Note on Group Varieties

by

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This note consists of two rather separated parts. In the first part (§ 1), we remark a property of homomorphisms of group varieties, and in the second part (§ 2 and the following), we prove that if a group variety $G$ contains a group subvariety $H$, there exists a non-singular variety $V$ which has $G$ as a group of transformations, and whose points are in one-to-one correspondence with the cosets of $H$ in $G$.

§ 1. Homomorphism of group variety.

**Theorem 1.** Let $G$ and $H$ be two group varieties (see (A)), and let $f$ be a function on $G$ with values in $H$, such that if $x$ and $y$ are two independent generic points of $G$ over a common field of definition for $G$, $H$ and $f$, we have $f(xy) = f(x) \cdot f(y)$. Then $f$ is defined at each point $a$ of $G$, and is a homomorphism of group $G$ into group $H$. Moreover, if the graph $A$ of $f$ has the projection $H$ on $H$, $f$ is a homomorphism onto $H$, i.e., for any point $b$ on $H$, there is a point $c$ of $G$ such that $f(c) = b$.

**Proof.** Let $k$ be a field of definition for $G$, $H$ and $f$, over which $a$ and $b$ are rational, and let $x$ and $y$ be two independent generic points of $G$ over $k$. Then we have $f(xy) = f(x) \cdot f(y)$ and so $f(x) = f(xy) \cdot f(y)^{-1}$.

Now the right hand side being defined at $x=a$, so is the left hand side at $x=a$.

Next we consider the second half of the theorem. As $f(x)$ is a generic point of $H$ over $k(y)$, $b$ is a specialization of $f(x)$ over $k(y)$. By the relation $f(xy) = f(x) \cdot f(y)$, $f(xy)$ has a unique specialization $b' = b \cdot f(y)$ over the above specialization, and $b'$ is a generic point of $H$ over $k$. Hence there is a generic point $z$ of $G$ over $k$, such that $f(z) = b'$, and we have $f(c) = b$ if we put $c = zy^{-1}$.

This simple remark is sometimes useful. For example the uniqueness of the group variety constructed from a variety with
normal law of composition ((A) theorem 15) is proved most naturally by our theorem.

We can also dispense with theorem 16 of (A), in the proofs of theorems 17 and 18 of (A) by using a remark similar to ours, as was pointed out in the seminary in Kyoto University.

§ 2. Preliminary propositions.

Lemma 1. Let \( \Sigma \) be a field, \((z)=\left(z_1, \ldots, z_N\right)\) a set of algebraic quantities over \( \Sigma \) and \((w)=\left(w_1, \ldots, w_N\right)\) \(N\) independent variables over \( \Sigma \). Then if \( t={}^\Sigma \sum_{j=1}^{N} w_j z_j \) is separable over \( \Sigma(w) \), \((z)\) is separable over \( \Sigma \).

Proof. Let \( II \) be the field composed of all the elements of \( \Sigma(z) \) which are separable over \( \Sigma \). As \( \Sigma(w) \) and \( \Sigma(z) \) are linearly disjoint over \( \Sigma \), \( II(w) \) is the field of all the elements of \( \Sigma(z,w) \) which are separable over \( \Sigma(w) \) (F-I, prop.13)(2). Then we have

\[
\sum_{j=1}^{N} w_j z_j = t \in II(w)
\]

or

\[
t = \frac{P(w)}{Q(w)}, \text{ where } P(W), Q(W) \in II[W].
\]

As \((w)\) are independent variables, the coefficients of each power product of \( w \) in both sides must be equal. From this we have \( z_j \in II \).

Lemma 2. Let \( W^r \) be a variety in \( S^r \), and let \( M \) be its Chow point(\(0\)). Then if we denote by \( k_0 \) the prime field contained in our universal domain, we have \( \text{def}(W) = k_0(M) \).

Proof. Put \( k = \text{def}(W), k_r = k_0(M) \). It is evident that \( k \supset k_r \). On the other hand it is known(\(0\)) that the necessary and sufficient condition for a point \( (x) \) of \( S^r \) to lie on \( W \) is that \( (x) \) annihilates a certain ideal in \( k[X] \).

If we take as \( (x) \) a generic point of \( W \) over \( k \), we see that \( W \) is algebraic over \( k_r \), and has no conjugate over \( k_r \), other than \( W \) itself. This implies that \( k \) is a purely inseparable extension of \( k_r \).

Now let \( \sum_{j=1}^{N} u_j x_j - v_r = 0 \) for \( i=1, \ldots, r \) be a generic set of linear equations over \( k \), and let \( L^{x,r} \) be the linear variety defined by this set of equations. Then if \( (z_1^{(r)}, \ldots, (z_r) \) are all the points of intersection of \( W \) and \( L \), they are algebraic over \( k(u,v) \) and diffe-
rent with each other. But if \( w_j \) \((j=1,\ldots,N)\) are independent variables over \( k(u, v) \), then, by the definition of Chow point,

\[
\Pi(T-\sum_{j=1}^{n} z_j w_j)
\]

is a polynomial in \( k_i(u, v, w) [T] \), so that \( \sum_{j} z_j w_j \) is separably algebraic over \( k_i(u, v, w) \). Then, by lemma 1, \( (z_j) \) is separably algebraic over \( k_i(u, v) \), and we have \( [k_i(u,z^0) : k_i(u)] = 1 \) and then \( [k_i(u,z^0) : k_i(u)] = 1 \), because \( k_i(u) \) and \( k_i(z^0) \) is linearly disjoint over \( k_i \). Now by F-I, prop. 18, \( z^0 \) has the locus \( W \) over \( k_i \), and therefore \( k = \text{def}(W) \subseteq k_i \).

From this lemma, we have

**Proposition 1.** Let an abstract variety \( V=\left[ V_s \mid S; T_{3s} \right] \) be defined over a field \( k \), and let \( W \) be its subvariety, then, among the fields of definition for \( W \) which contain \( k \), there exists the smallest one, and this is given by \( K = k(M_s) \), where \( M_s \) is the Chow point of an arbitrary representative \( W_s \) of \( W \).

**Proof.** If a field of definition for \( W \) contains \( k \), it must contain \( k \cup \text{def}(W_s) = k(M_s) = K \). Consider any other representative \( W_s \) of \( W \), then \( W_s \) and \( W \) correspond regularly by \( T_{3s} \), and \( W_s \) and \( T_{3s} \) are defined over \( K \). Therefore \( W_s \) is also defined over \( K \).

In the same way, we can verify all the conditions that \( W \) is defined over \( K \).

Our next lemma is a slight modification of F-VI, prop. 7, and is proved in the same way as that proposition.

**Lemma 3.** Let \( U^n, V^m \) be varieties in \( S^n, S^m \) respectively, \( W^a \) a simple subvariety of \( U \times V \), and let \( A^{n-r} \) be a linear variety such that \( V \cdot A \) is defined on \( S^m \) and \( = V_i \), where \( V_i \) is a subvariety of \( V \). If \( P=(x) \) is a simple point of \( U \), and \( L \) a linear variety in \( S^n \) defined by the set of equations \( F_i(x-x_i) = 0 \) \((i=1,\ldots,n)\), where \( F_i(x) \) form a uniformizing set of linear forms of \( U \) at \( P \), then a variety \( Z \) contained in \( (P \times V_i) \cap W \) is a component of \( (P \times V_i) \cap W \) if and only if it is a component of \( (L \times A) \cap W \), and if \( Z \) is a simple subvariety of \( U \times V_i \), it is a proper component of \( (L \times A) \cap W \) on \( U \times V_i \), if and only if it is a proper component of \( (L \times A) \cap W \) on \( S^n \times S^m \), and when that is so, we have

\[
i[W \cdot (P \times V_i), Z; U \times V] = j[W \cdot (L \times A), Z].
\]

**Lemma 4.** Let \( U^n, V^m \) be varieties respectively in \( S^n, S^m \), and
$W^{x\cdot r}$ be a simple subvariety of $U \times V$, with the projection $U$ on $U$. Let $k$ be a common field of definition for $U$, $V$ and $W$, and let $P=(x)$ be a generic point of $U$ over $k$. Further let a simple point $P^\prime=(x^\prime)$ on $U$ be given. If we denote by $A^{x\cdot r}$ the linear variety in $S^m$ defined by a generic set of linear equations $\sum_j u_j x_j - v_j = 0 \ (i=1, \ldots, r)$ over $k(x, x^\prime)$, then we have $V \cdot A = V_1$ where $V_1$ is a variety, and putting $k(u)=K_0, k(u, v)=K$ we have

(1) $(P \times V_1) \cap W$ is not empty, and each component is proper on $U \times V$ and hence reduces to a point. If we denote one of them by $P \times Q$, it is algebraic over $K(P)$, and is a generic point of $W$ over $K_0$. The points of $(P \times V_1) \cap W$ are the conjugates of $P \times Q$ over $K(P)$, and each point has multiplicity $[K(P, Q):K(P)]$, in this intersection.

(2) If $P^\prime \times V_1$ and $W$ has a proper point of intersection $P^\prime \times Q'$ of multiplicity $\mu$, every specialization $(Q'^{(1)}, \ldots, Q'^{(\mu)})$ of the complete set of conjugates $(Q^{(1)}, \ldots, Q^{(\mu)})$ of $Q$ over $K(P)$, extending the specialization $P \rightarrow P^\prime$ with respect to $K$, contains the point $Q'$ exactly $\mu$ times.

Proof. The first assertion about $V_1$ follows from Matsusaka's generalization of Zariski's lemma. The others are mere modifications of F-VI, th. 12, and can be proved in the same way as that theorem, except that we must use our lemma instead of F-VI, prop. 7.

Now let $G^\prime$ be a group variety and $H^\prime$ its group subvariety, both defined over a field $k$. We consider a representative $G_0$ in $S^m$ with a frontier $\gamma$. If a subvariety $W$ of $G$ has a representative in $G_0$, we denote it by $W_0$.

If $x$ is a generic point of $G$ over $k$, the transform $xH$ of $H$ by the left-translation by $x$ has a representative $(xH)_0$ in $G_0$. Let $M$ be its Chow point, then we have $k(M) \subset k(x)$, so that $M$ has a locus $U$ over $k$. $U$ is a variety in a projective space.

Now we prove the following proposition.

Proposition 2. Let $G$, $G_0$, $H$, $k$ and $x$ be as above, and let $x'$ be a point of $G$, such that $x'H$ has representative $(x'H)_0$ in $G_0$. Then if we denote by $M$ and $M'$, respectively, the Chow points of $(xH)_0$ and $(x'H)_0$, $M'$ is a specialization of $M$ over $x \rightarrow x'$ with respect to $k$ and thus a point of $U$.

Proof. For the proof, it is sufficient to show that if $A^{x\cdot r}$ is a linear variety in $S^m$ defined by a generic set of equations
Note on group varieties

\[ \sum_{i} u_i X_i - v_i=0 \quad (i=1, \ldots, r) \]

and if we put

\[ (xH)_0 \cdot A = \sum_{i=1}^{m} Q_i, \quad (x'H)_0 \cdot A = \sum_{i=1}^{m'} Q'_i. \]

then \( m=m' \), and except for a permutation, \((Q'_1, \ldots, Q'_{m'})\) is a specialization of \((Q_1, \ldots, Q_{m})\) over \( x \rightarrow x' \) with respect to \( K=k(u, v) \).

Let \( y \) be a generic point of \( H \) over \( k(x) \), and let \( Q \) be the representative of the point \( z=x \cdot y \) in \( G_o \), then \( x \times z \) and \( x \times Q \) have loci \( W, W_0 \) in \( G \times G, G \times S' \) respectively. It is easily seen that \( (x \times G) \cdot W=x \times (xH), (x' \times G) \cdot W=x' \times (x'H) \), and therefore we have

\[ (x \times G_o) \cdot W_0 = x \times (xH)_0 \quad (\text{mod. } G \times \mathfrak{F}) \]
\[ (x' \times G_o) \cdot W_0 = x' \times (x'H)_0 \quad (\text{mod. } G \times \mathfrak{F}) \]

on \( G \times G_o \)

where \((\text{mod. } G \times \mathfrak{F})\) means that in taking the intersection product we neglect all the components which are contained in \( G \times \mathfrak{F} \).

If we apply lemma 4 to \( G, G_o, W_0, x, x' \) and \( A \) we see that each \( Q_i \) is a specialization of one and only one \( Q_i \), over \( x \rightarrow x' \) with respect to \( K \). This implies \( m \geq m' \).

On the other hand, if \( Q_i \) is a representative of a point \( z_i \) of \( G \) in \( G_o \), and if we put \( z_i=x_iy_i \), each \( y_i \) is a generic point of \( H \) over \( k(x, x', u) \), because \( Q_i \) is one of \( (xH)_0 \) over the same field. \( y_i \) differs from each other, because \( Q_i \) is so. As \( x'H \) has a representative in \( G_o \), \( (v) \) have a uniquely determined specialization \( (v') \) over \( (x, y) \rightarrow (x', y') \) with respect to \( K_o=k(x', u) \), and \( (v') \) are independent variables over \( K_o \).

If we extend the generic specialization \( (x', v) \) of \( (x', v') \) over \( K_o \) to a generic specialization \( (x', v, y') \) of \( (x', v', y'), y_i, \ldots, y_{m'} \) are \( m \) distinct generic points of \( H \) over \( K_o \), and \( (x', v', y') \) is a specialization of \( (x, v, y) \) over \( K_o \). This specialization is uniquely extended to a specialization \( \tilde{Q}_i \rightarrow \tilde{Q}_i \) over \( K \), where \( \tilde{Q}_i \) is a representative of \( x'y_i \) in \( G_o \). The \( \tilde{Q}_i \) lie both on \( (x'H)_0 \) and \( A \), and are not contained in \( \mathfrak{F} \), so that we have \( m \leq m' \).

Combined with what was proved above, this completes our proof.

\section*{§ 3. Construction of homogeneous space.}

Let \( G^o \) be, as before, a group variety and \( H^o \) its group subvariety, both defined over a field \( k \). Let \( x \) be a generic point of \( G \) over \( k \). We consider a representative \( G_o \) of \( G \) in an affine space
$S'$, and denote by $M$ the Chow point of the representative $(xH)_s$ of $xH$ in $G_s$. Here we consider $M$ as a point of an affine space, by taking one of its representatives. As we have $k(M) \subseteq k(x)$, $M$ has a locus $U$ over $k$, and there is a function $\varphi$ on $G$ with values in $U$, defined by the relation $M = \varphi(x)$. We denote the graph of $\varphi$ by $\phi$.

If $z$ is a generic point of $G$ over $k(x)$, and if we put $\varphi(2z) = M'$, $k(M')$ is the smallest field of definition for $2zH$ containing $k$. But $zH = z(xH)$ is defined over $k(M, z)$, so that we have $k(M') \subseteq k(M, z)$. Hence $M'$ is a function of $z$ and $M$, this shall be denoted as $M' = zM$.

Conversely from $xH = z^{-1}(2zH)$ we have

$$k(z, M) = k(z, M') = k(z, zM).$$

From now on, we can follow Weil's method ((A), § V). Our construction will be done by successive lemmas.

**Lemma 5.** There exists a frontier $\mathcal{B}$ on $G \times U$, normally algebraic over $k$, such that if $t \times P \in G \times U - \mathcal{B}$, $tP$, $t^{-1}P$, $t^{-1}(tP)$ are defined. Moreover, $\mathcal{B}$ can be so chosen that $s \times U$ is not contained in $\mathcal{B}$ for any point $s$ of $G$.

The proof is similar to that of (A) lemma 7. In the following we shall fix a frontier $\mathcal{B}$ which satisfies lemma 5.

**Lemma 6.** If we put $(s \times U) \cap \mathcal{B} = s \times \mathcal{B}_s$ for $s \in G$ and $\mathcal{M} = \cap_{s \in G} \mathcal{B}_s$, there exist algebraic points $s_1, \ldots, s_m$ of $G$ over $k$, such that

$$\mathcal{M} = \cap_{s \in \mathcal{B}_s}. $$

Especially $\mathcal{M}$ is a frontier on $G$, normally algebraic over $K_s = k(s_1, \ldots, s_m)$

**Proof.** By lemma 5. $\mathcal{B}_s$ is a frontier on $G$, normally algebraic over $k(s)$. If $P$ is on $U - \mathcal{M}$, there is $s \in G$ such that $s \times P \notin \mathcal{B}$, and the $\mathcal{B}_P$ defined by $(G \times P) \cap \mathcal{B} = \mathcal{B}_P \times P$ is a frontier on $G$, so that there exists a point $s' \in G$, algebraic over $k$, such that $P \in U - \mathcal{B}_{s'}$. This shows that $\mathcal{M} = \cap_{s \in G}$, where $s$ runs over the set of all algebraic points of $G$ over $k$.

Then our assertion is contained in the well-known fact that a compact topology (Zariski topology) in which the closed sets of $U$ are the bunch of varieties algebraic over $k$, can be defined on $U$.

**Lemma 7.** Let $w$ be a point on $G$, and $M$ a generic point of $U$ over $k(w)$. Then the locus $T_w$ on $U \times U$, of the point $M \times wM$
Note on group varieties

over \( k(w) \) is a birational correspondence between \( U \) and \( U \) itself, and if \( P \times Q \) is a point of \( T_w \) such that \( P,Q \in U - \mathfrak{M} \) then \( T_w \) is birational at \( P \).

**Proof.** We repeat Weil's proof. Let \( u \) be a generic point of \( G \) over \( k(w, M, P, Q) \), then \( u^{-1}(uw)M \) is defined and equal to \( wM \). Therefore \( T_w \) is the locus of \( M \times u^{-1}(uw)M \) over \( k(w, w) \). As \( uw \) is a generic point of \( G \) over \( k(wP) \) and \( P \) is not on \( \mathfrak{M} \), \( (uw)P \) is defined. As \( P \times Q \) is a point of \( T_w \), it is a specialization of \( M \times wM \) over \( k(w,u) \), so that from the relation \( u(wM) = (uw)M \), we deduce \( uQ = (uw)P \). As \( u \times Q \notin \mathfrak{M}, u^{-1}(uQ) \) and therefore \( u^{-1}(uw)P \), is defined and \( Q = u^{-1}(uw)P \). This shows that the projection of \( T_w \) to the first factor is regular at \( P \). Also we can prove that the projection of \( T_w \) to the second factor is regular at \( Q \).

**Lemma 8.** If algebraic points \( t_1, \ldots, t_N \) of \( G \) over \( k \) are suitably chosen, for any \( t, z \in G \) and \( P \in U - \mathfrak{M} \), there exist a \( P \) such that \( (t,zt^{-1})P \) is defined and not on \( \mathfrak{M} \).

**Proof.** Consider the function \((uvw^{-1})M \) from \( G \times G \times G \times U \) into \( U \). The points where this function is not defined or its value is contained in \( \mathfrak{M} \), form a frontier \( \mathfrak{F} \), normally algebraic over \( k \). From lemma 6 we see that for any \( z, t \in G \), \( P \in U - \mathfrak{M}, G \times z \times t \times P \) is not contained in \( \mathfrak{F} \), and therefore if we put

\[
(s \times G \times G \times U) \cap \mathfrak{F} = s \times \mathfrak{F}, \quad \text{for} \ s \in G,
\]

we have

\[
\cap s \mathfrak{F} \subset G \times G \times G \times \mathfrak{M}.
\]

From this we can prove our lemma just as in lemma 6.

Now let \( K = K(t_1, \ldots, t_N) \) and let \( M \) be a generic point of \( U \) over \( K \). If we put \( V_a = U, \mathfrak{M}_a = \mathfrak{M}, \) for \( a = 1, \ldots, N \), and denote by \( T_{sa} \) the locus of \( t_a M \times t_a M \) in \( V_a \times V_a \) over \( K \), then by lemma 7 \([V_a ; \mathfrak{M}_a ; T_{sa}] \) defines an abstract variety \( V \), and there is a birational correspondence \( I \) between \( U \) and \( V \), defined over \( K \), by which \( M \) and \( \bar{M} = (t_a M) \) correspond to each other.

Let \( z \) and \( M \) be independent generic points of \( G \) and \( U \) over \( K \). We put \( \bar{M} = IM \) and \( I(zM) = z\bar{M} \), \( z\bar{M} \) is a function on \( G \times V \) into \( V \) defined over \( K \).

By lemma 8 \( z\bar{M} \) is everywhere defined on \( G \times V \). \( z \) and \( \bar{M} \) being independent generic points of \( G \) and \( V \) over \( K \), we denote by \( A \) the locus of \( z \times M \times z\bar{M} \) in \( G \times V \times V \) over \( K \), and we see from
the above result that the projection of $\bar{A}$ on the first and the second factors are everywhere regular. Therefore for any $s \in G$, $\bar{A} \cdot (s \times V \times V)$ is defined and is equal to $s \times \bar{T}$, where $\bar{T}$ is the everywhere biregular birational correspondence between $V$ and $V$ itself, defined over $K(s)$, by which a generic point $\bar{M}$ of $V$ over $K(s)$ corresponds to $s\bar{M}$.

We also see that $\bar{T} \cdot \bar{T}$ for any $s$, $t \in G$ (and that in the sense $\text{pr}_{13} \cdot [(\bar{T} \times V) \cdot (V \times \bar{T})] = \bar{T}$). This shows that $G$ is represented as a group of birational transformations on $V$.

Moreover, $G$ is a transitive group of transformations on $V$, because, if $N$ is a point of $V$, $L$ a field containing $K$, and if $y$ is a generic point of $G$ over $L(N)$, $yN$ is a generic point of $V$ over $L(N)$, and thus there is a transformation $f$ of $G$ which transforms any point of $V$ to a generic point of $G$ over any given field. As a consequence of transitivity, $V$ has no multiple points.

Corresponding to the function $\varphi$ on $G$ into $U$, we define $\bar{\varphi}$ by $\bar{\varphi}(x) = I \cdot \varphi(x)$, where $x$ is a generic point of $G$ over $K$. The graph of $\bar{\varphi}$ is denoted by $\bar{\Phi}$.

When $s \in G$ and $x$ is a generic point of $G$ over $K(s)$, we had

$$\varphi(sx) = s\varphi(x),$$

this, transformed by $I$, gives

$$\bar{\varphi}(sx) = s\bar{\varphi}(x)$$

Let $\bar{M}$ be a generic point of $V$ over $K(s,x)$, we consider the locus $R$, of $x \times \bar{M} \times sx < s\bar{M}$ in $G \times V \times G \times V$ over $K(s)$. $R$ is an everywhere biregular birational correspondence between $G \times V$ and itself, and the above formula shows that $R$ leaves $\bar{\Phi}$ invariant.

Now let $x'$ be a point of $G$, and $y$ a generic point of $G$ over $K(x')$, then $yx'$ is a generic point of $G$ over $K(x')$. Therefore $\bar{\varphi}(yx')$ is defined and we have

$$\bar{\varphi} \cdot (yx' \times V) = yx' \times \bar{\varphi}(yx').$$

This, transformed by $R_{\varphi}^{-1}$, gives

$$\bar{\varphi} \cdot (x' \times V) = x' \times y^{-1} \varphi(yx').$$

Therefore $\bar{\varphi}(x')$ is defined and
The last formula holds when we replace \( y \) by an arbitrary point \( s \) of \( G \), because both sides are defined at \( y=s \).

We have proved that \( \overline{\varphi} \) is everywhere defined on \( G \) and for any \( s, x' \in G \), \( s\overline{\varphi}(x')=\overline{\varphi}(sx') \).

Again consider the subvariety \( \Phi \) of \( G \times U \). If \( x \) is a generic point of \( G \) over \( K \), and \( M=\varphi(x) \), then \( \Phi \cap (G \times M) \) contains \((xH) \times M \), and \( M \) is the Chow point of \((xH)_0 \) in the sense of the beginning of this section. If \( y \times M \) is a generic point of any component of \( \Phi \cap (G \times M) \) with respect to \( K(M) \), \( y \) is a generic point of \( G \) over \( K \) and \( \varphi(y)=M \), i.e. \( M \) is also the Chow point of \((yH)_0 \). This implies \( yH=xH \), and \( xH \times M \) is the only component of \( \Phi \cap (G \times M) \).

Moreover \((xH) \times M \) is defined over \( K(M) \). Therefore, by F-VII, th. 12(i), we have

\[
\Phi \cdot (G \times M) = (xH) \times M,
\]

where \( x \) is such that \( \varphi(x)=M \).

When we go over to \( G \times V \), we obtain

\[
\Phi(G \times \bar{M}) = (xH) \times \bar{M} \quad \text{where } \bar{\varphi}(x)=\bar{M}.
\]

This shows that \( V \) has the dimension \( n-r \), and \( K(x) \) is regular over \( K(\bar{M})=K(\bar{\varphi}(x)) \).

Let \( \bar{N} \) be a point of \( V \), \( \bar{M} \) a generic point of \( V \) over \( K(\bar{N}) \) and \( y \) a point of \( G \) such that \( y\bar{M}=\bar{N} \), then as above

\[
\Phi \cdot (G \times \bar{M}) = (xH) \times \bar{M}
\]

where \( x \) is such that \( \bar{\varphi}(x)=\bar{M} \).

Transforming by \( R_x \), we see

\[
\Phi \cdot (G \times N) = y(xH) \times \bar{N},
\]

whence, for arbitrary \( \bar{N} \in V \), there exist points \( s \) such that \( \bar{\varphi}(s)=\bar{N} \), and for any one of them we have \( \Phi \cdot (G \times N) \) is defined and equal to \( sH \times \bar{N} \).

We resume the results obtained in the following theorem.

**Theorem 2.** Let \( G^\ast \) be a group variety, \( H^\ast \) its group subvariety,
both defined over a field $k$. Then there exist an algebraic extension $K$ of $k$, an abstract variety $V^n_r$ without multiple point, having $K$ as a field of definition, and a function $\varphi$ on $G$ onto $V$ defined over $K$, with the following properties:

(i) For any point $s$ of $G$, there is an everywhere biregular birational correspondence $T$, between $V$ and $V$ itself, defined over $K(s)$, and the correspondence $s \mapsto T_s$ is a representation of $G$. Moreover $G$ is transitive as a group of transformations of $V$. The image of a point $M$ by $T_s$ is denoted by $sM$.

(ii) $\varphi$ is everywhere defined, and for every $s, x \in G$

\[ \varphi(sx) = s\varphi(x). \]

(iii) For any point $N$ of $V$, points $x$ of $G$ such that $\varphi(x) = N$ exist, $\Phi = (G \times N)$ is defined and equal to $xH \times N$ for any such $x$.

§ 4. Next we consider the uniqueness of the above process.

Lemma 9. Let $k$ be a field, $(x)$ a set of quantities such that $k(x)$ is separably generated over $k$. Then $(x)$ has a finite separably algebraic specialization over $k$.

This differs but little from F-II, prop. 12, and is proved in the same way as that proposition.

Theorem 3. Let $G^n, H^r$ and $k$ be as in theorem 2, and let $V_1$ and $V_2$ be two varieties defined over $K_1$ and $K_2$ respectively, each having the properties of theorem 2 with respect to $K_1$. (Here $K_1$ are assumed to contain $k$, and not assumed to be algebraic extensions of $k$.) Then there exists an everywhere biregular birational correspondence $T$ between $V$ and $V_2$ defined over $K=K_1 \cup K_2$, such that $T\varphi_1(x) = \varphi_2(x)$ for all $x \in G$, where $\varphi_i$ means the function $\varphi$ in theorem 2 with reference to $V_i$.

Proof. Let $M_1$ be a generic point of $V_1$ over $K$. This corresponds to a coset $xH$ of $H$, and to this corresponds a generic point $M_2$ of $V_2$ over $K$. As $xH$ is defined over $K(M_1)$, $xH$ contains, by lemma 9, a point $s$, separably algebraic over $K(M_1)$, and $M_2 = \varphi_2(s)$ is separably algebraic over $K(M_1)$. Consider the conjugate $M_2^\sigma$ of $M_2$ over $K(M_1)$. This corresponds to the conjugate $(xH)^\sigma$ of $xH$ over $K(M_1)$. But as $xH$ is defined over $K(M_1)$, $(xH)^\sigma = xH$ and $M_2^\sigma = M_2$. As $M_2$ is separably algebraic over $K(M_1)$ and has no conjugate over $K(M_1)$ other than itself, it is rational over $K(M_1)$. Thus we have $K(M_2) = K(M_1)$. The converse being true, we have $K(M_1) = K(M_2)$, and there is a birational correspondence
Note on group varieties

65

T between $V_1$ and $V_2$, defined over $K$. That $T$ is everywhere bieguar follows from the fact that the points of $V_1$ and $V_2$ are in one-to-one correspondence, and that $V_1$ and $V_2$ have no multiple points. The last assertion is obvious.

§ 5. Factor group

Theorem 4. Let $G$ be a group variety, $H$ its normal group subvariety, both defined over $k$. Then the variety $V$ which was constructed in theorem 2 is a group variety, and the function $\phi$ of theorem 2 is a homomorphism from $G$ onto $V$, with the kernel $H$.

Proof. As $H$ is a normal subgroup, $T_\phi$ of theorem 2 depends only upon $\phi(x) = M$, so we write it as $T_\phi$. Let $M$ and $N$ be independent generic points of $V$ over $K$, and let $x$ and $y$ be such that $\phi(x) = M$, $\phi(y) = N$. Then $T_\phi \cdot N = xN$, and $K(T_\phi \cdot N)$ is the smallest field of definition for $xyH$ containing $K$. But $xH$ is defined over $K(M)$, and thus there is a point $s$ on $xH$, separably algebraic over $K(M)$. Then we have $xyH = s(yH)$, and as $yH$ is defined over $K(N)$, $xyH$ is defined over a seperably algebraic extension $K(s, M, N)$ of $K(M, N)$. Hence $T_\phi \cdot N$ is separably algebraic over $K(M, N)$. On the other hand, a conjugate of $sN$ over $K(M, N)$ is of the form $s^a N$, where $s^a$ is a conjugate of $s$ over $K(M)$. But as $sH$ is defined over $K(M)$, we have $s^a H = sH$, and $s^a N = sN$. This proves that $T_\phi \cdot N$ is rational over $K(M, N)$.

We consider $T_\phi \cdot N$ as a function defined on $V \times V$ with values in $V$ and write $T_\phi \cdot N = MN$. Then $MN$ is everywhere defined, because any specialization $(M, N) \to (M', N')$ over $K$ is uniquely extended to a specialization of $T_\phi \cdot N$, and $V$ has no multiple points.

From the relation $\phi(xy) = x\phi(y)$, we get

$$\phi(xy) = \phi(x) \phi(y)$$

This show that $\phi$ preserve the multiplication, and $V$, as a homomorphic image of a group, is itself a group.

Thus our theorem is proved.

§ 6. In (A) theorem 17, the construction of a factor group by finite subgroup of an Abelian variety is shown.

His method can be applied to the case of a group variety $G$ and its subgroup with finite number of components $\mathfrak{G} = \{a_H, \cdots, a_H\}$. Let $k$ be a field of definition for $G$, $H$ and $V = G/H$ over which all $a_H$ are rational, and let $x$ be a generic point of $V$ over $k$. Then the right-translations by the elements of $\mathfrak{G}$ define a group of automorphisms
of \( k(x) \) over \( k \). Let \( (y) \) be a set of quantities in \( k(x) \) such that \( k(y) \) is the field of all the elements which are invariant by this group. Then \( (y) \) has a locus \( U \) over \( k \), and we have a mapping \( \varphi \) from \( G \) to \( U \), defined by \( y = \varphi(x) \). Every element \( s \in G \) induces a birational correspondence \( T \), between \( U \) and \( U \) itself. In fact if \( x \) is a generic point of \( G \) over \( k(s) \), the left-translation by \( s \), considered as an automorphism of \( k(s, x) \) over \( k(s) \), leaves the field \( k(s, \varphi(x)) \) invariant, because left-translations and right-translations are commutative. From this, we can proceed as in §3, and thus our results in §§3-5 are extended to the case where \( G \) contains a subgroup \( \mathfrak{G} \), composed of a finite number of components.

We shall here refer that when a variety with normal composition is defined over \( k \) (together with its law of composition), we can construct a group variety birationally equivalent to it and defined over an algebraic extension of \( k \), by the method used in §3. This is stated in (A) n° 33.

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3) If \( W \) is a variety in a projective space, the coefficients of the associate form of \( W \) can be considered as homogeneous coordinates of a point in a projective space. This point is called the Chow point of \( W \). In our case we consider an affine variety \( W \) as a representative of a projective variety, and call the Chow point of the latter, the one of the former.
4) Cf. Van der Wearden: Einführung in die algebraische Geometrie, §38.