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On conformal mapping of multiply-connected domains.

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In the theory of conformal mapping of simply connected domains, we chiefly use the unit circle for canonical domain. On the other hand, in the case of multiply-connected domains, we utilize various types of domains for canonical one. For example, we use the concentric circular ring-domain (circular disc or whole plane) with slits of circular-arcs,(1) the concentric circular ring-domain (circular disc or whole plane) with slits of radial segments, the whole plane with parallel slits, the whole plane with slits of arcs of finite lengths on the logarithmic spirals $\arg z - k\log|z| = C$,(2) and so forth.

Hereafter we consider the domain of finite connectivity, and first, by the potential-theoretic method, we research the problem of conformal mapping of a given n-ply connected domain onto a band-domain parallel to the imaginary axis with slits also parallel to the imaginary axis.

1. Conformal mapping onto a parallel band-domain with parallel slits.

For simplicity we suppose that every boundary-component R_k $(k=1,\cdots,n)$ of a given n-ply connected domain B in the z-plane be Jordan curve.

Performing a finite number of suitable auxiliary mappings of simply connected domains, we can reduce the given domain B to one whose boundary-components are regular analytic curves. Such method has often been used in the mapping-theory of multiply connected domains. Thus we assume that all the curves $R_1, R_2, \ldots, R_n$ are regular analytic. Let z_1, z_2 be arbitrary two points on R_1, and R_1', R_1'' be two boundary-arcs of R_1 separated by them. More precisely described, R_1' is the part of R_1 from z_1 to z_2, when we go around along R_1 in the positive sense with respect to B; R_1'' is the remaining part of R_1. Let $U_1(z)$ be the harmonic measure of the boundary-arc R_1' with respect to B, i. e. $U_1(z)$ be harmonic in B satisfying the boundary conditions: $U_1(z)=1$ on R_1', $U_1(z)=0$ on R_1'' and $R_k(k=2,\dots,n)$. Next, let $U_k(z)\,(k=2,\dots,n)$ be the harmonic measure of the boundary-component $R_k(k=2,\dots,n)$ with respect to B, i. e. $U_k(z)$ be harmanic in B and $U_k(z)=1$ on R_k, $U_k(z)=0$ on $R_h(h\neq k)$. Furthermore, let $V_k(z)\,(k=1,2,\dots,n)$ be conjugate harmonic functions of $U_k(z)\,(k=1,2,\dots,n)$. In general, $V_k(z)\,(k=1,2,\dots n)$ are not single-valued and increase by the periodicity moduli $\omega_{k\nu}(k,\nu=1,2,\dots,n)$ after circling once along each boundary-component $R_\nu(\nu=1,2,\dots,n)$ in the positive sense. These moduli satisfy the relations

$$\sum_{\nu=1}^{n}\omega_{k\nu}=0 \quad (k=1,2,\dots,n). \tag{1}$$

Furthermore, it holds that for the determinant of $(n-1)$-th order $|\,\omega_{k\nu}\,|$

$$|\,\omega_{k\nu}\,|\neq 0^{(3)} \quad (k,\nu=2,\dots,n). \tag{2}$$

By means of these facts, we shall now prove that there exists a function

$$w=\Phi(z)=U(z)+iV(z), \tag{3}$$

single-valued, regular analytic in B, and satisfying the following three conditions:

(i) $U(z)$ is single-valued and harmonic in B,

(ii) $U(z)=1$ on R_1', $U(z)=0$ on R_1'' and $U(z)=\text{const.}$ on R_k $(k=2,\dots n)$,

(iii) the periodicity-moduli of $V(z)$ with respect to R_ν $(\nu=1,2,\dots,n)$ are zero, i. e.

$$\oint_{R_\nu} dV(z)=0 \quad (\nu=1,2,\dots,n). \tag{4}$$

Proof. We consider the function

$$U(z)+iV(z)=U_1(z)+iV_1(z)+\sum_{k=2}^{n}c_k(U_k(z)+iV_k(z)), \tag{5}$$

$c_k(k=2,\dots,n)$ being real constants. Then evidently (5) satisfies (i) and (ii). These constants can be, by (2), uniquely determined so as to satisfy the simultaneous equations

$$\sum_{k=2}^{n} c_k \omega_{k\nu} = -\omega_{1\nu} \quad (\nu=2,\dots,n). \tag{6}$$

Therefore

$$\oint_{K_\nu} dV(z) = 0 \quad (\nu=2,\dots,n).$$

By (1) and (6), we have

$$\sum_{k=2}^{n} c_k \omega_{k1} = -\omega_{11},$$

$$\therefore \quad \oint_{K_1} dV(z) = 0,$$

$$\therefore \quad \oint_{K_\nu} dV(z) = 0 \quad (\nu=1,2,\dots,n).$$

Thus we have proved that $w=\Phi(z)=U(z)+iV(z)$ is single-valued, and regular analytic in B, and also satisfies the above three conditions. (Q. E. D.)

Next, as a cons.quence of these conditions, the following properties of $w=\Phi(z)$ are derived:

$$\lim_{z\to z_1 (z\in \bar{B})} V(z) = -\infty, \quad \lim_{z\to z_2 (z\in \bar{B})} V(z) = +\infty. \tag{7}$$

Proof. In the function

$$w=U(z)+iV(z)=U_1(z)+iV_1(z)+\sum_{k=2}^{n} c_k (U_k(z)+iV_k(z)),$$

$U_1(z)$ is the harmonic measure of the boundary-arc R_1' and $V_1(z)$ its conjugate harmonic function. Now, we conformally transform the simply connected domain enclosed only by the boundary-curve R_1 and including the domain B, onto the upper half-plane in the x-plane such that R_1 corresponds to the real axis $\mathrm{Im}\, x=0$, $z=z_1$ to $x=0$ and $z=z_2$ to $x=1$. Let such mapping function be $x(z)$ and its inverse $z(x)$. Then we obtain

$$U_1(z(x))+iV_1(z(x)) = \frac{1}{i\pi}\log\frac{1}{-x} + \Psi_1(x)^{(4)}$$

in the neighborhood of $x=0$, (8)

$$= \frac{1}{i\pi}\log(x-1) + \Psi_2(x)$$

in the neighborhood of $x=1$, (8′)

where $\Psi_1(x)$ and $\Psi_2(x)$ are regular analytic in the neighborhood of $x=0$ and $x=1$ respectively, and the logarithms are restricted to assume their principal values. By (8) and (8′), we have

$$\lim_{x\to 0(\mathrm{Im}\,x\geq 0)} V_1(z(x)) = -\infty,\ \lim_{x\to 1(\mathrm{Im}\,x\geq 0)} V_1(z(x)) = +\infty,$$

$$\lim_{z\to z_1(z\in \bar{B})} V_1(z) = -\infty,\ \lim_{z\to z_2(z\in \bar{B})} V_1(z) = +\infty.$$

Since the real part of the function $\sum_{k=2}^{n} c_k(U_k(z)+iV_k(z))$ is zero everywhere on the boundary-curve R_1 which is supposed to be regular analytic, this function is regular analytic also on R_1 by the theorem of analytic continuation. Hence its imaginary part has a finite limit when $z\to z_1$, and also when $z\to z_2$. Thus we obtain by (5)

$$\lim_{z\to z_1(z\in \bar{B})} V(z) = -\infty,\ \lim_{z\to z_2(z\in \bar{B})} V(z) = +\infty. \qquad \text{Q. E. D.}$$

By means of the above properties of the function $w=\Phi(z)$, we shall now prove that *this function maps the domain B univalently onto the parallel band-domain* $0<\mathrm{Re}\,w<1$ *with slits parallel to the imaginary axis.*

Proof. By (ii) and (7), it follows that when the point z moves from z_1 to z_2 along R_1 in the positive sense and further returns to z_1 beyond z_2, the point $w=\Phi(z)$ moves from $-i\infty$ to $+i\infty$ along the straight line $\mathrm{Re}\,w=1$ and furthermore returns to $-i\infty$ along the straight line $\mathrm{Re}\,w=0$. In other words, w goes once round along the boundary of the parallel band-domain in the positive sense. On the other hand, if the point z goes once round along $R_\nu(\nu=2,\dots,n)$ in the positive sense, by (ii) and (iii), w moves along certain slit parallel to the imaginary axis and returns to its original position.

Let a be an arbitrary interior point of the parallel band-domain and not lying on any slit, i. e. $0<\mathrm{Re}\,a<1$ and $\Phi(z)\neq a$ on R_k $(k=2,\dots n)$. Let B_r be a sub-domain of B obtained by excluding the common parts of B and circular discs of radius r with centers at z_1 and z_2 respectively. Then, by (7), we can choose r sufficiently small such that the inequalities

$$\max_{|z-z_1|=r,\ z\in \bar{B}} V(z) < \mathrm{Im}\,a,\quad \min_{|z-z_2|=r,\ z\in \bar{B}} V(z) > \mathrm{Im}\,a \tag{9}$$

hold. Thus we obtain

$$\frac{1}{2\pi}\oint d\arg(\Phi(z)-a)=1, \tag{10}$$

the integration being taken along the boundary of B_r.

On the other hand, if a is any exterior point of the parallel band-domain, we have

$$\frac{1}{2\pi}\oint d\arg(\Phi(z)-a)=0. \tag{11}$$

The relations (10) and (11) hold, however small the radius r may be. Therefore, by means of principle of argument, we have proved that $w=\Phi(z)$ is univalent in B and the image of B is the parallel band-domain $0<\operatorname{Re} w<1$ with parallel slits. Q. E. D.

Finally, we shall prove that *if two points z_1 and z_2 are preassigned on R_1, then such mapping function can be uniquely determined except a translation parallel to the imaginary axis.*

Proof. Suppose that both $\Phi_1(z)$ and $\Phi_2(z)$ are such mapping functions. Performing an auxiliary mapping onto the half-plane in the x-plane, in the same way as (8) and (8′) are derived, we

$$\begin{aligned}\Phi_j(z(x)) &= \frac{1}{i\pi}\log\frac{1}{-x}+\varphi_j(x) \text{ in the neighborhood of } x=0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad (j=1,2) \quad (12) \\ &= \frac{1}{i\pi}\log(x-1)+\psi_j(x) \text{ in the neighborhood of } x=1,\end{aligned}$$

where the meanings of φ_j and ψ_j are same with $\Psi_1(x)$ in (8) and $\bar{\Psi}_2(x)$ in (8′) respectively. By (12), $\Phi_1(z)-\Phi_2(z)=\Phi_1(z(x))-\Phi_2(z(x))$ is regular at $x=0$ and $x=1$, and hence, as a function of z, also at $z=z_1$ and $z=z_2$. Further, we have

$$\begin{aligned}\operatorname{Re}\{\Phi_1(z)-\Phi_2(z)\} &= 0 && \text{on } R_1' \text{ and } R_1'', \\ &= \text{const.} && \text{on } R_k (k=2,\dots,n).\end{aligned}$$

Accordingly $\Phi_1(z)-\Phi_2(z)$ is single-valued and analytic in the domain B of finite connectivity, possessing the bounded boundary-values and its real part remains constant on each boundary-component. Hence, by a well-known theorem,[5] $\Phi_1(z)-\Phi_2(z)$, is identically constant in B and furthermore, since its real part is equal to zero on R_1, it must be

$$\Phi_1(z)-\Phi_2(z)=iC,$$

C being a real constant. This is just what is required. Q. E. D.

2. Example.

In this section, we shall, as an example, determine explicitly the mapping function of the circular ring-domain. Let the circular ring-domain B be $q<|z|<1$ and let $z_1=-1$, $z_2=+1$.

By Villat's formula,[6] the regular analytic function $\Omega(z)$ whose real part gives the solution of Dirichlet's problem such as

$$\mathrm{Re}\{\Omega(z)\}\to M(\varphi)\quad (z\to e^{i\varphi}),\ \mathrm{Re}\{\Omega(z)\}\to N(\varphi)\qquad (z\to qe^{i\varphi}),$$

is given in terms of elliptic functions by the following expression:

$$\Omega(z)=\frac{\omega_1}{\pi^2 i}\int_0^{2\pi} M(\varphi)\left\{\zeta\left(\frac{\omega_1}{\pi}(i\log z+\varphi)\right)-\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)\log z\right\}d\varphi$$

$$-\frac{\omega_1}{\pi^2 i}\int_0^{2\pi} N(\varphi)\left\{\zeta_3\left(\frac{\omega_1}{\pi}(i\log z+\varphi)\right)-\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)\log z\right\}d\varphi+iC,$$

$2\omega_1$ (real) and $2\omega_3$ (pure imaginary) being the primitive periods of Weierstrass's elliptic function $\wp(z)$, C being an arbitrary real constant and $q=e^{-\frac{\pi\omega_3}{i\omega_1}}$.

To find the function $\Phi(z)$ whose real part coincides with the harmonic measure $U_1(z)$ of R_1': $|z|=1$, $\pi<\arg z<2\pi$ with respect to B, we put $M(\varphi)=0\,(0<\varphi<\pi)$, $M(\varphi)=1\,(\pi<\varphi<2\pi)$ and $N(\varphi)=0$ $(0\leq\varphi<2\pi)$ in the above formula. Then we have

$$U_1(z)+iV_1(z)=\frac{\omega_1}{\pi^2 i}\int_\pi^{2\pi}\left\{\zeta\left(\frac{\omega_1}{\pi}(i\log z+\varphi)\right)-\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)\log z\right\}d\varphi$$

$$=\frac{1}{\pi i}\left[\log\sigma\left(\frac{\omega_1}{\pi}(i\log z+\varphi)\right)\right]_\pi^{2\pi}-\frac{\omega_1}{\pi i}\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)\log z$$

By fomulae of elliptic functions,

$$\sigma(u+2\omega_1)=-e^{2\eta_1(u+\omega)}\sigma(u),\ \sigma(u+\omega_1)=e^{\eta_1 u}\sigma(\omega_1)\sigma_1(u)$$

and $\wp(u)-e_1=\left(\frac{\sigma_1(u)}{\sigma(u)}\right)^2$, we have

$$U_1(z)+iV_1(z)=\frac{1}{\pi i}\log\frac{1}{\sqrt{\wp\left(\frac{i\omega_1}{\pi}\log z\right)-e_1}}-\frac{1}{2\pi i}\frac{\omega_1}{\omega_3}\log z+iC'$$

(C' : real const.).

Therefore the periodicity-modulus of $V_1(z)$ with respect to $|z|=q$ is $\frac{\omega_1}{i\omega_3}$. Next, the harmonic measure $U_2(z)$ of $|z|=q$ can easily be found:

$$U_2(z)=\frac{\log\frac{1}{|z|}}{\log\frac{1}{q}}=\frac{\log|z|}{\log q},$$

and hence

$$U_2(z)+iV_2(z)=\frac{\log z}{\log q}.$$

Hence the periodicity-modulus of $V_2(z)$ with respect to $|z|=q$ is $-\frac{2\pi}{\log q}=-\frac{2\omega_1}{i\omega_3}$. Thus

$$w=\Phi(z)=U_1(z)+iV_1(z)+\frac{1}{2}(U_2(z)+iV_2(z)),$$

or

$$w=\Phi(z)=\frac{1}{\pi i}\log\frac{1}{\sqrt{\wp\left(\frac{i\omega_1}{\pi}\log z\right)-e_1}}$$

is the required mapping function by the general argument discussed in the preceding section.

In fact we can also directly verify that this is the required function as the following shows. It can be decomposed into the functions:

$$\text{(I)}\ \zeta=\frac{i\omega_1}{\pi}\log z,\quad \text{(II)}\ t=\sqrt{\wp(\zeta)-e_1},\quad \text{(III)}\ w=\frac{1}{\pi i}\log\frac{1}{t}.$$

By (I) the upper half of the ring-domain $q<|z|<1$ is transformed onto a rectangle $-\omega_1<\text{Re}\,\zeta<0$, $-|\omega_3|\text{Im}\,\zeta<0$, by (II) this rectangle onto a right half of the lower half-plane $\text{Im}\,t<0$, and by (III) this quadrant onto a parallel band-domain $0<\text{Re}\,w<\frac{1}{2}$. Applying the inversion-principle to the composite function $w=\Phi(z)$ of (I), (II) and (III), we see at once that the circular ring-domain $q<|z|<1$ is conformally represented onto the parallel bond-domain $0<\mathrm{Re}\, w<1$ such that $-i\infty$ and $+i\infty$ correspond to $z_1=-1$ and $z_2=1$ respectively. Especially, to the inner circle $|z|=q$ corresponds a slit on the straight line $\mathrm{Re}\, w=\frac{1}{2}$ whose length is $\frac{1}{\pi}\log\sqrt{\frac{e_1-e_3}{e_1-e_2}}$ where $e_j=\wp(\omega_j)$ $(j=1, 2, 3,)$.

3. Conformal mapping of ring-domains.

In the present section, we deal with the position and the length of a slit in the conformal mapping of any ring-domain onto a parallel band-domain. Using the same notation as in § 1, we suppose that the boundary-components R_1 and R_2 of a given ring-domain B be both Jordan curves.

The ring-domain B whose modulus is $\log\frac{1}{q}(0<q<1)$, can be conformally represented onto a circnlar ring-domain in the t-plane $q<|t|<1$ such that $|t|=1$ and $|t|=q$ correspond to R_1 and R_2 respectively. Such mapping function is uniquely determined except a rotation around the origin $t=0$. Hence, we can determine the function in such a way that $t=e^{i\varphi}(0<\varphi<2\pi)$ and $t=1$ correspond to $z=z_1$ and $z=z_2$ respectively. Let us denote the circular-arcs $|t|=1(\varphi<\arg t<2\pi)$, $|t|=1(0<\arg t<\varphi)$, and $|t|=q(0\leq\arg t<2\pi)$ by Γ_1', Γ_1'' and Γ_2 respectively.

Using again the Villat's formula, we now find the mapping function $w=\Phi(t)$ of the ring-domain $q<|t|<1$ onto the parallel bnad-domain $0<\mathrm{Re}\, w<1$ with a slit such that $\mathrm{Re}\, w=1$ corresponds to Γ_1' and $\mathrm{Re}\, w=0$ to Γ_1''. We have

$$U_1(t)+iV_1(t)=\frac{\omega_1}{\pi^2 i}\int_\varphi^{2\pi}\left\{\zeta\left(\frac{\omega_1}{\pi}(i\log t+\varphi)\right)-\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi}\right)\log t\right\}d\varphi+iC$$

$$=\frac{1}{\pi i}\log\frac{\sigma\left(\frac{\omega_1}{\pi}(i\log t+2\pi)\right)}{\sigma\left(\frac{\omega_1}{\pi}(i\log t+\varphi)\right)}-\frac{\omega_1}{\pi^2 i}\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)(2\pi-\varphi)\log t+iC$$

$$=\frac{1}{\pi i}\log\frac{-e^{2\eta_1\frac{\omega_1}{\pi}i\log t}\sigma\left(\frac{\omega_1}{\pi}i\log t\right)}{\sigma\left(\frac{\omega_1}{\pi}(i\log t+\varphi)\right)}-\frac{\omega_1}{\pi^2 i}\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)(2\pi-\varphi)\log t+iC'$$

$$=\frac{1}{\pi i}\log\frac{-\sigma\left(\frac{\omega_1}{\pi}i\log t\right)}{\sigma\left(\frac{\omega_1}{\pi}(i\log t+\varphi)\right)}+\frac{\omega_1}{\pi^2 i}\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)\varphi\log t-\frac{1}{\pi i}\,\frac{\omega_1}{\omega_3}\log t+iC',$$

C and C' being real constants. Hence the periodicity-modulus of $iV_1(t)$ with respect to $|t|=q$ is

$$\frac{1}{\pi i}\left(-2\eta_1\frac{\omega_1}{\pi}\varphi\right)-\frac{2\omega_1}{\pi}\left(\frac{1}{2\omega_3}-\frac{\eta_1}{\pi i}\right)\varphi+\frac{2\omega_1}{\omega_3}=\frac{2\omega_1}{\omega_3}\left(1-\frac{\varphi}{2\pi}\right).$$

The periodicity-modulus of $iV_2(t)$ with respect to $|t|=q$ is $-2\frac{\omega_1}{\omega_3}$ (c. f. § 2). Accordingly, the required mapping function is given by the general argument of § 1, in the form

$$w=\Phi(t)=U_1(t)+iV_1(t)+\left(1-\frac{\varphi}{2\pi}\right)(U_2(t)+iV_2(t)),\tag{13}$$

or
$$w=\Phi(t)=\frac{1}{\pi i}\log\frac{-\sigma\left(\frac{\omega_1}{\pi}i\log t\right)}{\sigma\left(\frac{\omega_1}{\pi}(i\log t+\varphi)\right)}+\frac{\eta_1\omega_1}{\pi^3}\varphi\log t+iC'.\tag{14}$$

If the point t is situated on the inner circle $|t|=q$, then, putting $t=qe^{i\theta}(0\leq\theta<2\pi)$ in (14), we have

$$\Phi(qe^{i\theta})=\frac{1}{\pi i}\log\frac{-\sigma\left(\omega_3+\frac{\omega_1}{\pi}\theta\right)}{\sigma\left(\omega_3+\frac{\omega_1}{\pi}\overline{\theta-\varphi}\right)}+i\frac{\eta_1\omega_1}{\pi^3}\varphi\theta-\frac{\eta_1\omega_3}{\pi^2 i}\varphi+iC'$$

$$=\frac{1}{\pi i}\log e^{\frac{\eta_3\omega_1}{\pi}\varphi}\frac{\sigma_3\left(\frac{\omega_1}{\pi}\theta\right)}{\sigma_3\left(\frac{\omega_1}{\pi}\overline{\theta-\varphi}\right)}+1+i\frac{\eta_1\omega_1}{\pi^3}\varphi\theta-\frac{\eta_1\omega_3}{\pi^2 i}\varphi+iC'$$

$$=\frac{1}{\pi i}\log\frac{\sigma_3\left(\frac{\omega_1}{\pi}\theta\right)}{\sigma_3\left(\frac{\omega_1}{\pi}\overline{\theta-\varphi}\right)}+\left(1-\frac{\varphi}{2\pi}\right)+i\frac{\eta_1\omega_1}{\pi^3}\varphi\theta+iC'$$

$$=\frac{1}{\pi i}\log\frac{\vartheta_4\left(\frac{\theta}{2\pi}\right)}{\vartheta_4\left(\frac{\theta-\varphi}{2\pi}\right)}+\frac{2\eta_1\omega_1}{\pi i}\left\{\left(\frac{\theta}{2\pi}\right)^2-\left(\frac{\theta-\varphi}{2\pi}\right)\right\}^2+\left(1-\frac{\varphi}{2\pi}\right)$$

$$+i\frac{\eta_1\omega_1}{\pi^3}\varphi\theta+iC'$$

$$=\frac{1}{\pi i}\log\frac{\vartheta_4\left(\frac{\theta}{2\pi}\right)}{\vartheta_4\left(\frac{\theta-\varphi}{2\pi}\right)}+f(\varphi),$$

where $f(\varphi)$ is a function of φ independent of θ.

By a formula of the elliptic function

$$\vartheta_4\left(\frac{\theta}{2\pi}\right)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos\theta+q^{4n-2}),$$

we have

$$\mathrm{Im}\{\Phi(qe^{i\theta})\}=\frac{1}{\pi}\log\frac{\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos\overline{\theta-\varphi}+q^{4n-2})}{\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos\theta+q^{4n-2})}+f_1(\varphi), \tag{15}$$

where $f_1(\varphi)$ is a function of φ independent of θ. By (15), we get

$$\max_{0\leq\theta<2\pi}\mathrm{Im}\{\Phi(qe^{i\theta})\}\leq\frac{1}{\pi}\log\frac{\prod_{n=1}^{\infty}(1+q^{2n-1})^2}{\prod_{n=1}^{\infty}(1-q^{2n-1})^2}+f_1(\varphi), \tag{16}$$

$$\min_{0\leq\theta<2\pi}\mathrm{Im}\{\Phi(qe^{i\theta})\}\geq-\frac{1}{\pi}\log\frac{\prod_{n=1}^{\infty}(1+q^{2n-1})^2}{\prod_{n=1}^{\infty}(1-q^{2n-1})^2}+f_1(\varphi). \tag{17}$$

From (16) and (17), an inequality

$$\max_{0\leq\theta<2\pi}\mathrm{Im}\{\Phi(qe^{i\theta})\}-\min_{0\leq\theta<2\pi}\mathrm{Im}\{\Phi(qe^{i\theta})\}\leq\frac{1}{\pi}\log\frac{\prod_{n=1}^{\infty}(1+q^{2n-1})^4}{\prod_{n=1}^{\infty}(1-q^{2n-1})^4} \tag{18}$$

is derived.

By the formulae

$$\sqrt{e_1-e_2}=\frac{\pi}{2\omega_1}Q_0^2\,Q_3^4,\quad \sqrt{e_1-e_3}=\frac{\pi}{2\omega_1}Q_0^2\,Q_2^4,$$

$$Q_0=\prod_{n=1}^{\infty}(1-q^{2n}),\ Q_2=\prod_{n=1}^{\infty}(1+q^{2n-1}),\ Q_3=\prod_{n=1}^{\infty}(1-q^{2n-1}),$$

we have

$$\operatorname*{Max}_{0\leq\theta_1,\,\theta_2\leq 2\pi}|\operatorname{Im}\{\Phi(qe^{i\theta_1})\}-\operatorname{Im}\{\Phi(qe^{i\theta_2})\}|\leq\frac{1}{\pi}\log\sqrt{\frac{e_1-e_3}{e_1-e_2}}=\frac{1}{\pi}\log\frac{1}{k'} \tag{19}$$

k' denoting the complementary modulus of sn-function. It is obvious that the equality in (19) holds if and only if $\varphi=\pi$, $\theta_1=0$ (or π) and $\theta_2=\pi$ (or 0).

In the above described mapping, the ring-domain B is transformed onto a circular ring-domain $q<|t|<1$ such that $t=1$ and $t=e^{i\varphi}$ correspond to z_2 and z_1 respectively. In such a case, let the angle φ be called the "angular distance" measured from z_2 to z_1 with respect to B. Especially, let two points z_1 and z_2 be called to be "diametrically opposite" with respect to B, provided that $\varphi=\pi$. Then, we get the following theorem.

Theorem. *If any ring-domain B of a modulus* $\log\frac{1}{q}\ (q=e^{\frac{\pi\omega_3}{i\omega_1}})$ *is conformally represented onto a parallel band-domin* $0<\operatorname{Re} w<1$ *with a slit in such a way that* $w=-i\infty$ *and* $w=+i\infty$ *correspond to* $z=z_1$ *and* $z=z_2$ *on* R_1 *respectively, and the slit corresponds to another boundary-component* R_2 *of B, then the slit is situated on a straight line* $\operatorname{Re} w=1-\frac{\varphi}{2\pi}$ *and its length not greater than* $\frac{1}{\pi}\log\sqrt{\frac{e_1-e_3}{e_1-e_2}}$, *where* φ *is the angular distance measured from* z_2 *to* z_1 *with respect to B and* $e_j=\wp(\omega_j)\ (j=1,2,3)$. *The equality about the slit-length holds if and only if* z_1 *and* z_2 *are diametrically opposite with respect to B.*

Proof. It is obvious by (13) and (19). Q. E. D.

In the above argument, we have fixed the ring-domain B and selected two points z_1 and z_2 arbitrarily on the boundary-component R_1 of B. But in the following, we deform the boundary-arc R_1'' (or R_1') on R_1, z_1, z_2 being fixed and investigate the conformal mapping of such deformed domain onto the band-domain $0<\operatorname{Re} w<1$.

Then we obtain the following theorem.

Theorem. *Let any given ring-domain B be extended to a ring-domin B^* by substituting any Jordan curve R_1^* lying outside B for R_1'', while two points z_1, z_2 on R_1 and other boundary-arcs of B are kept fixed. Furthermore, let $w=f(z)$ be the mapping function of B onto the band-domain $0<\mathrm{Re}\, w<1$ such that $\mathrm{Re}\, w=1$ and $\mathrm{Re}\, w=0$ correspond to R_1' and R_1'' respectively and $w=\pm i\infty$ to z_2, z_1 respectively and $w^*=f^*(z)$ be the similar mapping function of B^* such that $\mathrm{Re}\, w^*=1$ and $\mathrm{Re}\, w^*=0$ correspond to R_1' and R_1^* respectively. Then an inequality*

$$\operatorname*{Re}_{z\in R_2} f(z) \leqq \operatorname*{Re}_{z\in R_2} f^*(z)$$

holds good. The equality holds if and only if $B\equiv B^*$.

Proof. Let the moduli of B and B^* $\log\frac{1}{q}$ and $\log\frac{1}{q^*}$ respectively. Then $\log\frac{1}{q}\leqq\log\frac{1}{q^*}$ since $B\subset B^*$, and hence $q\geqq q^*$. The equality holds if and only if B^*[7].

We now represent B^* conformally onto the circular ring-domain $q^*<|\zeta^*|<1$, and let $\zeta^*=G(z)$ be such mapping function, whereby the boundary-arcs R_1' and R_1^* are transformed into circular-arcs γ' and δ' on $|\zeta^*|=1$. At the same time, the sub-domain B of B^* are represented on a certain ring-subdomain of the circular ring-domain. Next, we represent B conformally onto the circular ring-domain $q<|\zeta|<1$, and let $\zeta=F(z)$ be such mapping function,whereby the boundary-arcs R_1' and R_1'' are transformed into circular-arcs γ and δ on $|\zeta|=1$ respectively. Therefore, by the composite function $\zeta^*=G\{F^{-1}(\zeta)\}$, the circular ring-domain $q<|\zeta|<1$ are represented onto the ring-subdomain of the domain $q^*<|\zeta^*|<1$ in such a way that the boundary-arc γ on $|\zeta|=1$ is transformed into γ' on $|\zeta^*|=1$ and the inner circle $|\zeta|=q$ into $|\zeta^*|=q^*$. Hence, by a theorem due to Prof. Y. Komatu,[8] we obtain

$$\gamma\leqq\gamma',$$
$$\therefore\quad \delta\geqq\delta',$$

where the equality hold if and only if $q=q^*$, therefore $B\equiv B^*$. Since δ ane δ' are the angular distances measured from z_2 to z_1 with respect to B and B^* respectively, we obtain, by the preceding theorem, the required inequality.

Similarly, if we extend the boundary-arc R_1', instead of R_1'', outside the domain B, we have an inequality

$$\operatorname*{Re}_{z\in R_2} f(z) \geqq \operatorname*{Re}_{z\in R_2} f^*(z).$$

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References.

1) G. Julia, Leçon sur la représentation conforme des aires multiplement connexes, (1934) p. 46.

2) Y. Komatu, Theory of conformal mapping (Japanese ;=Tôkakushazô-ron) II. (1949) pp. 189—218.

3) G. Juiia, loc. cit. p. 54.

4) R. Nevanlinna, Eindeutige analytische Funktionen, (1936) p. 35.

5) G. Julia, loc. cit. p. 48 Lemma.

6) H. .Villat, Le probleme de Dirichlet dans une aire annulaire. Rend. di Palermo 33(1912) p. 149. A brief proof of the Villat's formula is found in Y. Komatu, Sur la représentation de Villat pour les fonctions analytiques définies dans un anneau circulaire concentrique. Proc. Imp. Acad. Tokyo 21 (1945) pp. 94—96.

7) O. Teichmüller, Uutersuchungen über konforme und quasikonforme Abbildung. Deutsch. Math. 3 (1938) p. 626.

8) Y. Komatu, Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten. Proc. Phys-Math. Soc. (1943) pp. 33—36.