MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES, A
Vol. XXVII, Mathematics No. 2, 1952.

Some Applications of Bochner's Method to Riemannian Manifolds.

By

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The well-known treatise of S. Bochner [1, 2] is based upon Green's theorem and Ricci's identity. Let V_n be a compact, orientable Riemannian manifold, whose metric is given by the positive definite quadratic form:

$$ds^2 = g_{ij} dx^i dx^j.$$

Hereafter, unless otherwise stated, we shall denote by V_n a Riemannian manifold as above mentioned.

If we put, for r-tensors $\varphi_{i_1 \dots i_r}$ and $\psi_{i_1 \dots i_r}$,

$$(\varphi \cdot \psi) = \varphi_{i_1 \dots i_r} \psi^{i_1 \dots i_r},$$

$$(\varphi' \cdot \psi') = \varphi_{i_1 \dots i_r ; j} \psi^{i_1 \dots i_r ; j},$$

and denote by $\varDelta \varphi_{i_1 \dots i_r}$ the Laplacian of $\varphi_{i_1 \dots i_r}$; i. e.

$$\varDelta \varphi_{i_1 \dots i_r} = \varphi_{i_1 \dots i_r ; j ; k} \, g^{jk},$$

we have clearly

$$\frac{1}{2} \cdot \varDelta (\varphi \cdot \varphi) = (\varDelta \varphi \cdot \varphi) + (\varphi' \cdot \varphi').$$

And Green's theorem gives that

$$\int_{V_n} \varDelta (\varphi \cdot \varphi) dv = 0;$$

where dv is n-dimensional volume element. The other hand, we define operator D and its dual D^* as follows:

$$D \xi_{i_1 \dots i_{p+1}} = \delta^{a_1 \dots a_{p+1}}_{i_1 \dots i_{p+1}} \xi_{a_1 \dots a_p ; a_{p+1}},$$

$$D^* \xi_{i_1 \dots i_{p-1}} = \xi_{i_1 \dots i_{p-1} j ; k} \, g^{jk}.$$

In above definitions $\xi_{i_1 \dots i_p}$ is a skew-symmetric p-tensor and $\delta^{b_1 \dots b_r}_{a_1 \dots a_r}$

is generalized Kronecker's delta. A tensor $\xi_{i_1\ldots i_p}$ is called to be harmonic p-tensor, if $D\xi_{i_1\ldots i_{p+1}}=0$ and $D^*\xi_{i_1\ldots i_{p-1}}=0$ [3]. In this case we see readily

$$p!\cdot\Delta\xi_{i_1\ldots i_p}=-\frac{p}{2}\,\delta^{jka_3\ldots a_p}_{i_1i_2i_3\ldots i_p}\xi^{lm}{}_{a_3\ldots a_p}\underset{(p)}{H}_{jk\,lm}\quad(p\geqq 2),\tag{0.1}$$

$$\Delta\xi_i=R_{ij}\xi^j\quad(p=1)\;;\tag{0.2}$$

where by definition

$$\underset{(p)}{H}_{ijkl}=(p-1)R_{ijkl}-\frac{1}{2}\,(g_{ik}R_{jl}-g_{il}R_{jk}+R_{ik}g_{jl}-R_{il}g_{jk}),\tag{0.3}$$

and R_{ijkl} is the curvature tensor of V_n, i. e.

$$R^h_{i\cdot kl}=\frac{\partial\Gamma^h_{ik}}{\partial x^l}-\frac{\partial\Gamma^h_{il}}{\partial x^k}+\Gamma^a_{ik}\Gamma^h_{al}-\Gamma^a_{il}\Gamma^h_{ak},$$

$$R_{ijkl}=g_{jh}R^h_{i\cdot kl},\quad R_{ij}=R^h_{i\cdot jh}=g^{ab}R_{iajb}.$$

In order to obtain (0.1) and (0.2) we depend upon the following process, which is first effectively made use of S. Bochner. For an arbitrary tensor $\eta_{i_1\ldots i_r}$ we have the Ricci's identity:

$$\eta_{i_1\ldots i_r;j;k}-\eta_{i_1\ldots i_r;k;j}=-\sum_{s=1}^{r}\eta_{i_1\ldots\underset{(s)}{l}\ldots i_r}R_{i_s\cdot jk}^{\;\;l}$$

Hence, if $\eta_{i_1\ldots i_r}$ is harmonic, we have $D^*\eta_{i_1\ldots i_{r-1}}=0$, so that

$$\eta_{ia_2\ldots a_r;j;k}g^{ik}=\eta_{la_2\ldots a_r}R^l_j-\sum_{s=2}^{r}\eta_{ia_2\ldots\underset{(s)}{l}\ldots a_r}R_{a_s\cdot jk}^{\;\;l}g^{ik}$$

It follows from (0.1) or (0.2)

$$(\Delta\xi\cdot\xi)=-\frac{p}{2}\,\xi^{jka_3\ldots a_p}\xi^{lm}{}_{a_3\ldots a_p}\underset{(p)}{H}_{jklm}\quad(p\geqq 2),$$

$$(\Delta\xi\cdot\xi)=R_{ij}\xi^i\xi^j\quad(p=1).$$

This equations and Green's theorem is fundamental for many beautiful theorems in the paper of I. Mogi [4], which is written afresh systematically from papers of S. Bochner [1, 2]. Though this method is seemed to be most impressive adapting to the harmonic tensor, we shall obtain some interesting results for a certain type of tensor. The first section of this paper is a small attempt applying this method to the imbedding problem of Riemannian manifold. In remaining sections we give some additional results to the papers of I. Mogi [4] and Y. Tomonaga [5].

1. Application to the imbedding problem

A Riemannian manifold V_n can be imbedded in an Euclidean space of dimensionality $(n+1)$, if and only if, there exists the symmetric tensor H_{ij}, which is called the second fundamental tensor of V_n, such that the Gauss's and Codazzi's equations are satisfied, say

$$R_{ijkl}=H_{ik}H_{jl}-H_{il}H_{jk}, \tag{1.1}$$

$$H_{ij\,;\,k}-H_{ik\,;\,j}=0. \tag{1.2}$$

Paying our attention to the Codazzi's equation (1.2), let us generalize that type of tensor. We call a symmetric p-tensor $\xi_{i_1\ldots i_p}\,(\geqq 2)$ to be of Codazzi type, if the differential equation

$$\xi_{ia_2\ldots a_p\,;\,j}-\xi_{ja_2\ldots a_p\,;\,i}=0 \tag{1.3}$$

is satisfied. Making use of the method of S. Bochner, we calculate the Laplacian of above $\xi_{i_1\ldots i_p}$ as follows:

$$\begin{aligned}\Delta\xi_{i_1\ldots i_p}&=\xi_{a\,i_2\ldots i_p\,;\,i_1\,;\,b}\,g^{ab}\\&=g^{ab}\Big(\xi_{ai_2\ldots i_p\,;\,b\,;\,i_1}-\xi_{ci_2\ldots i_p}R_{a\cdot i_1 b}^{\ c}-\sum_{r=2}^{p}\xi_{ai_2\ldots\underset{(r)}{c}\ldots i_p}R_{i_r\ \cdot i_1 b}^{\ \ c}\Big)\\&=g^{ab}\xi_{abi_3\ldots i_p\,;\,i_2\,;\,i_1}+\xi_{ci_2\ldots i_p}R^{c}{}_{i_1}-g^{ab}\sum_{r=2}^{p}\xi_{ai_2\ldots\underset{(r)}{c}\ldots i_p}R_{i_r\ \cdot i_1 b}^{\ \ c}.\end{aligned}$$

Hence, if a restriction

$$(g^{ab}\xi_{abi_3\ldots i_p})_{;\,c\,;\,d}=0 \tag{1.4}$$

is subjoined, we have finally

$$(\Delta\xi\cdot\xi)=\xi^{abc_3\ldots c_p}\xi^{ij}{}_{c_3\ldots c_p}\underset{(p)}{M}_{aibj}\quad (p\geqq 2)\,; \tag{1.5}$$

where we put

$$\underset{(p)}{M}_{aibj}=-(p-1)R_{aibj}+\frac{1}{2}(g_{ai}R_{bj}+g_{bj}R_{ai}). \tag{1.6}$$

Define the positive-definiteness of M_{aibj} following I. Mogi [4]. If, for any symmetric tensor η^{ij},

$$\underset{(p)}{M}_{aibj}\eta^{ab}\eta^{ij} \underset{*}{\geq} 0,$$

then $\underset{(p)}{M}_{aibj}$ is called to be positive-definite. In the above equation "$\underset{*}{\geq}$" means that the equality does not be satisfied at least one point of V_n. The equation (1.5) gives us the

Theorem 1. *If $\underset{(p)}{M}_{aibj}(p \geq 2)$ of V_n is positive-definite, there exists no $p(\geq 2)$-tensor of Codazzi type satisfying* (1.4).

Now we apply Theorem I to the second fundamental tensor H_{ij}. If V_n can be imbedded in an Euclidean space of dimensionality $(n+1)$, such that $(g^{ab}H_{ab})_{;i;j}=0$ and furthermore $\underset{(2)}{M}_{aibj}$ is positive-definite, then H_{ij} must vanish, and hence from (1.1) V_n must be Euclidean. This fact leads us to the

Theorem 2. *If V_n is not Euclidean and $\underset{(2)}{M}_{aibj}$ is positive-definite, then it is impossible that V_n is imbedded throughout in an Eucliden S_{n+1}, such that $(g^{ab}H_{ab})_{;i;j}=0$.*

In particular, if V_n is an minimal variety of S_{n+1}, the mean curvature $g^{ab}H_{ab}$ is equal to zero. Also, if V_n is an umbilical variety of S_{n+1}, we have $H_{ij}=\lambda g_{ij}$; where λ is constant in virtue of (1.2). Hence we have $H_{ij;k}=0$. Thus we have, as a consequence of Theorem 2, the

Corollary. *If V_n is not Euclidean and $\underset{(2)}{M}_{aibj}$ is positive-definite, then it is impossible that V_n is imbedded throughout in an Euclidean S_{n+1}, such that V_n is minimal or umbilical variety.*

2. On harmonic vectors

Y. Tomonaga [5] gave a sufficient condition that the covariant derivatives of any harmonic tensor vanish, provided that V_n *is symmetric, say,* $R_{hijk;l}=0$. That is, *if V_n is symmetric and $\underset{(p)}{T}_{abcijk}$ is positive-definite, then any harmonic tensor is covariant constant.* In this statement $\underset{(p)}{T}_{abcijk}$ is given by

$$\underset{(p)}{T}_{abcijk}=-\frac{p(p-1)}{2}R_{abij}g_{ck}+\frac{p}{2}(R_{ai}g_{bj}+R_{bj}g_{ai})g_{ck}$$
$$-p(g_{ai}R_{bjck}+g_{bj}R_{aick})+g_{ai}g_{bj}R_{ck},\ (p\geq 1),$$

(this form is slightly modified by the author) and the positive-definiteness of this tensor is defined as follows:

$$\underset{(p)}{T}_{abcijk}\eta^{abc}\eta^{ijk} \underset{*}{\geqq} 0\,;$$

where η^{ijk} is any tensor, which is skew-symmetric with respect to i and j. If V_n is compact, $\xi_{i_1\dots i_p;j}=0$ is equivalent to $\Delta\xi_{i_1\dots i_p;j}=0$ [4], so that the above theorem of Y. Tomonaga say the vanishing of Laplacian of any harmonic tensor. Hence if the assumptions of the above theorem are satisfied and if the one-dimensional Betti number does not vanish, then from (0.2) we have $R_{ij}\xi^j=0$. Thus we have the

Theorem 3. *If V_n is symmetric and $\underset{(p)}{T}_{abcijk}(p \geqq 1)$ is positive-definite and furthermore the one-dimensional Betti number of V_n does not vanish, then the determinant $|R_{ij}|$ is throughout equal to zero.*

Next, in the theorems of I. Mogi [4] for the conformally flat V_n, we must suppose that the dimension n of V_n is more than three, because in case of dimensionality two or three the conformal curvature tensor is identically equal to zero. Hence, in these cases, those theorems are satisfied without such a supposition. Thus we may expect more remarkable results for these cases.

Let V_3 be conformally flat Riemannian manifold of three dimensions and ξ_i be covariant constant (this is, of course, harmonic). Then from (0.2) we obtain $R_{ij}\xi^j=0$ and hence

$$(R_{ij;k}-R_{ik;j})\xi^i=0. \tag{2.1}$$

Conformal flatness of V_3 means the vanishing of the tensor C_{ijk} defined by

$$C_{ijk}=R_{ij;k}-R_{ik;j}-\frac{1}{4}(g_{ij}R_{;k}-g_{ik}R_{;j}).$$

Hence (2.1) is written as follows:

$$\xi_jR_{;k}-\xi_kR_{;j}=0,$$

from which we have easily

$$\xi_i=\lambda^{-1}R_{;i}\,. \tag{2.2}$$

Since ξ_i is covariant constant, we obtain from (2.2) by differentiating covariantly

$$R_{;i;j}=\frac{\partial \log \lambda}{\partial x^j}R_{;i}\,. \tag{2.3}$$

From this we see that λ in (2.2) is uniquely determined to within constant coefficient. The symmetry of $R_{;i;j}$ imposes

$$R_{;i;j}=\mu R_{;i}R_{;j}. \tag{2.4}$$

Putting together we obtain the

Theorem 4. *If V_3 is conformally flat space of three-dimensions and if there exists a covariant constant vector, then the scalar curvature R of V_3 satisfies* (2.4) *and any covariant constant vector is given by* (2.2) *and* (2.3), *which is uniquely determined to within constant coefficient.*

Thus, if we denote by s the number of harmonic vectors, whose covariant derivatives do not vanish, then the one-dimensional Betti number is equal to $s+1$.

3. Special harmonic tensor in Ruse's space

Consider the Ruse's space of recurrent curvature, say

$$R_{hijk;l}=R_{hijk}\lambda_l, \quad (\lambda_l\neq 0). \tag{3.1}$$

By the similar process as the deduction of Y. Tomonaga in proving the theorem stated at the begining of the last section, we obtain the form:

$$\begin{aligned}\Phi&=g^{bc}\xi_{a_1\dots a_p;d;b;c}\xi^{a_1\dots a_p;d}\\&=\xi^{abd_3\dots d_p;c}\xi^{ij}{}_{d_3\dots d_p;}{}^{k}\underset{(p)}{T}_{abcijk}+\xi^{abc_3\dots c_p}\xi^{ij}{}_{c_3\dots c_p;}{}^{k}\cdot\\&\left\{\frac{p}{2}(g_{ai}R_{bjkl}+g_{bj}R_{aikl})\lambda^l+\frac{p}{2}(R_{ai}g_{bj}+R_{bj}g_{ai})\lambda_k\right.\\&\left.-\frac{p(p-1)}{2}R_{abij}\lambda_k\right\}.\end{aligned} \tag{3.2}$$

Now suppose that *a harmonic tensor $\xi_{i_1\dots i_p}$ satisfies the following differential equation*:

$$\xi_{i_1\dots i_p;j}=\xi_{i_1\dots i_p}\lambda_j; \tag{3.3}$$

where λ_j is the same one as in (3.1). Substituting from (3.3), Φ is written in the form:

$$\begin{aligned}\Phi=\xi^{abc_3\dots c_p}\xi^{ij}{}_{c_3\dots c_p}&\left\{\frac{1}{2}R_{uv}\lambda^u\lambda^v(g_{ai}g_{bj}-g_{aj}g_{bi})\right.\\&\left.-(\lambda\cdot\lambda)\underset{(p)}{H}_{abij}\right\}.\end{aligned} \tag{3.4}$$

It is to be noted that $\underset{(p)}{H}_{abij}$ of (0.3) entrances into (3.4). On the other hand, from (3.1) and Bianchi's identity we have

$$R_{hijk}\lambda_l + R_{hjkl}\lambda_j + R_{hilj}\lambda_k = 0.$$

Contracting this equation by $g^{hj}g^{ik}\lambda^l$ gives

$$R_{uv}\lambda^u\lambda^v = \frac{1}{2}R\cdot(\lambda\cdot\lambda).$$

Therefore Φ takes the form

$$\Phi = (\lambda\cdot\lambda)\xi^{abc_3\dots c_p}\xi^{ij}{}_{c_3\dots c_p}\underset{(p)}{L}_{abij} \quad (p \geqq 2)\ ; \tag{3.5}$$

setting

$$\underset{(p)}{L}_{abij} = -\underset{(p)}{H}_{abij} + \frac{R}{4}(g_{ai}g_{bj} - g_{aj}g_{bi}). \tag{3.6}$$

Especially the case $p=1$ is more simple ; i. e.

$$\Phi = (\lambda\cdot\lambda)L_{ij}\xi^i\xi^j \tag{3.7}$$

$$(p=1)$$

$$L_{ij} = 2R_{ij} + \frac{R}{2}g_{ij}. \tag{8.8}$$

Since $(\lambda\cdot\lambda)$ is positive, (3.5) or (3.7) gives the

Theorem 5. *If V_n is Ruse's space of recurrent curvature and $\underset{(p)}{L}_{abij}(p \geqq 2)$ or $L_{ij}(p=1)$ is positive-definite, then there exists no harmonic p-tensor satisfying* (3·3)

The positive-definiteness of $\underset{(p)}{L}_{abij}$ and L_{ij} is given by

$$\underset{(p)}{L}_{abij}\eta^{ab}\eta^{ij} \underset{*}{\geqq} 0,$$

$$\underset{(p)}{L}_{ij}\eta^i\eta^j \underset{*}{\geqq} 0\ ;$$

where η^{ij} is any skew-symmetric tensor and η^i is any vector. From (3.2) ϕ is also written as

$$\Phi = \Delta\xi_{i_1\dots i_p;j}\,\xi^{i_1\dots i_p;j}.$$

Hence we have the

Theorem 6. *If V_n is Ruse's space of recurrent curvature and $\underset{(p)}{L}_{abij}(p \geqq 2)$ or $L_{ij}(p=1)$ is definite (positive or negative), then there exists no harmonic p-tensor satisfying* (3·3), *such that its Laplacian is covariant constant.*

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