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Maximum Principle for Analytic Functions on Open Riemann Surfaces.

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1. Let $\mathfrak{F}$ be a non-compact region on an open Riemann surface F, such that its relative boundary Γ_0 consists of a finite number of closed analytic curves on F. Now let $w(P)$ be a single-valued analytic function on $\mathfrak{F}$, satisfying a condition

$$(1) \qquad \overline{\lim_{\Gamma_0}}\, |w(P)| \leqq 1.$$

We consider an arbitrary compact ring domain $G \subset \mathfrak{F}$, whose boundary consists of Γ_0 and Γ, where Γ is composed of a finite number of closed analytic curves and separates Γ_0 from the ideal boundary $\mathfrak{J}$ of $\mathfrak{F}$. If we put

$$\operatorname*{Max}_{P \in \Gamma} |w(P)| \equiv M(\Gamma),$$

then we have

$$(2) \qquad \log|w(P)| \leqq \omega(P,\Gamma,G)\log M(\Gamma), \text{ for } P \in G,$$

where $\omega_G(P) \equiv \omega(P, \Gamma, G)$ denotes the harmonic measure of Γ with respect to G. Namely, since $\omega(P, \Gamma, G) \log M(\Gamma) - \log |w(P)|$ is single-valued, harmonic in $P \in G-S$ (where $S=E\{P;\ w(P)=0,\ P \in G+\Gamma_0+\Gamma\}$) and $\geqq 0$ for P on Γ_0, Γ and arbitrarily large in the neighborhood of S, hence we easily obtain (2) by use of the maximum principle for harmonic function in compact region.

2. We fix an arbitrary point $P_0 \in G$ and consider the level curve Γ^G: $\omega_G(P) = \omega_G(P_0)$. Then Γ^G consists of a finite number of closed analytic curves (occasionally with multiple points) on G and separates Γ_0 from Γ. Clearly it contains a curve passing through P_0. In following we shall denote the ring domain (on $\mathfrak{F}$) by $R(\Gamma, \Gamma')$ which is surrounded by two disjoint arbitrary boundaries Γ and Γ'. Let $R(\Gamma_0, \Gamma^G) \equiv G^*$, where Γ^G is homologous to Γ_0, then

$$\omega_G^* \equiv \omega(P, \Gamma, G)/\omega_G(P_0)$$

is clearly the harmonic measure Γ^G with respect to G^* and its

Dirichlet integral taken over G^* has the value

$$(3)\quad D_{G^*}[\omega_G^*]=D_{G^*}[\omega_G]/\omega_G^2(P_0)=\frac{1}{\omega_G^2(P_0)}\int_{\Gamma_G}\omega_G d\tilde{\omega}_G=\frac{1}{\omega_G(P_0)}\int_{\Gamma_0}d\tilde{\omega}_G$$

$$=D_G[\omega_G]/\omega_G(P_0),$$

where $\tilde{\omega}_G$ denotes the conjugate harmonic function of ω_G. Let r_G and μ_G denote the harmonic moduli※ of G^* and G respectively, then we have

$$(4)\qquad \log\,\mu_G=2\pi/D_G[\omega_G],\ \log\,r_G=2\pi/D_{G^*}[\omega_G^*].$$

From (2), (3) and (4) we get

$$\omega_G(P_0)=\log\,r_G/\log\,\mu_G$$

$$(5)\qquad \log|w(P_0)|\leqq\log\,r_G\frac{\log\,M(\Gamma)}{\log\,\mu_G}$$

We shall next prove that $\sup\log\,r_G<\infty$ (for $\Gamma\to\mathfrak{J}$).

3. Let $\hat{\gamma}^G$ be an analytic curve which connects the point P_0 to Γ and lies in a domain (neighboured at P_0) of $G-G^*$. e.g. $\hat{\gamma}^G$ is a level curve ($\tilde{\omega}_G=$constant) passing through P_0. Now we take a z-circle $V_{P_0}^{r_0}(\subset G)$ with center P_0 i.e. the image mapped on its local parameter circle $|z|<1$ is the disc K_{r_0}: $|z|<r_0<1$. Let γ^G denote an analytic arc which issues from P_0 and is contained in $\hat{\gamma}^G\cap V_{P_0}^{r_0}$. Let

$$\hat{G}\equiv R(\Gamma_0,\ \gamma^G+\Gamma)$$

If we consider the harmonic measure $\omega_{\hat{G}}\equiv\omega(P,\gamma^G+\Gamma,\hat{G})$, then we have for $P\in G^*$

$$\omega_{G^*}(P)\geqq\omega_{\hat{G}}(P)$$

and easily obtain

$$r_G\leqq r_G'=2\pi/\int_{\Gamma_0}\frac{\partial\omega_{\hat{G}}}{\partial n}ds,$$

※ When the function u is harmonic in a ring domain $R=R(\Gamma,\Gamma')$ with analytic boundaries Γ, Γ' and has the boundary value zero on Γ and $\log\,\mu_R$ on Γ', where constant μ_R is so chosen that $\int_\Gamma\frac{\partial u}{\partial n}ds=2\pi$, then μ_R is called the harmonic modulus of R. (see, L. Sario [5]). Then we note that $\log\mu_R=2\pi\lambda_R(\Gamma,\Gamma')$, where $\lambda_R(\Gamma,\Gamma')$ denotes an extremal distance between Γ and Γ' with respect to R(cf. V. Wolontis [6]).

where r_G' denotes the harmonic modulus of $\hat{G}$. Therefore, it is sufficient to prove that $\sup_{G\to\mathfrak{F}} \log r_G' < \infty$. Suppose now $\sup_{G\to\mathfrak{F}} \log r_G' = \infty$, then there exists a sequence of domains $\{G_n\}$ $n=1, 2,\cdots (G_n \to \mathfrak{F},\ G_n \supset G_0 \equiv G \supset V^1_{P_0})$, such that $\lim_{n\to\infty} \log r'_{G_n} = \infty$. Here, we shall use the following Lemma.

Lemma. If $U_1, U_2,\cdots$ is an infinite sequence of function all harmonic in a domain D on open Riemann surface and uniformly bounded in D, then foy any compact closed region B on D, there exists a subsequence taken from the given sequence which converges uniformly in B to a limit function harmonic in B.

Proof. Since B is closed compact region in D, there exists a covering of B with a finite number of z_i-circles $V^{r_0}_{P_i}$ $(i=1, 2, \cdots, n)$, where $r_0 (< \frac{1}{2})$ is so chosen that all $V^{2r_0}_{P_i} \subset D$. At first, since $\{U_j\}$ $(j=1, 2,\cdots)$ is uniformly bounded sequence in $|z_1| < 2r_0$, by usual Lemma in plane-domain (e.g. cf. Kellogg [1]) we take from $\{U_j\}$ a subsequence $\{U_{1p_1}\}$ $p_1=1, 2,\cdots$, which converges uniformly in $K_{r_0}^1 = (|z_1| < r_0)$. Next we take from $\{U_{1p_1}\}$ a subsequence $\{U_{2p_2}\}$ $p_2=1, 2,\cdots$, which converges uniformly in $K_{r_0}^2$. And so on. Then, the sequence $\{U_{np_n}\}$ $p_n=1, 2,\cdots$ obviously converges uniformly to a limit function harmonic in B. *q. e. d.*

Now, since Γ_0 is analytic, each point on Γ_0 has a definite neighbourhood, in which any one of harmonic measures $\omega_{\hat{G}_n}(P)$ can be harmonically continued across Γ_0 by the principle of reflection. Lét $\mathfrak{D}$ be a compact closed region on $\hat{G}_0 + \Gamma_0 - (V^{r_0}_{P_0} + B^{r_0}_{P_0})$ containing $B^1_{P_0}$ (where B^r_P denotes the boundary of V^r_P) and Γ_0. Since $\{\omega_{\hat{G}_n}\}$ are all harmonic and uniformly bounded in a domain $\supset \mathfrak{D}$, hence by above Lemma we can take a subsequence $\{\omega_{\hat{G}_{ni}}\}$ (for simplicity, we write again $\{\omega_{\hat{G}_n}\}$ in the following) from $\{\omega_{\hat{G}_n}\}$, which converges uniformly in $\mathfrak{D}$ to a limit function ω and therefore uniformly

$$\frac{\partial \omega_{\hat{G}_n}}{\partial n} \to \frac{\partial \omega}{\partial n} \text{ on } \Gamma_0,$$

where $\dfrac{\partial}{\partial n}$ denotes the inner normal with respect to G_n.
Since

$$D_{\hat{G}_n}[\omega_{\hat{G}_n}] = \int_{\Gamma_0} \frac{\partial \omega_{\hat{G}_n}}{\partial n} ds = 2\pi / \log r_{G_n}' \to 0 \text{ (for } n\to\infty),$$

hence $\int_{\Gamma_0} \frac{\partial \omega}{\partial n} ds = 0$. Moreover, as $\frac{\partial \omega}{\partial n} \geqq 0$ on Γ_0, therefore $\frac{\partial \omega}{\partial n} =$ 0 throughout Γ_0. i.e. $\tilde{\omega}$ (conjugate harmonic function of ω) is constant on Γ_0, thus the derivative of analytic function $\Omega = \omega + i\tilde{\omega}$ vanishes on Γ_0 and therefore everywhere in $\mathfrak{D}$. This happens only in the case, when Ω reduces to a constant and thus ω is equal to zero in $\mathfrak{D}$. Therefore for given $\varepsilon > 0$, there exists a large number n_0, such that for $n \geqq n_0$

$$\omega_{\hat{G}_n}(P) \leqq \varepsilon \qquad P \in B^1_{P_0}$$

Fix a number $N \geqq n_0$, such that

$$D_{\hat{G}_N}[\omega_{\hat{G}_N}] \equiv \delta_N < 4r_0^2(1-\varepsilon)/\pi \tag{6}$$

Since $\omega_{\hat{G}_N}$(we write simply ω_N) is single-valued, harmonic function in $\hat{G}_N - V^{r_0}_{P_0}$ which is equal to 1 on γ^{G_N} and $\leqq \varepsilon$ on $B^1_{P_0}$, hence the level curves $\hat{L}^N_\rho$: $\omega_N = \rho$ $(\varepsilon \leqq \rho \leqq 1)$ lying in $V^1_{P_0}$ surround always a curve γ^{G_N}. Therefore in local parameter disc $K^1_{P_0}$: $|z| < 1$, we have always

$$2r_0 \leqq \int_{L^N_\rho} |dz| \qquad (\varepsilon \leqq \rho \leqq 1), \tag{7}$$

where L^N_ρ denotes the image of $\hat{L}^N_\rho$ on the z-plane. Put $\Omega_N = \omega_N + i\tilde{\omega}_N$ and consider Ω_N as another local parameter at P_0. Since

$$\int_{(L^N_\rho)} d\tilde{\omega}_N \leqq \int_{(L^N_\rho)+\Gamma} d\tilde{\omega}_N = \int_{\Gamma_0} d\tilde{\omega}_N = \delta_N .$$

where (L^N_ρ) denotes the image (on $\omega_N = \rho$) of L^N_ρ, hence by using the Schwarz's inequality to (7), we have

$$4\, r_0^2 \leqq \int_{(L^N_\rho)} d\tilde{\omega}_N \int_{(L^N_\rho)} \left|\frac{dz}{d\Omega_N}\right|^2 d\tilde{\omega}_N \leqq \delta_N \int_{(L^N_\rho)} \left|\frac{dz}{d\Omega_N}\right|^2 d\tilde{\omega}_N \tag{8}$$

Integrating (8) from ε to 1 with respect to $\rho (= \omega_N)$ then we obtain

$$4\, r_0^2 (1-\varepsilon) \leqq \delta_N \int_\varepsilon^1 \int_{(L^N_\rho)} \left|\frac{dz}{d\Omega_N}\right|^2 d\tilde{\omega}_N \, d\omega_N \leqq \pi \delta_N$$

i.e.

$$\delta_N \geqq 4\, r_0^2 (1-\varepsilon)/\pi > 0$$

which contradicts to (6). q.e.d.

4. From (5), thus we have

$$\log|w(P_0)| \leqq (\sup_{\Gamma\to\mathfrak{J}} \log r_G) \varlimsup_{\Gamma\to\mathfrak{J}} \frac{\overset{+}{\log} M(\Gamma)}{\log \mu_G}.$$

Suppose now that the ideal boundary $\mathfrak{J}$ of $\mathfrak{F}$ has zero harmonic measure, then $\log \mu_G \to \infty$ (for $\Gamma\to\mathfrak{J}$) and conversely. hence we can conclude finally the following theorem by the usual approximation and limiting process.

Theorem. Let F be an open Riemann surface with two disjoint boundaries Γ_0 and $\mathfrak{J}$, such that the harmonic measure of $\mathfrak{J}$ is zero, i.e. there exists a finite number of closed analytic curves Γ' on F, separating Γ_0 from $\mathfrak{J}$, and for this Γ', $\omega(P,\mathfrak{J},R(\Gamma',\mathfrak{J}))\equiv 0$. Let *$w(P)$ be a single-valued analytic function on F satisfying*

$$\varlimsup_{\Gamma_0}|w(P)| \leqq m,$$

then, if

$$\varlimsup_{\Gamma\to\mathfrak{J}} \frac{\overset{+}{\log} M(\Gamma)}{\log \mu_G} = 0,$$ *where μ_G denotes the harmonic modulus of $G=R(\Gamma'', \Gamma)$, ($\Gamma''=\Gamma_0$ if Γ_0 is analytic),*

the function $w(P)$ is bounded, such that $|w(P)|\leqq m$ for $P \in F$ (Maximumprinciple holds).

Corollary. Let F be a Riemann surface with null boundary. Now, let $w(P)$ be a single-valued analytic function bounded in F, then $w(P)$ reduces to a constant.

Proof. For an arbitrary point $P_0 \in F$, $|w(P)-w(P_0)| < \varepsilon$, $P \in V^{\delta}_{P_0}$ Take $\Gamma_0 \equiv B^{\delta}_{P_0}$, then by the theorem $|w(P)-w(P_0)| \leqq \varepsilon$, $P \in F$, q.e.d.

Remark. Let $w(z)$ be a regular function in $z \neq \infty$, and $\varepsilon = \operatorname{Max}_{\Gamma_0:|z|=\delta} |w(z)-w(0)|$, $M(r) \equiv \operatorname{Max}_{\Gamma:|z|=r} |w(z)|$. Since $\log \mu_G = \log \frac{r}{\delta}$ if $\varliminf_{r\to\infty} \frac{\log M(r)}{\log r} \left(= \varliminf_{r\to\infty} \frac{T(r)}{\log r}\right) = 0$, then we have by the theorem $|w(z) - w(0)| \leqq \varepsilon$, for $z \neq \infty$, i.e. $w(z) \equiv$ const.

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