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Wegner estimate for a generalized alloy type potential

By

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Abstract

W.Kirsch and I.Veselić proved a generalized Wegner estimate for Schrödinger operators with generalized alloy type potentials at negative energies for each fixed position of impurities. In this paper, a similar estimate is proven treating also the position of impurities as random variables.

1. Introduction

In this paper, we will give a Wegner estimate and state the properties of the exponential localization of eigenfunctions for a Schrödinger operator,

$$H^\omega := -\Delta + V^\omega(x) \text{ with } V^\omega(x) := -\sum_{i\in\mathbf{N}} f_i^{\omega_1} u(x-\xi_i^{\omega_2}), \tag{1.1}$$

where u is a nonnegative continuous function with a compact support, $\{f_i^{\omega_1}$, $i\in\mathbf{N}$, $\omega_1\in\Omega_1\}$ are independently and identically distributed random variables obeying the uniform distribution on the interval [0, 1], and $\{\xi_i^{\omega_2},\ i\in\mathbf{N},\omega_2\in\Omega_2\}$ is a Poisson point process independent of $\{f_i^{\omega_1}\}$ with the Lebesgue measure as its intensity. We write $\omega=(\omega_1,\omega_2)$. For any $a\in\mathbf{R}^d$ and $L>0$, we set $\Lambda_L(a)=\{x\in\mathbf{R}^d : |x_i-a_i|<L/2 \text{ for } 1\le i\le d\}$ and $\Lambda_L:=\Lambda_L(0)$. For simplicity we assume $\operatorname{supp} u\subset\Lambda_1$.

The investigations of the localization of eigenfunctions of the Schrödinger operators H^ω were begun by P.W.Anderson [1]. It has been discussed mainly about potential energies called alloy type potentials as $\sum_{i\in\mathbf{Z}^d} f_i^{\omega_1}u(x-i)$. Recently, Kirsch and Veselić proved a general form of the Wegner estimate used to prove the localization for the potential energies called generalized alloy type potentials [8]. Since positions of impurities in the lattice are considered as random variables, these potential energies are regarded as a liquid crystal type. In [8], they proved the Wegner estimate for each fixed position for impurities. In this paper, we will prove the Wegner estimate treating also the position of impurities as random variables for a typical example of the generalized alloy

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type potential energy defined in (1.1). Based on this estimate we will next use the variable energy multiscale analysis [6] to obtain the results on the Anderson localization as the strong Hilbert-Schmidt dynamical localization.

The main theorem of this paper is the following:

Theorem 1.1. *For any $L > 0$, let H_L^ω be the restriction of the operator H^ω to $L^2(\Lambda_L)$ under the Dirichlet boundary condition and P_L^ω be its spectral projection. Then we have the following: for any $0 < \varepsilon < 1$ and $\delta > 0$, there exists $C_{\varepsilon,\delta} > 0$ such that*

$$\mathbf{E}[\operatorname{Tr} P_L^\omega((E-\eta, E+\eta))] \le C_{\varepsilon,\delta} L^{2d} \eta^\varepsilon,$$

for any $L > 1$, $E < 0$ and $\eta > 0$ satisfying $E + 2\eta \le -\delta$.

Remark 1. We can prove Theorem 1.1 under weaker assumptions on $\{f_i^{\omega_1}, i \in \mathbf{N}, \omega_1 \in \Omega_1\}$. For example, it is enough that the conditional probability of each $f_i^{\omega_1}$ with respect to other random variables has a bounded density as in Assumption 1 (iv) in [8]. However for the proof of localization by the multiscale analysis, we need extra assumptions on the correlations of $\{f_i^{\omega_1}, i \in \mathbf{N}, \omega_1 \in \Omega_1\}$. For example, it is enough that they are independently and identically distributed. These extensions are straightforward. Therefore we choose our assumption for the sake of simplicity.

The organization of this paper is as follows. In Section 2 we give the basic properties of the Schrödinger operators. In Section 3 we prove the main theorem. In Section 4 we modify Germinet and Klein's theory on the multiscale analysis. Finally, in Section 5 we state the results of Germinet and Klein's theory on the strong dynamical localization.

2. The basic properties of the Schrödinger operators H^ω

In this section, we will prove the essential self-adjointness of the Schrödinger operators H^ω based on Kirsch-Veselić [8].

Lemma 2.1. *For any $j \in \mathbf{Z}^d$, let $\mathcal{L}_\omega(j) = \sharp\{i \in \mathbf{N} | \xi_i^\omega \in \Lambda(j)\}$, where $\Lambda(j) = \Lambda_1(j)$. Then, for almost all ω, there exists a finite constant $C(\omega)$ such that*

$$\mathcal{L}_\omega(j) \le \|j\|_\infty^2 + C(\omega) \text{ for any } j \in \mathbf{Z}^d, \tag{2.1}$$

where $\|j\|_\infty := \sup_{1\le i\le d} |j_i|$ for $j = (j_i)_{i=1}^d$.

Proof. We require only showing that

$$\mathbf{P}\ \{\text{for infinitely many } j \in \mathbf{Z}^d, \mathcal{L}_\omega(j) > \|j\|_\infty^2\} = 0. \tag{2.2}$$

In fact, when (2.2) holds, there exists some $\Omega' \subset \Omega$ such that $\mathbf{P}(\Omega') = 1$, and every $\omega \in \Omega'$ has a finite set $\Gamma(\omega) \subset \mathbf{Z}^d$ satisfying

$$\mathcal{L}_\omega(j) \le \|j\|_\infty^2, \text{ for all } j \in \mathbf{Z}^d \setminus \Gamma(\omega).$$

Then, (2.1) holds with $C(\omega) = \sharp\{i \in \mathbf{N} | \xi_i^\omega \in \bigcup_{j\in\Gamma(\omega)} \Lambda(j)\}$.

On the other hand, by Chebyshev's inequality, we have

$$\begin{aligned}\sum_{j\in\mathbf{Z}^d} e^{-\|j\|_\infty^2} \mathbf{E}(e^{\mathcal{L}_\omega(j)}) &\geq \sum_{j\in\mathbf{Z}^d} e^{-\|j\|_\infty^2} \cdot e^{\|j\|_\infty^2} \cdot \mathbf{P}[\omega | \mathcal{L}_\omega(j) > \|j\|_\infty^2] \\ &= \sum_{j\in\mathbf{Z}^d} \mathbf{P}[\omega | \mathcal{L}_\omega(j) > \|j\|_\infty^2].\end{aligned}$$

Since $\mathbf{E}(e^{\mathcal{L}_\omega(j)}) = e^{(e-1)|\Lambda(j)|} = e^{e-1}$, the left hand side is dominated by $\sum_{j\in\mathbf{Z}^d} \exp(-\|j\|_\infty^2 + e - 1)$, which is finite.

Therefore, we have $\sum_{j\in\mathbf{Z}^d} \mathbf{P}[\omega | \mathcal{L}_\omega(j) > \|j\|_\infty^2] < \infty$, from which we have (2.2). $\square$

Proposition 2.1. *For almost all ω, the Schrödinger operator H^ω is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof. By Lemma 2.1, for almost all ω, we have finite constants C_1, $C_2(\omega)$ such that

$$V_\omega \geq -C_1\|x\|_\infty^2 - C_2(\omega)$$

(cf. [8]). Consequently, by Faris-Lavine theorem [10], H^ω is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. $\square$

In the rest of the paper we denote the unique self-adjoint extension by the same symbol H^ω .

By Proposition 2.1 and Proposition V.3.1. in [3], the measurability of the self-adjoint operator H^ω in ω is obtained. Moreover $\{H^\omega\}_{\omega\in\Omega}$ is an ergodic family of self-adjoint operators. Therefore, by Theorem (5.34) in [5], the spectrum $\sigma(H^\omega)$ satisfies that $\sigma(H^\omega) = \mathbf{R}$ for almost all ω.

3. The proof of the main theorem

In this section, we will prove the Wegner estimate, Theorem 1.1, using the method in [8] and the theory of the spectral shift function.

By the method in [8], we have

(3.1)
$$\begin{aligned}&\mathbf{E}^{\omega_1}[\mathrm{Tr} P_L^\omega((E-\eta, E+\eta))] \\ &\quad\leq \mathbf{E}^{\omega_1}\left\{\sum_{j\in\Lambda_L^+} \frac{1}{\delta}\int_{-3\eta/2}^{3\eta/2} dt\, \mathrm{Tr}\left[\rho\left(H_0^{L,j} - E + t\right) - \rho\left(H_1^{L,j} - E + t\right)\right]\right\},\end{aligned}$$

In (3.1) $\mathbf{E}^{\omega_1}$ is the expectation with respect to the randomness of $\omega_1, \Lambda_L^+ = \{k \in \mathbf{N} | \Lambda_L \cap \operatorname{supp} u_k(\cdot - \xi_k^{\omega_2}) \neq \emptyset\}$, and ρ is a smooth monotone increasing function satisfying $0 \leq \rho \leq 1$, $\rho(t) = 0$ for $t \leq -\eta/2$, $\rho(t) = 1$ for $t \geq \eta/2$ and ρ'/η^{-1} is bounded. Moreover $H_0^{L,j}$ and $H_1^{L,j}$ are the operators obtained by replacing $f_j^{\omega_1}$ by 0 and 1, respectively, in the definition of H_L^ω.

Now we use the following proposition on spectral shift functions ([2] Theorem 2.1, [14] Chapter 8 §3 Theorem 3 and Theorem 6):

Proposition 3.1. *Let A_1 and A_0 be self-adjoint operators such that $A_1 - A_0 \in \mathcal{I}_{1/p}$ for $p > 1$, where $\mathcal{I}_{1/p}$ is the family of compact operators of the super trace class, which we define as follows: we say that $A \in \mathcal{I}_{1/p}$ if for some $p > 1$, $|||A|||_{1/p} := \left(\sum_j \mu_j(A)^{1/p}\right)^p < \infty$, where $\mu_j(A)$ denotes the j-th singular value of A.*

Then, there exists some $\xi(\cdot; A_1, A_0) \in L^p(\mathbf{R})$ such that for $\phi \in C^\infty(\Gamma)$ where $\Gamma \subset \mathbf{R}$: a compact interval which contains $\sigma(A_0)$ and $\sigma(A_1)$,

$$\mathrm{Tr}[\phi(A_1) - \phi(A_0)] = \int_\Gamma \xi(\lambda; A_1, A_0)\phi'(\lambda)d\lambda,$$

and

$$\|\xi\|_p \leq |||A_1 - A_0|||_{1/p}^{1/p}.$$

We set $A_1 = (H_1^{L,j} + M_{\omega_2})^{-\ell}$ and $A_0 = (H_0^{L,j} + M_{\omega_2})^{-\ell}$ where $M_{\omega_2} = 2\sup_{x\in\Lambda_L}\sum_{i\in\mathbf{N}} u(x-\xi_i^{\omega_2})+1$. Then, by Proposition 5.1 in [2], $A_1 - A_0 \in \mathcal{I}_{1/p}$ for any $p > 1$ and $2\mathbf{N}+1 \ni \ell > dp/2+2$. Moreover, for any $J \in C_o^\infty(\mathbf{R}^d)$, $|||J \times A_k^{1/\ell}|||_{\ell/p} \leq |||J \times (-\Delta+1)^{-1}|||_{\ell/p} < \infty$ (cf. [11] Theorem 2.13 and Theorem 4.1) and the operator norms of $A_k^{1/\ell}, (\partial/\partial x^i)A_k^{1/\ell}, A_k^{1/\ell}(\partial/\partial x^i)$ and $(\partial/\partial x^i)A_k^{1/\ell}(\partial/\partial x^j)$ are bounded by $1 (k = 0$ or 1 , $1 \leq i,j \leq d)$. Therefore $|||A_1 - A_0|||_{1/p} \leq C$, where C is independent of ω_2 and L.

Then noting $\sigma(A_1)$ and $\sigma(A_0) \subset [0,1]$ we have

$$\begin{aligned}
&\mathrm{Tr}[\rho(H_0^{L,j} - E + t) - \rho(H_1^{L,j} - E + t)] \\
&\quad = \mathrm{Tr}[\mu(A_0) - \mu(A_1)] \\
&\quad = \int_{[0,1]} \frac{\partial\mu(\lambda)}{\partial\lambda}\xi(\lambda)d\lambda \qquad (3.2)\\
&\quad \leq C\left(\int_{[0,1]} |\frac{\partial\mu}{\partial\lambda}(\lambda)|^{p'}d\lambda\right)^{1/p'},
\end{aligned}$$

where p' is $1/p + 1/p' = 1$, $\mu(\lambda) = \rho((1/\lambda)^{1/\ell} - M_{\omega_2} - E + t)$ and ξ is the spectral shift function for A_1 and A_0. By changing the variable as $\lambda = \gamma^{-\ell}$ we can show that the right hand side of (3.2) is dominated by

$$\sup_{-\eta/2\leq\gamma-M_{\omega_2}-E+t\leq\eta/2} |\gamma|^{(1+\ell)(p'-1)/p'}\left[\int_{\mathbf{R}} |\rho'(\gamma)|^{p'}d\gamma\right]^{1/p'}. \tag{3.3}$$

We may assume that $|E| \leq M_{\omega_2}$. In fact if

$$-\sup_{x\in\Lambda_L} |V^\omega(x)| > E + \eta, \tag{3.4}$$

it follows $[E-\eta, E+\eta] \subset \sigma(H_L^\omega)^c$. Since $2\sup_{x\in\Lambda_L}|V^\omega(x)| \le M_{\omega_2}$ and $E+\eta < E/2$, a sufficient condition for (3.4) is $-M_{\omega_2} > E$.

Thus $[E-\eta, E+\eta] \cap \sigma(H_L^\omega) \neq \emptyset$ implies $|E| \le M_{\omega_2}$. We here note that the restriction $|E| \le M_{\omega_2}$ does not affect the estimate (3.1), since M_{ω_2} is independent of ω_1. Therefore, the first factor of (3.3) is bounded by $M_{\omega_2}^{(1+\ell)(1-1/p')}$.

By using also

$$\int_{\mathbf{R}} |\rho'(\gamma)|^{p'} d\gamma \le \int_{\mathbf{R}} |\rho'(\gamma)| d\gamma \sup(\rho')^{(p'-1)},$$

we can show that the second factor of the right hand side of (3.3) is dominated by $\eta^{(1/p'-1)}$. Therefore, the second factor of (3.2) is dominated by

$$\eta^{(1/p'-1)} M_{\omega_2}^{(1+\ell)(1-1/p')}. \tag{3.5}$$

Consequently, we obtain

$$\mathbf{E}\{\mathrm{Tr} P_L^\omega((E-\eta, E+\eta))\} \le \mathbf{E}\left\{\sum_{j\in\Lambda_L^+} \eta^{1/p'} C_1 (M_{\omega_2})^N\right\},$$

where $N = (\ell+1)(1-1/p')$. Since

$$M_{\omega_2} \le C_2 \sum_{k\in\mathbf{N}} \chi_{\Lambda_1}(x - \xi_k^{\omega_2}) + 1,$$

this is dominated by

$$\begin{aligned}
&\mathbf{E}\left\{\sum_{j\in\Lambda_L^+} \eta^{1/p'} \left(\sup_{x\in\Lambda_L} 2\sum_{k\in\mathbf{N}} \chi_{\Lambda_1}(x-\xi_k^{\omega_2}) + 1\right)^N\right\} \\
&\qquad \le C_3 \eta^{1/p'} \mathbf{E}\left\{(\sharp\Lambda_L^+)\left(\sup_{x\in\Lambda_L} \sharp\{i\in\mathbf{N} | \xi_i^{\omega_2} \in \Lambda_1(x)\} + 1\right)^N\right\}.
\end{aligned}$$

By the Schwarz inequality, this is dominated by

$$\eta^{1/p'} \mathbf{E}\left[(\sharp\Lambda_L^+)^2\right]^{\frac{1}{2}} \times \mathbf{E}\left[\left(\sup_{x\in\Lambda_L} \sharp\{i\in\mathbf{N} | \xi_i^{\omega_2} \in \Lambda_1(x)\} + 1\right)^{2N}\right]^{\frac{1}{2}}.$$

The first factor is dominated by

$$\mathbf{E}\left[(\sharp\{i\in\mathbf{N} | \xi_i^{\omega_2} \in \Lambda_{L+1}\})^2\right]^{\frac{1}{2}} \le C_4 L^d,$$

and the second factor is dominated by

$$\begin{aligned}
&\sum_{a\in\Lambda_L\cap\mathbf{Z}^d} \mathbf{E}\left[\left(\sup_{x\in\Lambda_1(a)} \sharp\{i\in\mathbf{N} | \xi_i^{\omega_2} \in \Lambda_1(x)\} + 1\right)^{2N}\right]^{\frac{1}{2}} \\
&\le \sum_{a\in\Lambda_L\cap\mathbf{Z}^d} \mathbf{E}[(\sharp\{i\in\mathbf{N} | \xi_i^{\omega_2} \in \Lambda_2(a)\} + 1)^{2N}]^{\frac{1}{2}} \\
&\le C_5 L^d.
\end{aligned}$$

By all these, we obtain the theorem.

4. Multiscale analysis

In [6] Germinet and Klein gave the theory of the bootstrap multiscale analysis in an abstract setting under several conditions. In this section we show our model satisfies these conditions in a weakened form. We use the following definitions in [4].

Definition 4.1. Given $\theta > 0$, $E \in \mathbf{R}$, $x \in \mathbf{Z}^d$, and $L \in 6\mathbf{N}$, we say that the box $\Lambda_L(x)$ is (θ, E)-suitable for ω if $E \notin \sigma(H^\omega_{L,x})$ and

$$\| \Gamma_{L,x}(H^\omega_{L,x} - E)^{-1}\chi_{L/3,x} \| \leq \frac{1}{L^\theta},$$

where $H^\omega_{L,x}$ is the restriction of the operator H^ω to $L^2(\Lambda_L(x))$ under the Dirichlet boundary condition and, $\Gamma_{L,x}$ and $\chi_{L,x}$ are characteristic functions of $\overline{\Lambda}_{L-1}(x) \setminus \Lambda_{L-3}(x)$ and $\Lambda_L(x)$ respectively.

Definition 4.2. Given $m > 0$, $E \in \mathbf{R}$, $x \in \mathbf{Z}^d$, and $L \in 6\mathbf{N}$, we say that the box $\Lambda_L(x)$ is (m, E)-regular for ω if $E \notin \sigma(H^\omega_{L,x})$ and

$$\| \Gamma_{L,x}(H^\omega_{L,x} - E)^{-1}\chi_{L/3,x} \| \leq \exp\left(-\frac{mL}{2}\right).$$

Based on the paper by Fischer, Leschke and Müller [4], we will give the initial length scale estimate under our setting. By the Combes-Thomas estimate (Lemma A.1 in [4]) we have

$$\begin{aligned} &\| \Gamma_L(H^\omega_L - E)^{-1}\chi_{L/3} \| \\ &\leq \frac{\sqrt{\{(L-1)^d - (L-3)^d\}(L/3)^d}}{2^{(d+1)/4}(\pi\delta)^{(d-1)/2}}(V^\omega_0 - E)^{(d-3)/4} \\ &\quad \times \left(1 + \frac{d^2}{8\delta\sqrt{2(V^\omega_0 - E)}}\right)\exp\left(-\delta\sqrt{2(V^\omega_0 - E)}\right) \end{aligned} \tag{4.1}$$

for all $E < V^\omega_0 := \operatorname{ess\,inf}_{x\in\Lambda_L} V^\omega(x)$, where $\delta := (2L-9)/6$, $\Gamma_L := \Gamma_{L,0}, \chi_L := \chi_{L,0}$, and $H^\omega_L := H^\omega_{L,0}$. To control V^ω_0, we use the following:

Proposition 4.1. *For all $\eta > 0$, there exist finite positive constants C_1 and C_2 such that*

$$\mathbf{P}\left[\sup_{x\in\Lambda_L} |V^\omega(x)| > \eta\right] \leq C_1 L^d \exp(-C_2\eta\log\eta), \tag{4.2}$$

for any $L \in \mathbf{N}$.

Proof. Noting that $0 \le f_i^{\omega_1} \le 1$ and $\operatorname{supp} u \subset \Lambda_1(0)$,
we have

$$\begin{aligned}
\mathbf{P}\left[\sup_{x\in\Lambda_L} |V^\omega(x)| > \eta\right] &\le \mathbf{P}\left[\sup_{x\in\Lambda_L} \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_1(x)\} > \eta/\|u\|_\infty\right] \\
&\le \sum_{a\in\Lambda_L\cap\mathbf{Z}^d} \mathbf{P}\left[\sup_{x\in\Lambda_1(a)} \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_1(x)\} > \eta/\|u\|_\infty\right] \\
&\le L^d\mathbf{P}\left[\sup_{x\in\Lambda_1} \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_1(x)\} > \eta/\|u\|_\infty\right] \\
&\le L^d\mathbf{P}[\mathcal{L}'_{\omega_2} > \eta/\|u\|_\infty],
\end{aligned}$$

where $\mathcal{L}'_{\omega_2} := \sharp\{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_2\}$.

Since $\mathcal{L}'_{\omega_2}$ obeys the Poisson distribution, for $N \in \mathbf{N}$

$$\mathbf{P}[\mathcal{L}'_{\omega_2} > N] = \sum_{n=N}^{\infty} e^{-2^d} \frac{2^{dn}}{n!} = \frac{\gamma(N, 2^d)}{\Gamma(N)},$$

where γ is the incomplete gamma function and Γ is the gamma function. By estimating the integral representation of the gamma functions, we obtain

$$\mathbf{P}[\mathcal{L}'_{\omega_2} > N] \le C_1 \exp(-C_2 N \log N)$$

Using this formula, we obtain (4.2). □

By (4.1) and Proposition 4.1, we can prove the initial length scale estimate as follows:

Proposition 4.2 (Initial length scale estimate)**.** *For all $L_0 \in 6\mathbf{N}$ and $0 < \theta$, there exists $E_0 < 0$ such that*

$$\mathbf{P}\{\omega : \Lambda_{L_0} \textit{ is } (\theta, E)\textit{-suitable}\} > 1 - 841^{-d} \textit{ for all } E \le E_0.$$

Proof. On the event $V_0^\omega > E$, we set $V_0^\omega - E = \Delta E$ where $V_0^\omega := \inf_{x\in\Lambda_{L_0}} V^\omega$. Then, by (4.1), we have

$$\begin{aligned}
&\| \Gamma_{L_0}(H_{L_0}^\omega - E)^{-1}\chi_{L_0/3} \| \\
&\qquad \le C_1 \times L_0^{d/2}(\Delta E)^{(d-3)/4} \times \left(1 + \frac{3d^2}{2L_0\sqrt{2\Delta E}}\right) \exp(-C_2 L_0 \sqrt{2\Delta E}).
\end{aligned}$$

For this to be dominated by $L_0^{-\theta}$ for all $L_0 \in 6\mathbf{N}$, it should hold that

$$L_0^{(\theta+d/2)}(\Delta E)^{(d-3)/4} \times \left(1 + \frac{3d^2}{2L_0\sqrt{2\Delta E}}\right) \exp(-C_2 L_0 \sqrt{2\Delta E}) \le 1/C_1.$$

If we take $C_3 > 0$ sufficiently large, then this inequality holds whennever $\Delta E \ge C_3$. Consequently, we have only to take E_0 so that

$$\mathbf{P}[V_0^\omega \ge E_0 + C_3] \ge 1 - 841^{-d}.$$

This is possible by Proposition 4.1. □

The condition on the average number of eigenvalues is satisfied in the following form:

Proposition 4.3 (Number of eigenvalues). *For any compact interval I, there exists a finite constant C_I such that*

$$\mathbf{E}[\mathrm{Tr}[P_L^\omega(I)]] \leq C_I L^{2d} \quad \textit{for all } L \in 2\mathbf{N}. \tag{4.3}$$

Proof. We dominate the spectral projection by the heat semigroup $\exp(-tH_L^\omega)$ generated by H_L^ω:

$$\mathrm{Tr}[P_L^\omega(I)] \leq e^b\, \mathrm{Tr}[\exp(-H_L^\omega)],$$

where $b = \sup I$. By Mercer's theorem, we have

$$\mathrm{Tr}[\exp(-H_L^\omega)] = \int_{\Lambda_L} \exp(-H_L^\omega)(x,x)dx,$$

where $\exp(-H_L^\omega)(x,y),\ x,y \in \Lambda_L$, is the integral kernel of $\exp(-H_L^\omega)$. By the Feynman-Kac formula [3], we have

$$\exp(-H_L^\omega)(x,y) \leq \frac{1}{4\pi} \exp(-\inf_{x\in\Lambda_L} V_\omega(x)).$$

By Proposition 4.1, there is a finite positive constant C' such that

$$\mathbf{E}[\exp(-\inf_{x\in\Lambda} V_\omega(x))] \leq C' L^d.$$

Consequently, we obtain (4.3). □

The random fields $V^\omega(x)|_{\Lambda_L(y)}$ and $V^\omega(x)|_{\Lambda_{L'}(y')}$ are independent if $d(\Lambda_L(y),\ \Lambda_{L'}(y'))>1$. This means that the condition on the independence at distance is satisfied in our setting(cf. [13] p59 (IAD)). The Simon-Lieb inequality in our setting is as follows: for all compact interval I, there exists $\gamma_I \in (0,\infty)$ such that for all $L, \ell', \ell'' \in 2\mathbf{N}$, $y, y' \in \mathbf{Z}^d$ which satisfy $\Lambda_{\ell''}(y) \sqsubset \Lambda_{\ell'}(y') \sqsubset \Lambda_L$, and $E \in I - \sigma(H_L^\omega) - \sigma(H_{\ell',y'}^\omega)$,

$$\begin{aligned} \| \Gamma_L (H_L^\omega - E)^{-1} \chi_{\ell'',y} \| \leq \gamma_I (1 + \sup_{x \in \Lambda_{\ell'}(y')} |V^\omega(x)|) \| \Gamma_{\ell',y'} (H_{\ell',y'}^\omega - E)^{-1} \chi_{\ell'',y} \| \\ \times \| \Gamma_L (H_L^\omega - E)^{-1} \chi_{\ell',y'} \|, \end{aligned}$$

where $\Lambda_{\ell'}(y') \sqsubset \Lambda_L(x)$ denotes $\Lambda_{\ell'}(y') \subset \Lambda_{L-3}(x)$. In the inequality in Germinet and Klein theory, the term $\sup |V^\omega(x)|$ does not appear. However, this term $\sup |V^\omega(x)|$ is controlled in our setting using Propposituion 4.1(cf. [13]). Now the conditions in Germinet and Klein theory [6] are satisfied in a weakened form. For this situation, their theory is extended as follows(cf. [13]):

Proposition 4.4 (Bootstrap multiscale analysis [6]). *For any $\delta > 0$ and $\theta > 2d/\varepsilon$, there exists $\tilde{L} \in 6\mathbf{N}$ satisfying the following: if $\mathbf{P}[\omega : \Lambda_L$ is (θ, E_0) – suitable$] > 1 - 1/841^d$ holds for some $\tilde{L} \leq L \in 6\mathbf{N}$ and $E_0 \leq -\delta$, then there exists $\delta_0 > 0$ such that for any $0 < \zeta < 1$ and $1 < \alpha < \zeta^{-1}$, there exist $L_0 \in 6\mathbf{N}$ and $m_\zeta > 0$ satisfying*

$$\mathbf{P}[R(m_\zeta, L_k, I(E_0, \delta_0), x, y)] \geq 1 - \exp(-L_k^\zeta)$$

for any $k \in \mathbf{Z}^+$ and $x, y \in \mathbf{Z}^d$ with $\|x - y\|_\infty > L_k + 1$, where $L_{k+1} = [L_k^\alpha]_{6\mathbf{N}} := \max\{N \in 6\mathbf{N} : N \leq L_k^\alpha\}$ and $I(E_0, \delta_0) = [E_0 - \delta_0, E_0 + \delta_0] \cap (-\infty, -\delta]$, for an interval I, we set $R(m,\ L,\ I,\ x,\ y) := \{\omega :$ for all $E \in I$, $\Lambda_L(x)$ or $\Lambda_L(y)$ is (m, E)-regular$\}$. (see [13]).

Proof. This theorem is proven by extending the four theorems in Section 5 in [6]. In the proof of Theorem 5.1, as in [13], we require s satisfies

$$(p + 2d)/\varepsilon < s \text{ and } s < \theta.$$

Moreover, in the definition of the event $\mathcal{F}_{L,\ell}$ in [6], we add the condition on $\sup_{\Lambda_L} |V^\omega|$ as follows:

$$\begin{aligned}\mathcal{F}_{L,\ell} = \{\omega :&\text{there exist } n\ (\theta,\ E_0)\text{-non suitable boxes}\\ &\{\Lambda_\ell(y_i)\}_{i=1}^n, \text{ where } n \geq S+1, \text{ in } C_{L,\ell} \text{ such that}\\ &\mathrm{dist}(\Lambda_\ell(y_i), \Lambda_\ell(y_j)) > 1 \ for\ i \neq j\}\\ &\cup \{\omega : \mathrm{dist}(\sigma(H^\omega_{\ell',x}), E_0) \leq t_L \text{ for some } x \in \Xi'_{L,\ell}\\ &\text{and } \ell' = (7k + 2/3)\ell\ (1 \leq k \leq S)\}\\ &\cup \{\omega : \mathrm{dist}(\sigma(H^\omega_{L,x}), E_0) \leq t_L\}\\ &\cup \{\omega : \sup_{\Lambda_L} |V^\omega| \geq \log L\},\end{aligned}$$

where

$$\begin{aligned}\Xi_{L,\ell} &:= \Lambda_L \cap (\ell/3)\mathbf{Z}^d \subset \mathbf{Z}^d,\\ C_{L,\ell} &:= \{\Lambda_\ell(y) :\ y \in \Xi_{L,\ell}, \Lambda_\ell(y) \sqsubset \Lambda_L\},\\ \Xi'_{L,\ell} &:= \Lambda_L \cap (\ell/6)\mathbf{Z}^d \subset \mathbf{Z}^d.\end{aligned}$$

The rest of the proof is same as in [6, 13]. □

For the application to the Anderson localization, we need also conditions on the generalized eigenfunctions. These are also satisfied in a form which is enough for our purpose (cf. [7], [13]).

5. Dynamical localization

In this section, we will state the results on the Anderson localization obtained by the direct application of Germinet and Klein theory [6] on the basis of the results of Section 4:

Proposition 5.1 (Decay of kernel). *We take a compact interval I such that* $\sup I \leq E_0$, *where E_0 is the negative number given in Proposition* 4.2.

Then for all $0 < \zeta < 1$ *there exists some C_ζ, such that for all $x,y \in \mathbf{Z}^d$,*

$$\mathbf{E}\left[\sup_{f \in G} |||\chi_{1,x} f(H^\omega)\mathbf{P}^\omega(I)\chi_{1,y}|||_2^2\right] \leq C_\zeta(\exp(-\|x-y\|_\infty^\zeta),$$

where G is the set of all Borel measurable functions such that $\| f \|_\infty \leq 1$ and $\mathbf{P}^\omega(I)$ is the restriction of the projection operator of H^ω to the energy region I.

From this, we obtain the following:

Corollary 5.1.

1. (Strong Hilbert-Schmidt dynamical localization) *We take a compact interval I as in the last proposition. Then we have*

$$\mathbf{E}\left[\sup_t ||||x|^q e^{-itH^\omega}\mathbf{P}^\omega(I)\chi_0|||_2^2\right] < \infty$$

for any q>0.

2. (Semi Uniformly Localized Eigenfunction) *We take a compact interval I as in the last proposition. For any $\epsilon > 0$ there exists m_ϵ and for a.e. ω there are constants $C_{\epsilon,\omega}$, $\tilde{C}_\omega \in (0,\infty)$ and $\{x_{n,\omega}\}_{n\in\mathbf{N}} \subset \mathbf{Z}^d$, such that, if we let $\{\phi_{n,\omega}\}_{n\in\mathbf{N}}$ be the normalized eigenfunctions of H_ω with energy $E_{n,\omega}$ in I, we have*

$$\| \chi_{1,x}\phi_{n,\omega} \|_2 \leq C_{\epsilon,\omega} \exp\{m_\epsilon(\log\|x_{n,\omega}\|_\infty)^{1+\epsilon}\}\exp\{-m_\epsilon\|x - x_{n,\omega}\|_\infty\}$$

and

$$\|x_{n,\omega}\|_\infty \geq \tilde{C}_\omega n^{1/(4\nu)}$$

for any $n \in \mathbf{N}$, $x \in \mathbf{Z}^d$ and $\nu > d/4$.

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References

[1] P. W. Anderson, *Absence of diffusion in certain random lattice*, Phys. Rev. **109** (1958), 1492–1505.

[2] J. M. Combes, P. D. Hislop and S. Nakamura, *The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some randam operators*, Comm. Math. Phys. **218** (2001), 113–130.

[3] R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators*, Birkhäuser, Boston, 1990.

[4] W. Fischer, H. Leschke and P. Müller, *Spectral localization by Gaussian random potentials in multi-dimensional continuous space*, J. Stat. Phys. **101** (2000), 935–985.

[5] A. Figotin and L. A. Pastur, *Spectra of random and almost-periodic operators*, Splinger-Verlag, Berlin, 1992.

[6] F. Germinet and A. Klein, *Bootstrap multiscale analysis and localization in random media*, Comm. Math. Phys. **222** (2001), 415–448.

[7] A. Klein, A. Koines and M. Seifert, *Generalized eigenfunctions for waves in inhomogenious media*, J. Funct. Anal. **190** (2002), 255–291.

[8] W. Kirsch and I. Veselić, *Wegner estimate for sparse and other generalized alloy type potentials*, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), 131–146.

[9] S. Nakamura, *A remark on the Dirichlet-Neumann decoupling and the integrated density of states*, J. Funct. Anal. **179** (2001), 136–152.

[10] M. Reed and B. Simon, *Methods of modern mathematical physics* (II), Academic Press, Inc. New York, 1975.

[11] B. Simon, *Trace ideals and their applications*, American Mathematical Society, 2005.

[12] P. Stollmann, *Caught by disorder*, Birkhäuser, Boston, 2001.

[13] N. Ueki, *Wegner estimates and localization for Gaussian random potentials*, Publ. RIMS. Kyoto Univ. **40** (2004), 29–90.

[14] D. R. Yafaev, *Mathematical scattering theory*, American Mathematical Society, 1992.