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On some properties of trajectories of the group-spaces

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Let C_{a_0} be a trajectory through the origin a_0^α in the group-space S of a continuous transformation group G_r. Transforming C_{a_0} by two transformations with same parameters b_0^α, one of which belongs to the second parameter-group and the other to the first, we obtain two trajectories $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ respectively through the same point b_0^α in S. In general they do not coincide with each other. Therefore it is a question that under what conditions they coincide: as curves or not only as curves but also point-wisely. We shall study, in this paper, these conditions and their meaning in the group theory. Nextly taking the case where they coincide with each other as cnrves, we study the condition that C_{a_0} may be considered as a closed curve. For these research we shall use the concept of connections. Namely, we treat S as the space of affine connection into which the so-called $(+)$-connection is induced.

The notations in a previous paper [1] will be used also here.

1. Let G_r be a continuous group of transformations with r parameters a^α and $\mathfrak{G}_r^{(+)}$ be the first parameter-group of G_r. Let

$$a_3^\alpha=\varphi^\alpha(a_1, a_2) \qquad (\alpha=1, \cdots, r)$$

be the equations of $\mathfrak{G}_r^{(+)}$ where a_2^α are considered as parameters. We represent, hereafter, these euations symbolically by

$$(1\cdot1) \qquad a_3=a_1\, a_2$$

The same equations, when a_1^α are considered as parameters instead of a_2^α, represent the equations of the second parameter-group $\mathfrak{G}_r^{(-)}$

of G_r. The fundamental epuations of $\mathfrak{G}_r^{(+)}$ and the coefficients of the $(+)$-connection are given by

$$(1\cdot 2)\qquad \frac{\partial a_3^\alpha}{\partial a_2^\beta}=A_b^\alpha(a_3)A_\beta^b(a)\qquad (b,\alpha,\beta=1,\cdots,r),$$

$$(1\cdot 3)\qquad L_{\beta\gamma}^\alpha=-A_\beta^b\frac{\partial A_b^\alpha}{\partial a^\gamma}\qquad (b,\alpha,\beta=1,\cdots,r)$$

respectively. We denote the space into which the $(+)$-connection is induced by $S^{(+)}$. Since the curvature tensor of $S^{(+)}$ is zero, $S^{(+)}$ is called "to be flat", following Eisenhart [2].

We shall denote, hereafter, by "a" or "a^α" not only the parameters a^α but also the point on S having them as coordinates, and accordingly "a_0" or "a_0^α" the representative of the identical transformation of G_r.

$$(1\cdot 4)\qquad \frac{\partial A_b^\alpha}{\partial a^\gamma}+L_{\beta\gamma}^\alpha A_b^\beta=0\qquad (b,\alpha,\beta,\gamma=1,\cdots,r),$$

the vectors $\vec{A}_b(a)$ $(b=1,\cdots,r)$, whose components are $A_b^1(a),\cdots,A_b^r(a)$, are absolutely parallel. Accordingly, when a curve C in $S^{(+)}$ is developed on the tangential space at any point on C, all the images of $\vec{A}_b(a)$ for any b are parallel to each other. We shall denote them, therefore, simply by $\vec{\mathfrak{A}}_b$.

A curve in S whose differential equations are

$$(1\cdot 5)\qquad \frac{da^\alpha}{dt}=u^bA_b^\alpha(a)\qquad (b,\alpha=1,\cdots,r)$$

is called a trajectory of S, where u^b are constants one of which does not vanish at least. From $(1\cdot 3)$, $(1\cdot 4)$ and the well known relations $A_b^\alpha A_\alpha^c=\delta_b^c$, we have

$$\frac{d^2a^\alpha}{dt^2}+L_{\beta\gamma}^\alpha\frac{da^\beta}{dt}\frac{da^\gamma}{dt}=0\qquad (\alpha,\beta,\gamma=1,\cdots,r),$$

so that any trajectory is a path of $S^{(+)}$. Since u^aA_af generates a one-parameter sub-group $\mathfrak{G}_1^{(+)}$ of $\mathfrak{G}_r^{(+)}$, the solutions of $(1\cdot 5)$ subjected to the initial conditions $a^\alpha(0)=a_0^\alpha$ represent the point a^α which are the parameters of the transformations of G_1 with the symbol u^aX_af.

2. We denote by T_a the transformation of G_r whose parameters are a^α. Let $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ be segments each of which is taken on two trajectories which may be coincident. These segments $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are called to be equipollent of the first, or second kind, if

$$T_a T_b^{-1} = T_{a'} T_{b'}^{-1}$$

or

$$T_a^{-1} T_b = T_{a'}^{-1} T_{b'}$$

Consequently, when $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are equipollent of the first kind, $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ are equipollent of the second kind, and vice-versa. Take a certain curve C_{a_0} which, we suppose, are represented by $a^\alpha(t)$, C_{a_0} being a general curve not necessarily a trajectory. When the point $a^\alpha(t)$ on C_{a_0} are transformed by the transformations of $\mathfrak{G}_r^{(+)}$ and $\mathfrak{G}_r^{(-)}$ with same parameters b_0^α, we have the points $b^\alpha(t)$ and $\bar{b}^\alpha(t)$ which are given by

$$b(t) = b_0\, a(t) \tag{2·1}$$

and

$$\bar{b}(t) = a(t)\, b_0 \tag{2·2}$$

respectively. Each of them describes a curve which we shall denote by $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ respectively. From (2·1), for any set of t_1 and t_2 of t we have the relations $T_{b(t_i)} = T_{a(t_i)}\, T_{b_0}$ $(i = 1, 2)$, consequently

$$T_{a(t_1)} T_{b(t_1)}^{-1} = T_{a(t_2)} T_{b(t_2)}^{-1}. \tag{2·3}$$

If a_0^α and b_0^α are able to be connected by a segment of a trajectory, then so are $a^\alpha(t)$ and $b^\alpha(t)$. Hence, from (2·3), all segments $\overrightarrow{a(t)b(t)}$ are equipollent of the second kind. Similarly, concerning to the curve $C_{b_0}^{(-)}$, all segments $\overrightarrow{a(t)\bar{b}(t)}$ are equipollent of the first kind.

When the curve C_{a_0} is a trajectory, the curves $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ obtained as above are called to be $(+)$ and $(-)$-parallel to C_{a_0} respectively. We will use these terminologies even when the curve C_{a_0} is not a trajectory.

Furthermore, take another point $b_0'^{\alpha}$. The point defined by

$$(2\cdot4) \qquad b'(t) = b_0' a(t)$$

describes a surve $C_{b_0'}^{(+)}$. As we have

$$b'(t) = (b_0' b_0^{-1}) b(t),$$

in virtue of (2·1) and (2·4), $C_{b_0'}^{(+)}$ is obtained in transforming $C_{b_0}^{(+)}$ by the transformation of $\mathfrak{G}_r^{(-)}$ with parameters $(b_0' b_0^{-1})^{\alpha}$. It is natural therefore to call them to be (+)-parallel to each other. Similarly $C_{b_0'}^{(-)}$ and $C_{b_0}^{(-)}$ which are (−)-parallel to are C_{a_0} respectively called to be (−)-parallel to each other. Now, we regard S as the space of connection $S^{(+)}$. We have shown in the paper [1] that the image $\overrightarrow{\delta P}$ of the infinitesimal vector $\overrightarrow{dP}$ from $a^{\alpha}(t)$ to $a^{\alpha}(t+dt)$ is given by

$$(2\cdot5) \qquad \overrightarrow{\delta P} = \left\{ A_{\alpha}^{b}(a(t)) \frac{da^{\alpha}}{dt} dt \right\} \vec{\mathfrak{A}}_b$$

on a certain tangential space, and the translation $\vec{\mathfrak{T}}(C)$ which transforms the image of the terminal point $a^{\alpha}(t_2)$ of C to that of the initial point $a^{\alpha}(t_1)$, is given by

$$(2\cdot6) \qquad \vec{\mathfrak{T}}(C) = -\left\{ \int_{t_1}^{t_2} A_{\alpha}^{b}(a(t)) \frac{da^{\alpha}}{dt} dt \right\} \vec{\mathfrak{A}}_b .$$

we have called this translation "the transformation attached to C".

Let P be a point on C_{a_0} whose coordinates are $a^{\alpha}(t)$ and Q be the corresponding point on $C_{b_0}^{(+)}$ whose coordinates are $b^{\alpha}(t)$, then $\overrightarrow{\delta P}$ is given by (2·5) and similarly $\overrightarrow{\delta Q}$ by

$$\overrightarrow{\delta Q} = \left\{ A_{\alpha}^{b}(b(t)) \frac{db^{\alpha}}{dt} dt \right\} \vec{\mathfrak{A}}_b .$$

we have used the same vectors $\vec{\mathfrak{A}}_b$ in (2·5) and (2·6), since taking a curve which meets C_{a_0} and $C_{b_0}^{(+)}$, we can describe the two developments of C_{a_0} and $C_{b_0}^{(+)}$ on a same tangential space. From (1·2) and (2·2), we get

$$\frac{db^{\alpha}}{dt}=A^{\alpha}_{c}(b(t))A^{c}_{\beta}(a(t))\frac{da^{\beta}}{dt}.$$

While $A^{b}_{\alpha}A^{\alpha}_{c}=\delta^{b}_{c}$, hence we have

$$\overrightarrow{\delta Q}=\left\{A^{b}_{\alpha}(a(t))\frac{da^{\alpha}}{dt}\right\}\vec{\mathfrak{A}}_{b}.$$

Comparing this result with (2·6) we have

Theorem1. *The transformations attached to two (+)-parallel curves are equal to each other.*

3. Let us research for the conditions that $C^{(+)}_{b_0}$ and $C^{(-)}_{b_0}$ which are (+) and (−)-parallel to a trajectory C_{a_0} coincide with each other, where b^{α}_{0} is any point in S. As C_{a_0} is a trajectory through the origin, it represents a certain one-parameter sub-group $\mathfrak{G}^{(+)}_{1}$ of $\mathfrak{G}^{(+)}_{r}$. When $C^{(+)}_{b_0}$ and $C^{(-)}_{b_0}$ coincide with each other as curves, we can find a certain t'(or t) for any t(or t') such that

$$(3\cdot1)\qquad b^{\alpha}(t)=\bar{b}^{\alpha}(t').$$

From (2·1) and (2·2), we have

$$(2\cdot2)\qquad b^{-1}_{0}a(t)b_{0}=a(t').$$

As b^{α}_{0} is any point in S, this means that $\mathfrak{G}^{(+)}_{1}$ is an invariant sub-group of $\mathfrak{G}^{(+)}_{r}$. Conversely, when $a^{\alpha}(t)$ is a trajectory of an invariant sub-group $\mathfrak{G}^{(+)}_{1}$ of $\mathfrak{G}^{(+)}_{r}$, we have (3·2) and consequently (3·1) for any b^{α}_{0}. Therefore we have:

Theorem 2. *Let $C^{(+)}_{b_0}$ and $C^{(-)}_{b_0}$ (through b^{α}_{0}) be (+) and (−)-parallel curve to a trajectory (through the origin). A necessary and sufficient condition that $C^{(+)}_{b_0}$ and $C^{(+)}_{b_0}$ always coincide independent of b^{α}_{0} is that C_{a_0} is a trajectory of a invariant sub-group $\mathfrak{G}^{(+)}_{1}$ of $\mathfrak{G}^{(+)}_{r}$.*

When G_r, consequentely $\mathfrak{G}^{(+)}_{r}$, is simple, it has no invariant sub-group. Hence we have:

Corollary. *When G_r is simple, that is to say, when the rank of the matrix $\| c^{d}_{ab} \|$ (a: columns; b, d: rows) is r, there are no pair of trajectories $C^{(+)}_{b_0}$ and $C^{(-)}_{b_0}$ which coincide with each other.*

Let G_1 be a one-parameter sub-group which is generated by a symbol $u^a X_a f$ where u^a are constants one of which does not vanish at least. When the rank of $\| u^a c_{ab}^d \|$ is $r-p (p \geqq 2)$ the equations

$$c_{ab}^d u^a v^b = 0 \qquad (a, b, d = 1, \cdots, r),$$

where v^b are unknowns, have $p-1$ systems of solutions $v_{(i)}^b (i=1, \cdots, p-1)$ which are not proportional to u^b. Combining each one of $v_{(i)}^a X_a f$ $(i=1, \cdots, p-1)$ with $u^a X_a f$, $(p-1)$ sets of two symbols are obtained. These $(p-1)$ sets generate $(p-1)$ Abelian sub-groups $G_{(i)2}$ respectively. The parameters of each $G_{(i)2}$ form 2-dimensional sub-space $S_{(i)2}$ of S. On each $S_{(i)2}$ two points are commutative, since $S_{(i)2}$ is a group-space of an Abelian group. Hence we have the next:

Theorem 3. *Let $u^a X_a f$ be a symbol of a one-parameter sub-group G_1 of G_r,* and C_{a_0} *be a trajectory of $\mathfrak{G}_1^{(+)}$, whose symbol is $u^a A_a f$. If the rank of the matrix $\| u^a c_{ab}^d \|$ is $r-p (r \geqq p \geqq 2)$, then there exist $p-1$ 2-dimensional varieties $S_{(i)2}$ $(i=1, \cdots, p-1)$ such that if b_0^a is any point on any one of $S_{(i)2}$ then $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$* coin*cide point-wisely.*

When a sub-group G_1 of G_r is commutative with all the transformotions of G_r, the G_1 is called exceptional. If the rank of the matrix $\| c_{ab}^d \|$ is $r-p (r \geqq p \geqq 1)$ there are p one-parameter sub-group $G_{(i)1} (i=1, \cdots, p)$ which are exceptional and these $G_{(i)1}$ form an Abelian sub-group G_p. Therefore we have:

Theorem 4. *If the rank of $\| c_{ab}^d \|$ is $r-p$ $(r \geqq p \geqq 1)$, then there exists a p-dimensional invariant variety S_p which has the next properties: if a trajectory C_{a_0} is taken in S_p, $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ coincide with each other as curves independent of b_0^a, and furthermore point-wisely when, and only when, b_0^a is in S_p.*

In the above theorem, when the rank of the matrix is zero, that is, when G_r is Abelian, S_p is S itself and $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ always coincide for any point b_0^a. This is evident from the fact that the group-space of an Abelian group with $(+)$-connection is regarded as an ordinary affine space and both of $\mathfrak{G}_r^{(+)}$ and $\mathfrak{G}_r^{(-)}$ are isomorphic to the group of affine translations.

4. In this section we research for the condition that a trajectory C_{a_0} is to be closed in S when $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ coincide with each other as curves.

It is well known that every one-parameter group is isomorphic to either the translation group or the toroidal group in 1-space. Since C_{a_0} is not only a trajectory through the origin but also a closed curve, it must represent a one-parameter sub-group which is isomorphic to the toroidal group in 1-space; Hence we ean choose a parameter t which defines the point $a^{\alpha}(t)\,(0 \leqq t \leqq 1)$ on C_{a_0}, so as to get the following relations:

$$(4\cdot1) \qquad \begin{cases} a(0)=a(1)=a_0, \\ a(t_1+t_2)=a(t_1)a(t_2)=a(t_2)a(t_1). \end{cases}$$

As $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ are represented by (2·1) and (2·2) respectively, we have also

$$(4\cdot2) \qquad \begin{cases} b(0)=b(1)=b_0, \\ b(t_1+t_2)=b_0a(t_1+t_2)=b(t_1)a(t_2)=b(t_2)a(t_1), \end{cases}$$

$$(4\cdot3) \qquad \begin{cases} \bar{b}(0)=\bar{b}(1)=b_0, \\ \bar{b}(t_1+t_2)=a(t_1+t_2)b_0=a(t_1)\bar{b}(t_2)=a(t_2)\bar{b}(t_1). \end{cases}$$

From the first of (4·2) and (4·3) we know that both of $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ are also closed.

When $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ are coincident as curves, the point $b^{\alpha}(t)$ and $\bar{b}^{\alpha}(t)$ desci ibe the curve in the same sense, when t increases. In fact, when b_0^{α} is chosen infinitely near a_0^{α}, by the relation

$$b(t)=b_0a(t)\doteqdot a(t)$$

$b^{\alpha}(t)$ is infinitely near $a^{\alpha}(t)$, and so is it for $\bar{b}^{\alpha}(t)$. Hence whenever we choose b_0^{α} sufficiently near a_0^{α}, $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ are described in the same sense.

Now we show that there exists a positive rational number $\frac{q}{p}(<1)$ such that, when $t=\frac{q}{p}$, $b^{\alpha}(t)$ does not coincide with $\bar{b}^{\alpha}(t)$. Because, if $b^{\alpha}(t)=\bar{b}^{\alpha}(t)$ for every positive rational number $\frac{q}{p}$, then $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ must coincide point-wisely, since the rational numbers are every-where dense and $b^{\alpha}(t)$ and $\bar{b}^{\alpha}(t)$ are continuous functions. This contradicts our supposition.

Therefore we can take a point $a^\alpha(t_1)$ on C_{a_0} where $t_1=\frac{q}{p}$ so that the corresponding point $b^\alpha(t_1)$ on $C_{b_0}^{(+)}$ and $\bar{b}^\alpha(t_1)$ on $C_{b_0}^{(-)}$ do not coincide. We take on C_{a_0} the point $a^\alpha(t_1)$, $a^\alpha(2t_1), \cdots, a^\alpha(pt_1)$ $(=a_0{}^\alpha)$. Let $b^\alpha(t_1), \cdots, b^\alpha(pt_1)(=b_0{}^\alpha)$ on $C_{b_0}^{(+)}$ and $\bar{b}(t_1), \cdots, \bar{b}^\alpha(pt_1)$ $(=b_0{}^\alpha)$ on $C_{b_0}^{(-)}$ be the corresponding points to them. Let $\overrightarrow{a(t_1)b(t_1)}$ and $\overrightarrow{a(t_1)\bar{b}(t_1)}$ be two segments of trajectories, then they are $(-)$ and $(+)$-parallel to $\overrightarrow{a_0b_0}$ respectively. Let γ_1 be a closed curve formed by four segments $\overrightarrow{a_0a(t_1)}$, $\overrightarrow{a(t_1)\bar{b}(t_1)}$, $\overrightarrow{\bar{b}(t_1)b_0}$ and $\overrightarrow{b_0a_0}$, oriented in the order indicated. Then from $(2\cdot6)$ we have

$$\vec{\mathfrak{T}}(\gamma_1)=-\left\{\int_{\gamma_1} A_\alpha^b da^\alpha\right\}\vec{\mathfrak{A}}_b.$$

Since $\overrightarrow{a_0b_0}$ and $\overrightarrow{a(t)\bar{b}(t_1)}$ are $(+)$-parallel, and also $\overrightarrow{a_0a(t_1)}$ and $\overrightarrow{b_0b(t_1)}$ from Theorem 1 we have

$$\vec{\mathfrak{T}}(\overrightarrow{a_0b_0})=\vec{\mathfrak{T}}(\overrightarrow{a(t_1)\bar{b}(t_1)}),$$

and

$$\vec{\mathfrak{T}}(\overrightarrow{a_0a(t_1)})=\vec{\mathfrak{T}}(\overrightarrow{b_0b(t_1)}).$$

Therefore we have

$$\vec{\mathfrak{T}}(\gamma_1)=-\left\{\int_{\overrightarrow{\bar{b}(t_1)b(t_1)}} A_\alpha^b d_a^\alpha\right\}\vec{\mathfrak{A}}_b.$$

As C_{a_0} is a trajectory, its differential equations are given by $(1\cdot5)$, where the constants u^α are suitably chosen so that the conditions $(4\cdot1)$ are obtained in this case, too. They are also the equations of $C_{b_0}^{(+)}$. suppose that the point $\bar{b}^\alpha(t_1)$ on $C_{b_0}^{(-)}$ is represented by $b^\alpha(t)$ when it is regarded as a point of $C_{b_0}^{(+)}$. Then $t_2 \neq t_2$ by the assumption. Therefore we have

$$\vec{\mathfrak{T}}(\gamma_1)=-\left\{\int_{t_2}^{t_1} A_a^b u^a A_a^\alpha dt\right\}\vec{\mathfrak{A}}_b=u^b(t_1-t_2)\vec{\mathfrak{A}}_b\neq 0.$$

Let γ_i be an oriented quadrilateral passing through its vertices $a^\alpha(t_{i-1})$, $a^\alpha(t_i)$, $\bar{b}^\alpha(t_i)$ and $\bar{b}^\alpha(t_{i-1})$ in this order. Then γ_i is $(+)$-parallel to γ_1. Therefore by Theorem 1 we have

$$\overrightarrow{\mathfrak{T}}(\gamma_t)=\overrightarrow{\mathfrak{T}}(\gamma_1) \qquad (t=2,\cdots,p).$$

Consider a route γ described by a moving point P as follows. Firstly, P makes q-circuits along C_{a_0}, starting from a_0^α and passing $a^\alpha(t_1)$, $a^\alpha(2t_1)$, $\cdots$, $a^\alpha(\overline{p-1}\,t_1)$ in this order, and then returning to a_0^α. Secondly, P moves from a_0^α to b_0^α along $\overrightarrow{a_0b_0}$. Thirdly, P makes q-circuits along $C_{b_0}^{(-)}$ in the opposite sense to that of C_{a_0}. Finaly P comes back from b_0^α along $\overrightarrow{b_0a_0}$.

As the route γ consists of $\gamma_1,\gamma_2,\cdots,\gamma_p$, we obtain

$$(4\cdot4) \qquad \overrightarrow{\mathfrak{T}}(\gamma)=p\overrightarrow{\mathfrak{A}}(\gamma_1) \quad (\neq 0).$$

On the other hand the route γ may be decomposed in C_{a_0}, $C_{b_0}^{(+)}$ and $\overrightarrow{a_0b_0}$. Since $C_{b_0}^{(+)}$ is $(+)$-parallel to C_{a_0}, both of the transformations attached to C_{a_0} and $C_{b_0}^{(+)}$ are equal by Theorem 1. Thus we have

$$(4\cdot5) \qquad \overrightarrow{\mathfrak{T}}(\gamma)=0.$$

The result $(4\cdot5)$ is contrary with that of $(4\cdot4)$. This contradiction is caused from either one of the two assumptions, (1) C_{a_0} is closed and (2) $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ do coincide as curves but not point-wisely. Consequently we have:

Theorem 5. *Let C_{a_0} be a trajectory through the origin a_0^α. If C_{a_0} is closed and $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ which are $(+)$ and $(-)$-parallel to C_{a_0} are coincident with each other, then they must coincide point-wisely.*

Theorem 6. *If $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ which are $(+)$ and $(-)$-parallel to C_{a_0} do coincide as curves but not point-wisely, then C_{a_0} is not closed.*

By Theorem 2, the condition that $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ coincide as curves is equivalent to the condition that $\mathfrak{G}_1^{(+)}$ which is represented by the trajectory C_{a_0} is an invariant sub-group of $\mathfrak{G}_r^{(+)}$, accordingly that G_1 whose parameter-group is $\mathfrak{G}_1^{(+)}$ is an invariant sub-group of G_r. The condition that $C_{b_0}^{(+)}$ and $C_{b_0}^{(-)}$ coincide point-wisely is equivalent to the condition that above $\mathfrak{G}_1^{(+)}$ (or G_1) is an exceptional group of $\mathfrak{G}_r^{(+)}$ (or G_r). Furthermore the condition that C_{a_0} is closed is equivalent to the condition that G_1 is isomorphic to the toroidal group in 1-space. Hence expressing Theorems 5 and 6 in the terminologies of the group theory, we may state as follows.

Theorem 7. *If G_1 is an invariant sub-group of G_r and isomorphic to the toroidal group in* 1*-space, then G_1 is exceptional in G_r.*

Theorem 8. *If G_1 is an invariant sub-group of G_r but not exceptional, then G_1 is isomorphic to the translation group in* 1*-space.*

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