C(\Omega)$ depends on Ω.

Lemma 3.2 (R. Temam [19]).
For all $\boldsymbol{u} \in \mathbb{H}_0^1(\Omega)$, the inequalities

$$(n=2) \quad \|\boldsymbol{u}\|_{\mathbb{L}^4(\Omega)}^2 \le 2^{\frac{1}{2}}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2,$$

$$(n=3) \quad \|\boldsymbol{u}\|_{\mathbb{L}^4(\Omega)}^2 \le 2\|\boldsymbol{u}\|_2^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_2^{\frac{3}{2}}$$

hold.

For any vector function $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$, we define

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \sum_{i,j=1}^{2} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

Then a following Lemma holds.

Lemma 3.3 (R. Temam [19]).
The inequality

$$|((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})| \le C\|\nabla\boldsymbol{u}\|_2\|\nabla\boldsymbol{v}\|_2\|\nabla\boldsymbol{w}\|_2 \quad (\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\in\mathbb{H}_0^1(\Omega))$$

and the equalities

$$\begin{aligned} ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) &= -((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{v}) &&(\boldsymbol{u}\in\boldsymbol{V}(\Omega),\ \boldsymbol{v},\ \boldsymbol{w}\in\mathbb{H}^1(\Omega)),\\ ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}) &= 0 &&(\boldsymbol{u}\in\boldsymbol{V}(\Omega),\ \boldsymbol{v}\in\mathbb{H}^1(\Omega)). \end{aligned}$$

hold.

Lemma 3.4 (K. Masuda [14], Lemma 2.5).
For any $\varepsilon>0$ and $\boldsymbol{w}_3\in C([0,T];\mathbb{L}^n(\Omega))$ $(n=2,3)$, there exist a constant M, an integer N and functions $\boldsymbol{\psi}_j\in\mathbb{L}^2(\Omega)(j=1,\dots,N)$ such that the inequality

$$\begin{aligned}\int_0^T |((\boldsymbol{w}_1\cdot\nabla)\boldsymbol{w}_2,\boldsymbol{w}_3)|dt \le \varepsilon\int_0^T (\|\nabla\boldsymbol{w}_1\|_2^2+\|\nabla\boldsymbol{w}_2\|_2^2+\|\boldsymbol{w}_1\|_2\|\nabla\boldsymbol{w}_2\|_2)dt\\ + M\sum_{j=1}^{N}\int_0^T |(\boldsymbol{w}_1,\boldsymbol{\psi}_j)|^2dt\end{aligned}$$

holds for all $\boldsymbol{w}_1,\ \boldsymbol{w}_2\in L^2((0,T);\boldsymbol{V}(\Omega))$.

We define a functional $\boldsymbol{r}$ from

(3.1) $$\boldsymbol{\varphi}\in\boldsymbol{V}(\Omega)\mapsto \nu(\nabla\boldsymbol{V}^P,\nabla\boldsymbol{\varphi})+((\boldsymbol{V}^P\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{\varphi}),$$

where $\boldsymbol{V}^P$ is defined in (2.6), (2.7) (2.8).

Lemma 3.5 (C. J. Amick [2], p. 490–p. 491).
The map $\boldsymbol{r}$ is linear and continuous functional on $\boldsymbol{V}(\Omega)$.

Therefore $\boldsymbol{r}\in(\boldsymbol{V}(\Omega))'$ satisfies

$$_{(V(\Omega))'}\langle\boldsymbol{r},\boldsymbol{\varphi}\rangle_{V(\Omega)} = \nu(\nabla\boldsymbol{V}^P,\nabla\boldsymbol{\varphi})+((\boldsymbol{V}^P\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{\varphi}) \quad (\boldsymbol{\varphi}\in\boldsymbol{V}(\Omega)).$$

4. Main Theorem

Firstly we define the weak and time periodic solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5).

Definition 4.1.
A measurable function $\boldsymbol{u} = \boldsymbol{u}(t,x)$ on $(0,T)\times\Omega$ is called a weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5), if and only if $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{V}^P$ belongs to $L^\infty((0,T);\boldsymbol{H}(\Omega)) \cap L^2((0,T);\boldsymbol{V}(\Omega))$ and $\boldsymbol{v}$ satisfies

$$\begin{aligned}&\int_0^T -(\boldsymbol{v},\boldsymbol{\varphi})\psi' + \{\nu((\boldsymbol{v},\boldsymbol{\varphi})) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) \\ &\qquad\qquad + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{\varphi}) + ((\boldsymbol{V}^P\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\}\psi dt \\ &= \int_0^T {}_{(V(\Omega))'}\langle \boldsymbol{F},\boldsymbol{\varphi}\rangle_{V(\Omega)}\psi dt \quad (\boldsymbol{\varphi}\in\boldsymbol{V}(\Omega),\ \psi\in C_0^\infty(0,T)),\end{aligned} \tag{4.1}$$

where $\boldsymbol{F}$ equals to $\boldsymbol{f}-\boldsymbol{r}$. Moreover the weak solution $\boldsymbol{u}$ is time periodic, if $\boldsymbol{v}$ is a weakly continuous function from $[0,T]$ to $\mathbb{L}^2(\Omega)$ and satisfies

$$\boldsymbol{v}(0) = \boldsymbol{v}(T) \quad \text{in} \quad \mathbb{L}^2(\Omega). \tag{4.2}$$

We call $\boldsymbol{u}$ "a time periodic weak solution".

Hereafter $\langle\cdot,\cdot\rangle$ represents ${}_{(V(\Omega))'}\langle\cdot,\cdot\rangle_{V(\Omega)}$.
Secondly we define a constant concerned with the Poiseuille velocity.

Definition 4.2.
We set

$$\sigma_i = \sup_{\boldsymbol{\varphi}\in V(\omega_i)} \frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi},\boldsymbol{V}_i^P)_{\omega_i}}{\|\nabla\boldsymbol{\varphi}\|^2_{\mathbb{L}^2(\omega_i)}} \quad (i=1,2) \tag{4.3}$$

and

$$\sigma = \max\{\sigma_1,\sigma_2\}. \tag{4.4}$$

Remark 1. The constant σ_i does not depend on the domain Ω. The constant σ_i depends only on the channel ω_i and the flux of the Poiseuille velocity. C. J. Amick [2] defined the constant (4.3) in two and three dimensional infinite channels. In two dimensional infinite channels H. Morimoto and H. Fujita [16] and H. Morimoto [17] used the symmetric version of σ.

We know that in two dimensional channels the following Proposition holds true.

Proposition 4.1 (C. J. Amick [2], H. Morimoto and H. Fujita [16]**).**
If $n=2$, the constant σ_i is positive and

$$\sigma_1 = \sigma_2$$

holds true.

For the proof of Proposition 4.1, see H. Morimoto and H. Fujita [16], p.461, Lemma 2. In the two dimensional channels, the constants σ_1 and σ_2 depend only on the flux α, that is to say,

$$\sigma = \sigma_1 (= \sigma_2).$$

holds true.

But in the three dimensional channels the above Proposition does not hold true. We obtain the following Proposition.

Proposition 4.2.
If $n = 3$, the constant σ_i is not negative and

$$a_1 \sigma_1 = a_2 \sigma_2$$

holds true.

Therefore in the three dimensional channels ω_1 and ω_2, the constants σ_1 and σ_2 depend not only on the flux α but also on the radii a_1 and a_2 of the channels ω_1 and ω_2 respectively. If $a_1 < a_2$ holds true, we obtain $\sigma_1 > \sigma_2$. For the proof, see the last section.

Remark 2. In the two and three dimensional channels the explicit value of the constants σ_1 and σ_2 are shown in the paper of C. J. Amick [3].

Remark 3 (Stationary results)**.**
Supposing $\sigma < \nu$ (kinematic viscosity), C. J. Amick [2] proved that there exists weak solutions of the stationary Navier-Stokes equations in the two and three dimensional infinite channels. In a two dimensional symmetric semi infinite channel H. Morimoto and H. Fujita [16] proved that there exists symmetric weak solutions of the stationary Navier-Stokes equations with a certain symmetric Dirichlet boundary condition satisfying (GOC) using the symmetric version of σ. Supposing $\sigma < \nu$ (where σ is the symmetric version), in a two dimensional symmetric infinite channel H. Morimoto [17] obtained symmetric weak solutions of the stationary Navier-Stokes equations with a certain symmetric Dirichlet boundary condition satisfying (GOC).

Using the constant σ, we obtain the following result.

Theorem 4.1.
Suppose that $\boldsymbol{f}$ belongs to $L^2((0,T);(\boldsymbol{V}(\Omega))')$ and $\sigma < \nu$.
Then there exists a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5).

Remark 4. We understand that in the unbounded channels there exists stationary solutions and time periodic solutions of the Navier-Stokes equations under the same condition $\sigma < \nu$.

5. Proof of Theorem 4.1

5.1. Auxiliary Proposition

In this section we argue the proof of Theorem 4.1 for three dimensional domains. But in two dimensional domains the principle of the proof is similar to three dimensional domains. Before we state the proof, we present the following Propositions. For the proofs of these Propositions, see C. J. Amick [2].

Proposition 5.1 (C. J. Amick [2], p. 495–p. 496)**.**
We suppose $\theta \in C^{\infty}(\mathbb{R})$ *satisfies* $0 \le \theta \le 1$, $\theta(t) = 1$ $(t < -1)$ *and* $\theta = 0$ $(t \ge 0)$. *For all* $\delta > 0$, *we set*

$$\theta_\delta(x) = \begin{cases} \theta(\delta x_1) & (x \in \omega_{10}) \\ \theta(-\delta x_1') & (x' \in \omega_{20}) \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $\varepsilon > 0$, *there exists an* $\boldsymbol{s} \in \mathbb{C}^{\infty}_{0,\sigma}(\Omega)$ *and a constant* C_0 *such that*

$$((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}^P) \le ((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{s}) + ((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}^P\theta_\delta^2)_{\omega_{10}\cup\omega_{20}} + (\varepsilon + C_0\delta)\|\nabla\boldsymbol{\varphi}\|_2^2 \quad (\boldsymbol{\varphi} \in \boldsymbol{V}(\Omega))$$

holds true.

Proposition 5.2 (C. J. Amick [2], Theorem 4.3)**.**

$$\lim_{\delta\to 0} \sup_{\boldsymbol{\varphi}\in V(\Omega)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}^P\theta_\delta^2)_{\omega_{10}\cup\omega_{20}}}{\|\nabla\boldsymbol{\varphi}\|_2^2} = \sigma$$

holds true.

5.2. Time periodic solutions in bounded domains

We suppose that $\{\Omega^n\}$ is a sequence of a smooth bounded domain of Ω and satisfies $\Omega^n \subset \Omega^{n+1}$ and $\cup_{n\in\mathbb{N}}\Omega^n = \Omega$. Let $\partial\Omega^n$ be a boundary of Ω^n. We may assume that Ω^1 containes the support of $\boldsymbol{s} \in \mathbb{C}^{\infty}_{0,\sigma}(\Omega)$ of Proposition 5.1. In the bounded domain Ω^n we consider time periodic solutions of the Navier-Stokes equations

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad & \text{in} \quad (0,T)\times\Omega^n, \\ \operatorname{div}\boldsymbol{u} = 0 \qquad & \text{in} \quad (0,T)\times\Omega^n, \\ \boldsymbol{u} = \boldsymbol{V}^P \qquad & \text{on} \quad (0,T)\times\partial\Omega^n, \\ \boldsymbol{u}(0) = \boldsymbol{u}(T) \qquad & \text{in} \quad \Omega^n. \end{aligned}$$

Let Γ_j^n $(j = 0, 1, \cdots, J)$ be conneted components of $\partial\Omega^n$. Then $\boldsymbol{V}^P$ satisfies

$$\int_{\Gamma_j^n} \boldsymbol{V}^P \cdot \boldsymbol{n}dS = 0 \quad (j = 0, 1, \cdots, J),$$

where $\boldsymbol{n}$ is the unit outward to $\partial\Omega^n$, that is to say, $\boldsymbol{V}^P$ satisfies (SOC) on $\partial\Omega^n$. Therefore there exists a smooth vector function $\boldsymbol{V}_n^{P,\varepsilon}$ in Ω^n satisfying

$$\begin{aligned} \boldsymbol{V}_n^{P,\varepsilon} &= \boldsymbol{V}^P \quad \text{on} \quad \partial\Omega^n, \\ |((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{V}_n^{P,\varepsilon})_{\Omega_n}| &< \varepsilon\|\nabla\boldsymbol{v}\|^2_{\Omega^n} \quad (\boldsymbol{v}\in\boldsymbol{V}(\Omega^n)). \end{aligned} \tag{5.1}$$

For the proof, see H. Fujita [6] or R. Temam [19], Lemma 1.8 in Chapter 2. Consequently we obtain $\boldsymbol{u}_n$ satisfying

$$\begin{aligned} &\boldsymbol{u}_n \in L^2((0,T);\mathbb{H}^1_\sigma(\Omega^n)) \cap L^\infty((0,T);\mathbb{L}^2(\Omega^n)), \\ &\boldsymbol{u}_n \textit{ is continuous on } [0,T] \textit{ in the weak topology of } \mathbb{L}^2(\Omega^n) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{u}_n,\boldsymbol{\varphi})_{\Omega^n} + \nu(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{\varphi})_{\Omega^n} + ((\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n,\boldsymbol{\varphi})_{\Omega^n} &= \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_n \ (\boldsymbol{\varphi}\in\boldsymbol{V}(\Omega^n)), \\ \boldsymbol{u}_n &= \boldsymbol{V}^P \quad \text{on} \quad (0,T)\times\partial\Omega, \\ \boldsymbol{u}_n(0) &= \boldsymbol{u}_n(T) \quad \text{in} \quad \mathbb{L}^2(\Omega^n), \end{aligned}$$

where $\langle\cdot,\cdot\rangle_n$ denotes the duality of $(\boldsymbol{V}(\Omega^n))'$ and $\boldsymbol{V}(\Omega^n)$. For the proof, see J. L. Lions [13], Section 6.2. We set

$$\boldsymbol{v}_n = \begin{cases} \boldsymbol{u}_n - \boldsymbol{V}^P & \text{in} \quad \Omega^n \\ \boldsymbol{0} & \text{in} \quad \textit{otherwise.} \end{cases}$$

We obtain

$$\begin{aligned} &\frac{d}{dt}(\boldsymbol{v}_n,\boldsymbol{\varphi}) + \nu((\boldsymbol{v}_n,\boldsymbol{\varphi})) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) \\ &\quad + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{\varphi}) + ((\boldsymbol{V}^P\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) = \langle\boldsymbol{F},\boldsymbol{\varphi}\rangle \quad (\boldsymbol{\varphi}\in\boldsymbol{V}(\Omega^n)), \end{aligned} \tag{5.2}$$

where $\boldsymbol{\varphi}\in\boldsymbol{V}(\Omega^n)$ are extended as a $\boldsymbol{0}$ function to Ω and $\boldsymbol{F} = \boldsymbol{f} - \boldsymbol{r}$.

We will prove that $\|\boldsymbol{v}_n(0)\|_2$ is a bounded sequence with respect to n. Now we set $\boldsymbol{\varphi} = \boldsymbol{v}_n$ in the equation (5.2). We obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \nu\|\nabla\boldsymbol{v}_n\|_2^2 + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{v}_n) = \langle\boldsymbol{F},\boldsymbol{v}_n\rangle. \tag{5.3}$$

Using Proposition 5.1, we obtain

$$\begin{aligned} ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{V}^P) \le ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{s}) &+ ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{V}^P\theta_\delta^2)_{\omega_{10}\cup\omega_{20}} \\ &+ (\varepsilon + C_0\delta)\|\nabla\boldsymbol{v}_n\|_2^2. \end{aligned}$$

Using Proposition 5.2, for small enough $\delta>0$ we have

$$((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{V}^P\theta_\delta^2) \le (\sigma+\varepsilon)\|\nabla\boldsymbol{v}_n\|_2^2.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + (\nu-\sigma-3\varepsilon-C_0\delta)\|\nabla\boldsymbol{v}_n\|_2^2 \le ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{s}) + C\|\boldsymbol{F}\|^2_{(V)'}$$

holds true, where the constant C is dependent on ε and we choose ε and δ such that $\nu - \sigma - 4\varepsilon - C_0\delta$ is greater than 0. It is easy to obtain from (5.2) the inequality

$$\begin{aligned}((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{s}) &= -\frac{d}{dt}(\boldsymbol{v}_n,\boldsymbol{s}) - \nu((\boldsymbol{v}_n,\boldsymbol{s}))\\ &\quad - ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{s}) - ((\boldsymbol{V}^P\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{s}) + \langle\boldsymbol{F},\boldsymbol{s}\rangle\\ &\le -\frac{d}{dt}(\boldsymbol{v}_n,\boldsymbol{s}) + \varepsilon\|\nabla\boldsymbol{v}_n\|_2^2 + C(\|\nabla\boldsymbol{s}\|_2^2 + \|\boldsymbol{F}\|_{(V)'}^2),\end{aligned}$$

where the constant C is dependent on ε. We set

$$K_1(t) = C(\|\nabla\boldsymbol{s}\|_2^2 + \|\boldsymbol{F}(t)\|_{(V)'}^2).$$

Using the Poincaré inequality, we have

$$\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \mu\|\boldsymbol{v}_n\|_2^2 \le -2\frac{d}{dt}(\boldsymbol{v}_n,\boldsymbol{s}) + 2K_1(t),$$

where

$$\mu = 2\frac{\nu-\sigma-4\varepsilon-C_0\delta}{C(\Omega)^2}.$$

For $\xi > 0$ (smaller than μ) multiplying by $e^{(\mu-\xi)t}$, then the inequality

$$\begin{aligned}&e^{(\mu-\xi)t}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \mu e^{(\mu-\xi)t}\|\boldsymbol{v}_n\|_2^2\\ &\quad\le -2e^{(\mu-\xi)t}\frac{d}{dt}(\boldsymbol{v}_n,\boldsymbol{s}) + 2K_1(t)e^{(\mu-\xi)t}\\ &\quad= -2\frac{d}{dt}\{(\boldsymbol{v}_n,\boldsymbol{s})e^{(\mu-\xi)t}\} + 2(\mu-\xi)e^{(\mu-\xi)t}(\boldsymbol{v}_n,\boldsymbol{s}) + 2K_1(t)e^{(\mu-\xi)t}\\ &\quad\le -2\frac{d}{dt}\{(\boldsymbol{v}_n,\boldsymbol{s})e^{(\mu-\xi)t}\} + \xi e^{(\mu-\xi)t}\|\boldsymbol{v}_n\|_2^2 + C(\|\boldsymbol{s}\|_2^2 + K_1(t))e^{(\mu-\xi)t}\end{aligned}\tag{5.4}$$

holds true, where the constant C depends only on μ and ξ. Setting

$$K_2(t) = C(\|\boldsymbol{s}\|_2^2 + K_1(t))e^{(\mu-\xi)t},$$

it follows from (5.4) that

$$\frac{d}{dt}(e^{(\mu-\xi)t}\|\boldsymbol{v}_n\|_2^2) \le -2\frac{d}{dt}\{(\boldsymbol{v}_n,\boldsymbol{s})e^{(\mu-\xi)t}\} + K_2(t).\tag{5.5}$$

Integrating (5.5) on $[0,T]$, then we have

$$\|\boldsymbol{v}_n(T)\|_2^2 e^{(\mu-\xi)T} \le \|\boldsymbol{v}_n(0)\|_2^2 - 2(\boldsymbol{v}_n(T),\boldsymbol{s})e^{(\mu-\xi)T} + 2(\boldsymbol{v}_n(0),\boldsymbol{s}) + K,$$

where

$$K = \int_0^T K_2(t)dt.$$

Since $\boldsymbol{v}_n$ is time periodic, for all $\eta > 0$ the inequality

$$\begin{aligned}\|\boldsymbol{v}_n(0)\|_2^2 e^{(\mu-\xi)T} \leq \|\boldsymbol{v}_n(0)\|_2^2 + (\eta\|\boldsymbol{v}_n(0)\|_2^2 + C\|\boldsymbol{s}\|_2^2)e^{(\mu-\xi)T} \\ + \eta\|\boldsymbol{v}_n(0)\|_2^2 + C\|\boldsymbol{s}\|_2^2 + K\end{aligned}$$

holds true, where C depend only on η. We set

$$\begin{aligned}H &= Ke^{-(\mu-\xi)T} + C\|\boldsymbol{s}\|_2^2(e^{-(\mu-\xi)T} + 1), \\ \gamma &= 1 - \eta - (1+\eta)e^{-(\mu-\xi)T}.\end{aligned}$$

Choosing $\eta > 0$ such that γ is greater than 0, we fainally reach

$$\|\boldsymbol{v}_n(0)\|_2^2 \leq \frac{H}{\gamma}. \tag{5.6}$$

Therefore $\{\boldsymbol{v}_n(0)\}_n$ is a bounded sequence of $\mathbb{L}^2(\Omega)$.

5.3. Weak limit

For the equation (5.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + (\nu - \varepsilon)\|\nabla \boldsymbol{v}_n\|_2^2 \leq |((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{V}^P, \boldsymbol{v}_n)| + C\|\boldsymbol{F}\|_{(V)'}^2. \tag{5.7}$$

The inequality

$$|((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{V}^P, \boldsymbol{v}_n)| \leq C(V^P)\|\boldsymbol{v}_n\|_2^2 \tag{5.8}$$

holds true, where the constant $C(V^P)$ depends only on the extended Poiseuille velocity $\boldsymbol{V}^P$. Using (5.8) and the Gronwall inequality and intergarating (5.7) from 0 to t $(\leq T)$, we obtain

$$\|\boldsymbol{v}_n(t)\|_2^2 \leq \frac{H}{\gamma}e^{C(V^P)T} + C\int_0^T e^{C(V^P)t}\|\boldsymbol{F}\|_{(V)'}^2 dt. \tag{5.9}$$

We define L_1 as the right hand side of (5.9). Intergarating (5.7) on $[0,T]$, we see that

$$\int_0^T \|\nabla \boldsymbol{v}_n\|_2^2 dt \leq \frac{1}{\nu - \varepsilon}\left(C(V^P)TL_1 + C\int_0^T \|\boldsymbol{F}\|_{(V)'}^2 dt\right) \tag{5.10}$$

holds true.

For any $\boldsymbol{\varphi} \in \mathbb{C}_{0,\sigma}^\infty(\Omega)$, there exists an $N \in \mathbb{N}$ (depending on the support of $\boldsymbol{\varphi}$) such that $(\boldsymbol{v}_n(t), \boldsymbol{\varphi})_{n\geq N}$ is uniformly bounded and equicontinuous on $[0,T]$

with respect to n because a culculation

$$
\begin{aligned}
&|(\boldsymbol{v}_n(t),\boldsymbol{\varphi})-(\boldsymbol{v}_n(s),\boldsymbol{\varphi})| \\
&\quad = \left|\int_s^t \frac{d}{d\tau}(\boldsymbol{v}_n(\tau),\boldsymbol{\varphi})d\tau\right| \\
&\quad \le \int_s^t \nu|((\boldsymbol{v}_n,\boldsymbol{\varphi}))|+|((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi})|+|((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{\varphi})| \\
&\qquad +|((\boldsymbol{V}^P\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi})|+|\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle|d\tau \\
&\quad \le \int_s^t (\nu\|\nabla\boldsymbol{v}_n\|_2+2^{\frac{1}{2}}\|\boldsymbol{v}_n\|_2^{\frac{1}{2}}\|\nabla\boldsymbol{v}_n\|_2^{\frac{3}{2}} \\
&\qquad +C(\Omega)C(V^p)\|\nabla\boldsymbol{v}_n\|_2+\|\boldsymbol{F}\|_{(V)'})\|\nabla\boldsymbol{\varphi}\|_2 d\tau \\
&\quad \le (L_2|t-s|^{\frac{1}{2}}+L_3|t-s|^{\frac{1}{4}})\|\nabla\boldsymbol{\varphi}\|_2,
\end{aligned}
$$

yields, where the constant L_2 depends on the right hand side of (5.9), (5.10) and $\boldsymbol{F}$ and the constant L_3 depends only on the right hand side of (5.9) and (5.10).

Since we know that the time periodic solution $\{\boldsymbol{v}_n\}$ is a bounded sequence in $L^\infty((0,T);\boldsymbol{H}(\Omega))\cap L^2((0,T);\boldsymbol{V}(\Omega))$ with respect to n, therefore there exists a subsequence $\{\boldsymbol{v}_{nk}\}_k$ of $\{\boldsymbol{v}_n\}$ and some $\boldsymbol{v}\in L^\infty((0,T);\boldsymbol{H}(\Omega))\cap L^2((0,T);\boldsymbol{V}(\Omega))$ such that

$$
\boldsymbol{v}_{nk}\to\boldsymbol{v} \quad \text{in} \quad \left\{\begin{array}{cc} L^\infty((0,T);\boldsymbol{H}(\Omega)) & \textit{weak star} \\ L^2((0,T);\boldsymbol{V}(\Omega)) & \textit{weakly}\end{array}\right. \quad (k\to\infty). \tag{5.11}
$$

For any $\boldsymbol{\varphi}\in\mathbb{C}^\infty_{0,\sigma}(\Omega)$, there exists a subsequence $\{\boldsymbol{v}_{nki}\}$ of $\{\boldsymbol{v}_{nk}\}$ such that

$$
\lim_{i\to\infty}(\boldsymbol{v}_{nki},\boldsymbol{\varphi})=(\boldsymbol{v},\boldsymbol{\varphi}) \tag{5.12}
$$

holds true by using *the Ascoli-Arzelà* Theorem. Let us prove (5.12) for any $\boldsymbol{\varphi}\in\mathbb{L}^2(\Omega)$. Since we know $\mathbb{L}^2(\Omega)=\boldsymbol{H}(\Omega)\oplus(\boldsymbol{H}(\Omega))^\perp$, we have $\boldsymbol{\varphi}=\boldsymbol{\varphi}_\sigma+\boldsymbol{\varphi}_p(\boldsymbol{\varphi}_\sigma\in\boldsymbol{H}(\Omega),\boldsymbol{\varphi}_p\in(\boldsymbol{H}(\Omega))^\perp)$. For any $\delta>0$ there exists a $\boldsymbol{\varphi}^\delta_\sigma\in\mathbb{C}^\infty_{0,\sigma}(\Omega)$ such that

$$
\|\boldsymbol{\varphi}^\delta_\sigma-\boldsymbol{\varphi}_\sigma\|_2<\delta
$$

holds true because $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ is dence in $\boldsymbol{H}(\Omega)$. We have

$$
\begin{aligned}
|(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi})| &\le |(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi}_\sigma-\boldsymbol{\varphi}^\delta_\sigma)|+|(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi}^\delta_\sigma)| \\
&\le 2L_1\delta+|(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi}^\delta_\sigma)|.
\end{aligned} \tag{5.13}
$$

because $\boldsymbol{v}_n$ is bounded in $L^\infty((0,T);\mathcal{H}^S(\Omega))$. We can choose a subsequence $\{\boldsymbol{v}_{nk}\}_k$ of $\{\boldsymbol{v}_n\}_n$ such that the second term of the right hand side of (5.13) goes to 0. For any $\boldsymbol{\varphi}\in\mathbb{L}^2(\Omega)$ there exists a subsequence $\{\boldsymbol{v}_{nk}\}$ such that $(\boldsymbol{v}_{nk},\boldsymbol{\varphi})$ converges to $(\boldsymbol{v},\boldsymbol{\varphi})$ uniformly on $[0,T]$.

5.4. Time periodic solution

Let $\boldsymbol{\varphi}$ be $\mathbb{C}^{\infty}_{0,\sigma}(\Omega)$. Then there exists an $N \in \mathbb{N}$ (depending on the support of $\boldsymbol{\varphi}$) such that (5.2) holds true for $\boldsymbol{\varphi}$ and $n \geq N$. Mulitiplying (5.2) by $\phi \in C^{\infty}_{0}(0,T)$ and integrating on $[0,T]$, we obtain

$$\begin{aligned}\int_0^T &-(\boldsymbol{v}_n,\boldsymbol{\varphi})\phi' + \{\nu((\boldsymbol{v}_n,\boldsymbol{\varphi})) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) \\ &\qquad + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{V}^P,\boldsymbol{\varphi}) + ((\boldsymbol{V}^P\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi})\}\phi dt \\ = &\int_0^T \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle\phi dt.\end{aligned} \tag{5.14}$$

We know that there exists a subsequence $\{\boldsymbol{v}_{nk}\}$ such that the the left hand side of (5.14) except the nonliner term converges to

$$\int_0^T -(\boldsymbol{v},\boldsymbol{\varphi})\phi' + \nu\{((\boldsymbol{v},\boldsymbol{\varphi})) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{V}^{P,\alpha},\boldsymbol{\varphi}) + ((\boldsymbol{V}^{P,\alpha}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\}\phi dt$$

using (5.11). We will prove

$$\int_0^T ((\boldsymbol{v}_{nk}\cdot\nabla)\boldsymbol{v}_{nk},\boldsymbol{\varphi})\phi dt \to \int_0^T ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\phi dt \quad (k\to\infty). \tag{5.15}$$

Since

$$\begin{aligned}&\int_0^T ((\boldsymbol{v}_{nk}\cdot\nabla)\boldsymbol{v}_{nk},\boldsymbol{\varphi})\phi dt - \int_0^T ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\phi dt \\ &\quad = \int_0^T ((\boldsymbol{v}_{nk}-\boldsymbol{v})\cdot\nabla\boldsymbol{v}_{nk},\boldsymbol{\varphi})\phi dt - \int_0^T (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{v}_{nk}-\boldsymbol{v})\phi dt.\end{aligned} \tag{5.16}$$

We consider the first term of the right hand side of (5.16). By Lemma 3.4, for any $\delta>0$ there exist a constant M, an integer N and $\boldsymbol{\psi}_l \in \mathbb{L}^2(\Omega)(l=1,\dots,N)$ such that the inequality

$$\begin{aligned}&\left|\int_0^T ((\boldsymbol{v}_{nk}-\boldsymbol{v})\cdot\nabla\boldsymbol{v}_{nk},\phi\boldsymbol{\varphi})dt\right| \\ &\quad \leq \delta\int_0^T (\|\nabla\boldsymbol{v}_{nk}-\nabla\boldsymbol{v}\|_2^2 + \|\nabla\boldsymbol{v}_{nk}\|_2^2 + \|\boldsymbol{v}_{nk}-\boldsymbol{v}\|_2\|\nabla\boldsymbol{v}_{nk}\|_2)dt \\ &\qquad + \sum_{l=1}^N \int_0^T |(\boldsymbol{v}_{nk}-\boldsymbol{v},\boldsymbol{\psi}_l)|^2 dt.\end{aligned} \tag{5.17}$$

holds true. Since the time periodic solution $\{\boldsymbol{v}_{nk}\}$ is a bounded sequence in $L^\infty((0,T);\boldsymbol{H}(\Omega))\cap L^2((0,T);\boldsymbol{V}(\Omega))$ with respect to n, there exists a constant M_1 independent of n such that

$$\delta\int_0^T (\|\nabla\boldsymbol{v}_{nk}-\nabla\boldsymbol{v}\|_2^2 + \|\nabla\boldsymbol{v}_{nk}\|_2^2 + \|\boldsymbol{v}_{nk}-\boldsymbol{v}\|_2\|\nabla\boldsymbol{v}_{nk}\|_2)dt \leq M_1\delta$$

holds true. Next we turn to estimate the first term of the right hand side of (5.16). Since we know that $\phi \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}$ belongs to $L^2((0,T);\mathbb{L}^2(\Omega))$, we obtain the decomposition

$$\phi(t)\boldsymbol{v}(t) \cdot \nabla \boldsymbol{\varphi} = \boldsymbol{\Phi}_\sigma(t) + \boldsymbol{\Phi}_p(t) \quad (\boldsymbol{\Phi}_\sigma(t) \in \boldsymbol{H}(\Omega), \boldsymbol{\Phi}_p(t) \in (\boldsymbol{H}(\Omega))^\perp).$$

It is easy to see that $\boldsymbol{\Phi}_\sigma$ belongs to $L^2((0,T);\boldsymbol{H}(\Omega))$ and the equality

$$\int_0^T (\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v}_{nk} - \boldsymbol{v})\phi dt = \int_0^T (\boldsymbol{\Phi}_\sigma, \boldsymbol{v}_{nk} - \boldsymbol{v}) dt.$$

holds true. Consequently it follows from (5.16) that

$$\begin{aligned} &\left| \int_0^T ((\boldsymbol{v}_{nk} \cdot \nabla)\boldsymbol{v}_{nk}, \boldsymbol{\varphi})\phi dt - \int_0^T ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\phi dt \right| \\ &\quad \leq M_1\delta + \sum_{l=1}^N \int_0^T |(\boldsymbol{v}_{nk} - \boldsymbol{v}, \boldsymbol{\psi}_l)|^2 dt + \left| \int_0^T (\boldsymbol{\Phi}_\sigma, \boldsymbol{v}_{nk} - \boldsymbol{v}) dt \right|. \end{aligned} \tag{5.18}$$

As k goes to infinity, the thrid term of the right hand side of (5.18) converges to zero. We choose a subsequence $\{\boldsymbol{v}_{nki}\}_{i\in\mathbb{N}}$ such that the second term of the right hand side of (5.18) converges to zero. Therefore we obtain (5.15) for the subsequence $\{\boldsymbol{v}_{nki}\}_{i\in\mathbb{N}}$. We know that (5.14) holds true for the subsequence $\{\boldsymbol{v}_{nki}\}_{i\in\mathbb{N}}$. Therefore $\boldsymbol{v}$ satisfies

$$\begin{aligned} &\int_0^T -(\boldsymbol{v}, \boldsymbol{\varphi})\phi' + \{\nu((\boldsymbol{v}, \boldsymbol{\varphi})) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) \\ &\qquad + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{V}^P, \boldsymbol{\varphi}) + ((\boldsymbol{V}^P \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\}\phi dt \\ &= \int_0^T \langle \boldsymbol{F}, \boldsymbol{\varphi} \rangle \phi dt \quad (\boldsymbol{\varphi} \in \mathbb{C}_{0,\sigma}^\infty(\Omega), \phi \in C_0^\infty(0,T)). \end{aligned}$$

Since $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ is dence in $\boldsymbol{V}(\Omega)$, we obtain

$$\begin{aligned} &\int_0^T -(\boldsymbol{v}, \boldsymbol{\varphi})\phi' + \{\nu((\boldsymbol{v}, \boldsymbol{\varphi})) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) \\ &\qquad + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{V}^P, \boldsymbol{\varphi}) + ((\boldsymbol{V}^P \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\}\phi dt \\ &= \int_0^T \langle \boldsymbol{F}, \boldsymbol{\varphi} \rangle \phi dt \quad (\boldsymbol{\varphi} \in \boldsymbol{V}(\Omega), \phi \in C_0^\infty(0,T)). \end{aligned} \tag{5.19}$$

Namely if we set $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V}^P$, then $\boldsymbol{u}$ is a weak solution.

Lastly we make sure that the weak solution $\boldsymbol{u}$ is time periodic. Since it is obvious that for any $\boldsymbol{\varphi} \in \mathbb{L}^2(\Omega)$ there exists a subsequence $\{\boldsymbol{v}_{nk}\}_k$ such that (5.12) holds true with respect to $\boldsymbol{\varphi}$, therefore we obtain

$$(\boldsymbol{v}(0) - \boldsymbol{v}(T), \boldsymbol{\varphi}) = (\boldsymbol{v}(0) - \boldsymbol{v}_{nk}(0), \boldsymbol{\varphi}) + (\boldsymbol{v}_{nk}(T) - \boldsymbol{v}(T), \boldsymbol{\varphi}) \to 0 \quad (k \to \infty).$$

Consequently, we see that $\boldsymbol{u}$ is certainly time periodic. □

6. Proof of Proposition 4.2

In this section we prove Proposition 4.2.

Firstly, we prove σ_i is not negative. Without loss of generality, we may prove it in the channel ω_1. Let $\boldsymbol{\varphi}$ be $\boldsymbol{V}(\omega_1)$. Set $\boldsymbol{\xi}(x) = \boldsymbol{\varphi}(-x)$. Then $\boldsymbol{\xi}$ belongs to $\boldsymbol{V}(\omega_1)$. We obtain

$$\frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_1^P)_{\omega_1}}{\|\nabla\boldsymbol{\varphi}\|^2_{\mathbb{L}^2(\omega_1)}} = -\frac{((\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\xi}, \boldsymbol{V}_1^P)_{\omega_1}}{\|\nabla\boldsymbol{\xi}\|^2_{\mathbb{L}^2(\omega_1)}}$$

by a direct calculation. Therefore we get the inequality

$$-\sigma_1 \le \frac{((\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\xi}, \boldsymbol{V}_1^P)_{\omega_1}}{\|\nabla\boldsymbol{\xi}\|^2_{\mathbb{L}^2(\omega_1)}} \le \sigma_1.$$

Therefore σ_1 is not a negative number.

Secondly, we prove $a_1\sigma_1 = a_2\sigma_2$. Without loss of generality, we may assume that the channels ω_1 and ω_2 are centered on the x_1-axis. Let $\boldsymbol{\varphi}$ be $\boldsymbol{V}(\omega_1)$. We set

$$\boldsymbol{\psi}(x) = \frac{a_1}{a_2}\boldsymbol{\varphi}\left(\frac{a_1}{a_2}x\right).$$

Then $\boldsymbol{\psi}$ belongs to $\boldsymbol{V}(\omega_2)$ and we obtain the equations

$$((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_1^P)_{\omega_1} = \frac{a_2^2}{a_1^2}((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi}, \boldsymbol{V}_2^P)_{\omega_2},$$
$$\|\nabla\boldsymbol{\varphi}\|^2_{\mathbb{L}^2(\omega_1)} = \frac{a_2}{a_1}\|\nabla\boldsymbol{\psi}\|^2_{\mathbb{L}^2(\omega_2)}.$$

Therefore the equation

$$a_1\frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_1^P)_{\omega_1}}{\|\nabla\boldsymbol{\varphi}\|^2_{\mathbb{L}^2(\omega_2)}} = a_2\frac{((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi}, \boldsymbol{V}_2^P)_{\omega_2}}{\|\nabla\boldsymbol{\psi}\|^2_{\mathbb{L}^2(\omega_1)}}$$

holds true. This proves $a_1\sigma_1 = a_2\sigma_2$. □

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Department of Mathematics
Meiji University
1-1-1 Tama-ku, Kawasaki
Japan, 214-0038
e-mail: teppeik@isc.meiji.ac.jp

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