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On the existence of a curve connecting given points on an abstract variety

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In the course of study in algebraic geometry, we are frequently encountered to treat the following problem. Let V be an abstract variety, and P, Q be two points on V, then does there exist an irreducible curve connecting these two points? It may seem to be almost self-evident, but it seems to us that there is no any proof in the literature. In this note we shall answer the above in the following generalized form.

THEOREM. Let V^n be an abstract variety, and $U_i^{s_i}(i=1, \cdots, m)$ be finite number of subvarieties of dimensions s_i respectively, such that $s=\max (s_i) < n-1$. Then there exists an irreducible subvariety of V containing all U_i, of any dimension r such that $s+1 \leqq r \leqq n-1$. Moreover there exists such one which is algebraic over any common field of definition for V and U_i $(i=1, \cdots, m)$.

First we shall prove the theorem in the case when V is a projective model, and then go into the general case.

LEMMA 1. Let V^n be a projective model, and $P_i(i=1, \cdots, m)$ be arbitrary points on V. Then there exists an irreducible subvariety of V, containing all P_i, of any dimension r such that $1 \leqq r \leqq n-1$. Moreover let k be a field of definition for V, then there exists such one which is algebraic over $k(P_1, \cdots, P_m)$.

PROOF. It is sufficient to treat the case $r=n-1$. First we shall assume that V is normal. Let t be an integer satisfying the following condition. Let Q be an arbitrary point of V, different from any of P_i, there exists a hypersurface of order $t-1$, containing all P_i, but not Q. Such integer surely exists, e.g., $t=m+1$. Put $\mathfrak{A}=\sum P_i$ then the linear system $\Sigma_{\mathfrak{A}}$ which consists of the intersections of V with all hypersurfaces of order t containing all points in $\mathfrak{A}$, will be shown to be noncomposite with the pencils. In fact, let C be a hyperplane section of V, then there exists the linear system $\sum_{\mathfrak{A}}-C$ on V. Moreover from the choice of an integer t, $\sum_{\mathfrak{A}}$ cannot have any fixed component. Hence by the theorem of Bertini on the linear system[1] the generic member of $\sum_{\mathfrak{A}}$ is irreducible. When V is not normal, we construct a normal projective model $\bar{V}$[2]. Since the correspondence T between V and $\bar{V}$ has no fundamental point, the transform of $\mathfrak{A}$ in $\bar{V}$ is also a set of finite number of points. Let them be $\bar{\mathfrak{A}}$. Then we can construct a subvariety $\bar{U}$ of $\bar{V}$ containing all points in $\bar{\mathfrak{A}}$. Then $T^{-1}(\bar{U})$ also contains all P_i and irreducible. To prove the last part of the lemma, it will be sufficient to remark that the linear system $\sum_{\mathfrak{A}}$ is defined over $k(\mathfrak{A})$ and the correspondence can be defined over the algebraic closure of k. Then the remaining part follow from Prop. 1 of Matsusaka (3). *q. e. d.*

LEMMA 2. Theorem holds for a projective model V in L^N.

PROOF. Without any restriction we can assume that $s=s_i$ $(i=1, \cdots, m)$. Let k be a common field of definition for V and U_i, and H^{N-s} be a generic hinear variety over k defined by the equations

$$\sum_{j=0}^{N} u_{ij}X_j=0 \qquad (i=1, \cdots, s)$$

where (u_{ij}) are $s(N+1)$-independent variables over k. Put $H.V=\bar{V}^{n-s}$ and $H.U_i=\sum_j P_{ij}$. Without loss of generalities we can suppose that all U_i have representatives in the affine representative S of L where $X_0=1$. Put $K=k(u_{ij}, i=1, \cdots, s; j=1, \cdots, N)$ and $K_1=K(u_{i0}, i=1, \cdots, s)$. Then P_{ij} are generic points of U_i over K and $\bar{V}$ is defined over K_1. Let $\bar{U}^{r-s}$ be a subvariety of $\bar{V}$ algebraic over K_1 containing all $\{P_{ij}\}$, and P a generic point of $\bar{U}$ over $\bar{K}_1$. Then since $\bar{U}$ is on $\bar{H}$ we have

$$\dim_K(P)=\dim_K(K_1)+\dim_{K_1}(P)=s+(r-s)=r$$

Let U^r be the locus of P over $\bar{K}$. We shall show that U contains all U_i as its subvarieties. Let Q be any point in U_i. Then $P_{ij}\to Q$ is a specialization over $\bar{K}$. Moreover $P\to P_{ij}$ is a specialization over $\bar{K}_1$, hence a fortiori, over $\bar{K}$. Thus any point of U_i is get be the specialization of P over $\bar{K}$, and Q is on U. Let $\mathfrak{M}$ be the locus

1) Cf. Zariski (7) and Matsusaka (2).

2) Cf. Zariski (8).

of $c(U)$ of $\bar{k}$, where $c(U)$ is the Chow-point of U[3] and w be the point of $\mathfrak{M}$, rational over $\bar{k}$, such that the corresponding cycle W in L is an irreducible variety[4]. Then W will be seen to satisfy all the requirements in the lemma, since the inclusion relation are preserved by the specialization of cycles[5]. Thus the lemma is proved. *q. e. d.*

As is well known the variety in a multiply projective space can be transformed by a biregular birational correspondence into a projective model, hence we have

LEMMA 3. Theorem holds for a variety embedded in a multiply projective space.

THE PROOF OF THE THEOREM. Let V be an abstract variety given by $V=[\mathrm{V}_\alpha, \mathfrak{F}_\alpha; \mathrm{T}_{\beta\alpha}]$, V_α $(\alpha=1, \cdots, s)$ be the representatives of V, S_α the ambiant affine spaces of V_α, and M_α the representatives of a generic point M of V over a field of definition k for V and U_i. Then since $k(\mathrm{M}_1 \times \cdots \times \mathrm{M}_s) = k(M)$ is a regular extension of k, $\mathrm{M}_1 \times \cdots \times \mathrm{M}_s$ has a locus T over k. Now taking S_α as a representative of a projective space L_α, we have a projective model $\bar{V}_\alpha$ in L_α, having V_α as a representative in S_α. Similarly we have $\tilde{T}$ in $\prod_\alpha L_\alpha$, which has the representative T in the representative $\prod_\alpha \mathrm{S}_\alpha$ of $\prod_\alpha L_\alpha$. Suppose that U_i has the representative $\mathrm{U}_{i\alpha}$ in V_α, and $\bar{U}_{i\alpha}$ be the subvariety of $\bar{V}_\alpha$ such that $\bar{U}_{i\alpha}$ has the representative $\mathrm{U}_{i\alpha}$ in V_α. Since $\tilde{T}$ is complete there exists a subvariety $\tilde{U}_i$ of $\tilde{T}$ with the projection $\bar{U}_{i\alpha}$ in $\bar{V}_\alpha$. Moreover we can find such one among those which is algebraic and $\dim(\tilde{U}_i) = \dim(U_i)$. Then by Lemma 3, there exists a subvariety $\tilde{U}'$ of $\tilde{T}$ containing all $\tilde{U}_i$ and algebraic over k. Let U be a representative of $\tilde{U}$ in $\prod_\alpha \mathrm{S}_\alpha$. Then we see that the projection U_α of U on V_α is not contained in $\mathfrak{F}_\alpha$, since U_α contains $\mathrm{U}_{i\alpha}$. Thus U determines a subvariety U of V algebraic over k, which will be seen to satisfy all the conditions in the theorem. *q. e. d.*

In the case of a projective model we can say further as follows.

Let V be a projective model and $\mathfrak{A} = \sum U_i$ is an unmixed

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3) By the main theorem on associated forms, any positive cycle in a projective space can be represented as point in a suitable projective space. We call it briefly the Chow-point of the cycle. Cf. V. d. Waerden (6).

4) Cf. Matsusaka (3), Prop. 1.

5) For the theory of specialization of cycles in a projective space, see Matsusaka (1), or P. Samuel (4).

V-cycle[6], then there exists a subvariety W of V such that $\mathfrak{A}$ is also a W-cycle.

First we shall show the existence of such W when $\mathfrak{A}$ is of dimension zero. The general case will follow immediately from it. For this purpose we must add the following condition on the choice of an integer t in Lemma 1. Let P be any point in $\mathfrak{A}$, then there exists a hypersurface of order $t-1$, not containing P, but contains all points in $\mathfrak{A}$ other than P. Under this condition, the generic member of the linear system $\sum_{\mathfrak{A}}$ contains any point P_i in $\mathfrak{A}$ as a simple point. The proof is as follows. Let H_1 be a hypersurface of order $t-1$, containing all points P_i for $2 \leqq i \leqq m-1$, and does not pass through P_1, and H_2 be hyperplane containing P_1 which is transversal to V at P_1. Put $H=H_1+H_2$. Then $H.V$ contains only one component, say C, containing P_1, and P_1 is a simple point of C.[7] Hence the generic member of $\sum_{\mathfrak{A}}$ contains P_1 as a simple point. Using again the similar argument in Prop. 1 of Matsusaka (3), we see that such member can be found among those which are algebraic over $k(\mathfrak{A})$.

The corresponding results for abstract varieties are still unsolved.

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6) This means that all U_i are simple subvarieties of V and all U_i have the same dimensions. Cf. Chap. VII of Weil (5).

7) Cf. Chap. IV of Weil, l. c.

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