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On the Varieties of the Classical Groups in the Field of Arbitrary Characteristic

by

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Let $\boldsymbol{K}$ be a universal domain over the prime field ρ of characteristic p, then the matrix of degree $n(n\geq 1)$ with coefficients in $\boldsymbol{K}$

$$\sigma=(x_i^{(j)})(1\leq i,\ j\leq n)$$

can be considered as a point of an n^2-space in our algebraic geometry. Since the equation

$$\det\ (\sigma)=0$$

in the n^2 coefficients of σ is absolutely irreducible, it defines over ρ a variety of n^2-1 dimension in the n^2-space. If we take out this variety as a frontier from the space, the abstract variety so obtained forms the *general linear group* $GL(n, \boldsymbol{K})$ by matrix multiplication. Moreover since the group operation

$$(\sigma,\ \tau)\rightarrow\sigma\cdot\tau^{-1}$$

in $GL(n, \boldsymbol{K})$ is a function, which is everywhere defined on the product variety $GL(n, \boldsymbol{K})\times GL(n, \boldsymbol{K})$, the group $GL(n, \boldsymbol{K})$ is the so-called *group variety* in the recent terminology.[1]

We shall now define the *special linear group* $SL(n, \boldsymbol{K})$ and the *special orthogonal group* $SO(n, \boldsymbol{K})$ in $\boldsymbol{K}$ by the equations

$$\det\ (\sigma)=1$$

and $${}^t\sigma\cdot\sigma={}^t\sigma\cdot I_n\cdot\sigma=I_n,\quad \det\ (\sigma)=1$$

respectively, where I_n means the unit matrix of degree n. More-

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We shall use freely the results and terminology of Weil's book: *Foundations of algebraic geometry*, Am. Math. Soc. Colloq., Vol. 29 (1946).

1) See A. Weil, *Variétés Abéliennes et courbes algébriques*, Act. Sc. et Ind., n^0 1964 (1948), § II.

over we shall define the *symplectic group* $Sp(n, \boldsymbol{K})$ in $\boldsymbol{K}$ by the equation

$$ {}^t\sigma \cdot J_n \cdot \sigma = J_n, $$

where J_n is the n-fold sum of the following matrix

$$ J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

These "classical groups" are clearly subgroups and at the same time bunches in the general linear groups. In such a case we have the following useful lemma.

Lemma. *Let* $\mathfrak{g}$ *be a subgroup and a bunch in a group variety* $\mathfrak{G}$, *then the component of the identity in* $\mathfrak{g}$ *is a group variety* $\mathfrak{g}_0$. *Therefore* $\mathfrak{g}_0$ *is a normal subgroup with finite index in* $\mathfrak{g}$, *which is equal to the number of components of* $\mathfrak{g}$.

This lemma is well known if $\mathfrak{G}$ is an abelian variety (Cf. loc. cit. 1)-§ VI, *prop.* 8) and can be proved by the same reasoning. We note that the component $\mathfrak{g}_0$ (in Weil's terminology) is also the "connected component" of the identity in $\mathfrak{g}$ with respect to Zariski's topology.

Theorem I. *The classical groups are connected and hence they are group varieties all defined over the prime field* ρ.

Since our theorem is almost evident for $SL(n, \boldsymbol{K})$, we shall consider the other cases. We note thereby that the case $SL(n, \boldsymbol{K})$ can be treated also by the same method.[2]

(*Case* 1) $\mathfrak{G} = GL(n, \boldsymbol{K})$, $\mathfrak{g} = SO(n, \boldsymbol{K})$.[3]

We first define the scalar product of two column vectors x and y in n-space by

$$ (x, y) = {}^t x \cdot y. $$

Let σ be a generic point of $GL(n, \boldsymbol{K})$ over ρ, then the well-known process of "Schmidt normalization" is prossible for the n column vectors of σ. Otherwise there exists an integer $\nu (1 \leq \nu \leq n)$ such that the process is possible for the first $\nu - 1$ vectors but is impossible for the ν-th vactor. In such a case we see readily that the Gramian

2) In Weyl's book: *Classical groups*, Princeton (1939), the case of characteristic 0 is discussed by a different method.

3) In § II-p. 18, loc. cit. 1), this part of the theorem is stated as "*on vérifie sans paine....*".

of these ν vectors must vanish. However this is absurd since this Gramian has clearly the specialization 1 over ρ.

Now we can prove our theorem by examining more closely the Schmidt process. However we shall prefer to follow an idea essentially due to Hurewicz.

Since our theorem is trivial for $n=1$, we shall assume by induction that $n\geq 2$. In this case the equation

$$\sum_{i=1}^{n} X_i^2 - 1 = 0$$

is absolutely irreducible and defines a variety Q^{n-1} over ρ. Thereby the case $\rho = G.F.(2)$ $(p=2)$ is exceptional and we must take the linear equation

$$\sum_{i=1}^{n} X_i - 1 = 0$$

for Q^{n-1}. Now consider the projection π from $\mathfrak{G}$ to the space of the first column vectors, then by the possibility of the Schmidt process the projection of at least one and hence every coset of $\mathfrak{g}$ modulo $\mathfrak{g}_0$ is Q^{n-1}. However if two matrices σ_1 and σ_2 in $\mathfrak{g}$ have the same projection on Q^{n-1}, they belong to the same coset modulo a subgroup $\mathfrak{h}$ of the matrices of the from

$$\left(\begin{array}{c|c} \boxed{1} & 0 \\ \hline 0 & \boxed{} \end{array}\right).$$

Since $\mathfrak{h}$ is isomorphic with $SO(n-1, \boldsymbol{K})$, it is connected by induction assumption. Therefore $\mathfrak{h}$ is contained in $\mathfrak{g}$, and σ_1 and σ_2 belong to the same coset of $\mathfrak{g}$ modulo $\mathfrak{g}_0$. On the other hand since the generic point of Q^{n-1} over ρ is a projection of at least one point in each coset of $\mathfrak{g}$ modulo $\mathfrak{g}_0$, by what we have just remarked $\mathfrak{g}$ has only one coset; and hence $\mathfrak{g}$ is connected.

(*Case* 2) $\mathfrak{G} = GL(2n, \boldsymbol{K})$, $\mathfrak{g} = Sp(n, \boldsymbol{K})$.

In this case we define the skew product of two column vectors

$$x = {}^t(x_1\ x_1' \cdots x_n\ x_n'),\ y = {}^t(y_1\ y_1' \cdots y_n\ y_n')$$

by

$$[x, y] = \sum_{i=1}^{n} (x_i\ y_i' - x_i'\ y_i).$$

Then a similar method as Schmidt process can be applied to the $2n$ column vectors of a generic point of $GL(2n, \boldsymbol{K})$ and we obtain a matrix in $Sp(n, \boldsymbol{K})$. On the other hand the equation

$$\sum_{i=1}^{n}(X_i Y_i' - X_i' Y_i) - 1 = 0$$

is absolutely irreducible and defines a variety N^{4n-1} over ρ. Now consider the projection π from $\mathfrak{G}$ to the space of the first two column vectors, then the projection of any coset of $\mathfrak{g}$ modulo $\mathfrak{g}_0$ is N^{4n-1}. The rest of the proof is the same as in the previous case.

A direct consequence of our theorem I is the following well-known corollary.

Corollary 1. *If the universal domain $\boldsymbol{K}$ is the field of all complex numbers, the classical groups are connected in the usual topology.*

Moreover we can calculate the dimensions of the classical groups by induction on n.

Corollary 2. *The dimensions of the classical groups are as follows*

$$dim\ SL(n, \boldsymbol{K}) = n^2 - 1, \quad dim\ SO(n, \boldsymbol{K}) = n(n-1)/2$$

$$dim\ Sp(n, \boldsymbol{K}) = 2n^2 + n.$$

We shall now prove the following theorem.

Theorem II. *The classical groups are rational varieties. More precisely the field $\rho(\sigma)$ of the generic point σ of each one of them over ρ is a purely transcendental extension over ρ.*

Since our theorem is trivial for $SL(n, \boldsymbol{K})$, we shall consider the other cases.

(*Case* 1) $SO(n, \boldsymbol{K})$.

If ρ is not the field $G.F.(2)$, the "Cayley parametrization" gives a birational correspondence over ρ between $SO(n, \boldsymbol{K})$ and the linear variety of all skew-symmetric matrices of degree n in $\boldsymbol{K}$. The theorem follows immediately from this fact.

On the other hand if $\rho = G.F.(2)$, the condition for a matrix σ in $GL(n, \boldsymbol{K})$ to be in $SO(n, \boldsymbol{K})$ can be expressed successively by the linear equations in the column vectors of σ. Thereby if σ is a generic point of $SO(n, \boldsymbol{K})$ over ρ, we conclude from the dimension of $SO(n, \boldsymbol{K})$ and from the number of successive linear equations that its column vectors are "general solutions" at each step. Therefore our theorem holds also in this case.

(*Case* 2) $Sp(n, \boldsymbol{K})$.

In this case if we remark that the field $\rho(a, x, y)$ of a generic point (x, y) of the variety with the equation

$$\sum_{i=1}^{\nu}(X_i Y_i' - X_i' Y_i) - a = 0$$

over $\rho(a)$ is purely transcendental extension over $\rho(a)$, the argument is similar as in the latter part of the above proof.

In concluding this paper we shall discuss about a criticism of Prof. Akizuki. He has given me the following example.

Consider the case of $SO(3, \boldsymbol{K})$ for $p=2$, then we can not find a matrix in $SO(3, \boldsymbol{K})$ with the column vector

$${}^t(1 \ 1 \ 1).$$

Therefore although the projection of $SO(3, \boldsymbol{K})$ by π (in the sense of algebraic geometry) is Q^2, the "point-set theoretical" projection is not the whole Q^2. This phenomenon will be precised in the following general theorem.

Theorem III. *The point-set theoretical projection of $SO(n, \boldsymbol{K})$ by π is Q^{n-1} in general. The only exception arises when*

$$\rho = G.F.(2), \quad n \equiv 1 \ (\text{mod. } 2)$$

and only for the point

$${}^t(1 \ 1 \cdots\cdots 1).$$

For the variety N^{4n-1} there does not arise any exceptional case.

We shall now prove this theorem by examining the Schmidt process somewhat closely.

(*Case* 1) $SO(n, \boldsymbol{K})$.

Let σ be a matrix of the from

$$\sigma = (x^{(1)} \ x^{(2)} \cdots \ x^{(n)}) \quad (x^{(1)} = e),$$

where e is a given point of Q^{n-1} and $x^{(2)}, \cdots, x^{(n)}$ are algebraically independent vectors over $\rho(e)$. It is clear that e is an "exceptional" point if and only if the Schmidt process is impossible for σ from $x^{(1)}$ to $x^{(n)}$. Moreover the set of exceptional points is invariant under the transformation

$${}^t(e_1 \ e_2 \cdots e_n) \to {}^t(\pm e_1 \pm e_2 \cdots \pm e_n).$$

Now if the process is impossible at the ν-th step $(2 \leq \nu \leq n)$, the

Gramian of $x^{(1)},\cdots, x^{(\nu)}$ must vanish. Then by specializing $x^{(1)},\cdots, x^{(\nu)}$ to the following type of vectors

$$ {}^t(0 \cdots 0\ 1\ 0 \cdots 0) $$

over $\rho(e)$, we see readily that e must be of the form

$$ e=(\pm\varepsilon\ \pm\varepsilon\ \cdots\ \pm\varepsilon) $$

with the supplementary conditions

$$ n\cdot\varepsilon^2=1,\quad n\equiv\nu-1 \pmod{p}. $$

Therefore if $p=2$, we have

$$ e={}^t(1\ 1\ \cdots\ 1),\quad n\equiv1 \pmod{2}; $$

and this is actually exceptional.

In the general case of $p\geq3$ we can assume by our previous remark that e is of the form

$$ e={}^t(\varepsilon\ \varepsilon\ \cdots\ \varepsilon). $$

Since $\nu\leq n$ and at the same time $n\equiv\nu-1 \pmod{p}$, we have

$$ \nu\leq n-p+1\leq n-1 $$

even in the case of $p=2$. Then by specializing $x^{(2)},\cdots, x^{(\nu)}$ first to

$$ \begin{array}{l} {}^t(\varepsilon\ -\varepsilon\ \ 0\ \cdots\cdots\ 0) \\ {}^t(\varepsilon\ \ 0\ -\varepsilon\ \cdots\cdots\ 0) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ {}^t(\underbrace{\varepsilon\ \ 0\ \ 0\cdots-\varepsilon}_{\nu}\cdots0) \end{array} $$

and next to

$$ \begin{array}{l} {}^t(\varepsilon\ -\varepsilon\ \ 0\ \ 0\ \cdots\ \cdots\ 0) \\ {}^t(0\ \ 0\ \ \varepsilon\ -\varepsilon\ \cdots\ \cdots\ 0) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ {}^t(\underbrace{0\ \ 0\ \ \varepsilon\ \ 0\ \cdots-\varepsilon}_{\nu+1}\cdots 0), \end{array} $$

we conclude the incompatible congruences

$$ \nu\equiv0 \pmod{p},\quad \nu\equiv1 \pmod{p}. $$

Thereby the assumption $p \geq 3$ is essential for the second congruence.

(*Case* 2) $Sp(n, \boldsymbol{K})$.

This case can be treated similarly, hence we shall omit its detail.

We conclude from this theorem that the varieties Q^{n-1} and N^{4n-1} are *algebraic homogeneous spaces* with the "structure groups" $SO(n, \boldsymbol{K})$ and $Sp(n, \boldsymbol{K})$ respectively in general.

Appendix

Let $A(n, \boldsymbol{K})$ be the space of non-singular matrices σ of degree n in $\boldsymbol{K}$ such that ${}^t x \cdot \sigma \cdot x = 0$ for every vector x in $\boldsymbol{K}$. It is clear that $A(n, \boldsymbol{K})$ is roughly speaking the space of skew-symmetric non-singular matrices of degree n in $\boldsymbol{K}$; hence it is present only for $n = 2\nu$. On the other hand the group $GL(n, \boldsymbol{K})$ operates transitively on $A(n, \boldsymbol{K})$ by

$$A(n, \boldsymbol{K}) \ni \sigma \longrightarrow {}^t\tau \cdot \sigma \cdot \tau \quad (\tau \in GL(n, \boldsymbol{K}));$$

hence $A(n, \boldsymbol{K})$ is a homogeneous rational variety

$$A(n, \boldsymbol{K}) = GL(n, \boldsymbol{K}) / Sp(\nu, \boldsymbol{K}).$$

In the same way let $S(n, \boldsymbol{K})$ be the space of non-singular symmetric matrices of degree n in $\boldsymbol{K}$, then $S(n, \boldsymbol{K})$ is certainly a homogeneous rational variety

$$S(n, \boldsymbol{K}) = GL(n, \boldsymbol{K}) / O(n, \boldsymbol{K})$$

for $p \neq 2$. Thereby the "isotropy group" $O(n, \boldsymbol{K})$ is the *orthogonal group* of degree n in $\boldsymbol{K}$. We shall show that

the only exceptional case arises when

$$\rho = G.F.(2), \quad n \equiv 0 \pmod{2};$$

and then $GL(n, \boldsymbol{K})$ *operates transitively on* $S(n, \boldsymbol{K})$-$A(n, \boldsymbol{K})$

$$S(n, \boldsymbol{K}) - A(n, \boldsymbol{K}) = GL(n, \boldsymbol{K}) / O(n, \boldsymbol{K}).$$

In fact the well-known process of "diagonalization" for the symmetric matrix will be useless if and only if it is contained in $A(n, \boldsymbol{K})$. Therefore every matrix in $S(n, \boldsymbol{K})$ is equivalent under some operation in $GL(n, \boldsymbol{K})$ to one of the following matrices

$$\begin{pmatrix} I_a & 0 \\ 0 & J_b \end{pmatrix} \quad (a+2b=n).$$

However we can see readily that the matrix of degree 3

$$\begin{pmatrix} 1 & 0 \\ 0 & J_1 \end{pmatrix}$$

is equivalent to I_3. Therefore if $a \neq 0$, the above matrix is equivalent to I_n.; which clearly implies our assertion.