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On cyclic points of group-spaces

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In the group-space S of a continuous group G_r of transformations, we have considered two kinds of repères $\mathfrak{R}(a)$ and $\bar{\mathfrak{R}}(a)$ at every point a in one of the previous papers [1]. Though they can be chosen so that they coincide at the origin a_0, they do not coincide at other points in general. Moreover, let C_a be a path through a point a in S. Through another point b there exist two paths $C_b^{(+)}$ and $C_b^{(-)}$ which are $(+)$-parallel and $(-)$-parallel to C_a respectively. These paths are also not coincident in general except the case when G_r is abelian. In this paper we define that a point b is a cyclic point with respect to a point a when $C_b^{(+)}$ and $C_b^{(-)}$ are coincident with one another not only as curves but also pointwisely. We shall show that a necessary and sufficient condition for the existence of such cyclic points is that the relative position between the two kinds of repères at a point is coincident with the one at another certain point. The notations in the previous papers ([1], [2]) will be used also here.

1. In the group-space S of a continuous group G_r of transformations with r parameters a^α's, let a be a point whose coordinates are $(a^1, \cdots, a^r)$ and T_a a transformation of G_r with parameters $a^\alpha(a^1, \cdots, a^r)$. We denote by a_0 the point which represents the identical transformation G_r. When a point a is transformed to b by the transformation of the first parameter-group $\mathfrak{G}_r^{(+)}$ of G_r with certain constant parameters $p^\alpha(p^1, \cdots, p^r)$, we have

$$T_b = T_p T_a \tag{1·1}$$

Similarly, if b' is a point transformed from a' by the same transformation, we have

$$T_{b'} = T_p T_{a'} \tag{1·2}$$

Then if we have the relation

$$(1\cdot3) \qquad T_{b'}T_b^{-1}(=T_p(T_{a'}T_a^{-1})T_p^{-1})=T_{a'}T_a^{-1}$$

for any a and a', let us call the point p a *cyclic point of the group-space with respect to the origin* a_0. When G_r is abelian, the relation (1·3) is always satisfied by any p. Hereafter we say, however, p is a cyclic point when and only when it is isolated, that is, in a suitable neighbourhood of p there exists no point other than p itself which satisfies the relation (1·3).

Now, we assume that the group-space under consideration has a cyclic point p with respect to a_0. When a and a' are taken so sufficiently near that there is only one path aa' which passes both of them, the path bb' is also determined uniquely by b and b'. Since the vectors $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ are equipollent of the first kind ([3], cf. p. 4) from (1·3), the two paths aa' and bb' are (+)-parallel to each other. On the other hand from (1·1) and (1·2) we obtain

$$T_b^{-1}T_{b'}=T_a^{-1}T_{a'}$$

This shows that the vectors $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ are equipollent of the second kind. Let C_a be a path which passes a point a, and $C_b^{(+)}$ and $C_b^{(-)}$ be the paths through b, being (+)-parallel and (−)-parallel to C_a respectively. As a result of the above we have:

THEOREM 1. *Let p be a certain point in the group-space, C_a be any path through an arbitrary point a and b be the point obtained from a by the transformation of the first parameter-group of G_r with the parameters p^α's. A necessary and sufficient condition that there exists a cyclic point p with respect to the origin is that $C_b^{(+)}$ and $C_b^{(-)}$ are coincident with one another not only as curves but also point-wisely.*

2. Let $\vec{A}_b(a)$'s and $\vec{\bar{A}}_b(a)$'s, where $b=1, \cdots, r$, be contravariant vectors whose components are $(A_b^1(a), \cdots, A_b^r(a))$ and $(\bar{A}_b^1(a), \ldots, \bar{A}_b^r(a))$ respectively. Moreover, let $\mathfrak{R}(a)$ and $\bar{\mathfrak{R}}(a)$ be the repères attached to a, the former being composed of the vertex a and r vecters $\vec{A}_1(a), \cdots, \vec{A}_r(a)$ and the latter the same vertex and r vectors $\vec{\bar{A}}_1(a), \cdots, \vec{\bar{A}}_r(a)$. Since in general $\vec{A}_b(a_0)$ and $\vec{\bar{A}}_b(a_0)$ for each $b=1, \cdots, r$ are chosen so that they coincide, the repères $\mathfrak{R}(a_0)$ and $\bar{\mathfrak{R}}(a_0)$ coincide. We shall show that $\mathfrak{R}(p)$ and $\bar{\mathfrak{R}}(p)$ coincide when the group-space has a cyclic point p with respect to a_0.

In general p is not so close to a_0 that there may be many paths which connect the points a_0 and p. We choose any one of them and denote it by D. The vector $\vec{A}_b(p)$ is obtained from $\vec{A}_b(a_0)$ by $(+)$-parallel displacement, that is, after developping D along itself by the $(+)$-connection on the tangent space at a_0, the image of $\vec{A}_b(p)$ is obtained from $\vec{A}_b(a_0)$ by parallel displacement from a_0 to the image of p. Accordingly if C_{a_0} is the path to which $\vec{A}_b(a_0)$ is tangent then $\vec{A}_b(p)$ is tangent to $C_p^{(+)}$. Similarly $\vec{\bar{A}}_b(p)$ is tangent to $C_p^{(-)}$. Being p is a cyclic point, $\vec{A}_b(p)$ and $\vec{\bar{A}}_b(p)$ are determined by two adjacent points on $C_p^{(+)}$ and $C_p^{(-)}$ respectively corresponding to a_0 and a_0+da on C_{a_0} which determine the sense of $\vec{A}_b(a_0)$ $(=\vec{\bar{A}}_b(a_0))$. As $C_p^{(+)}$ and $C_p^{(-)}$ are coincident not only as curves but also point-wisely, the senses of $\vec{A}_b(p)$ and $\vec{\bar{A}}_b(p)$ also coincide. It follows that:

THEOREM 2. *If there exists a cyclic point p with respect to a_0 in the group-space where $\mathfrak{R}(a_0)$ and $\bar{\mathfrak{R}}(a_0)$ are chosen so that they coincide, then $\mathfrak{R}(p)$ and $\bar{\mathfrak{R}}(p)$ also coincide.*

3. Conversely, we assume that $\mathfrak{R}(p)$ coincides with $\bar{\mathfrak{R}}(p)$ at a certain point p other than a_0. We shall show that from this assumption p becomes a cyclic point with respect to the origin. The path C_{a_0} is represented by

$$\frac{da^\alpha}{dt}=e^a A_a^\alpha(a), \tag{3·1}$$

where e^a's are constants and t is a parameter chosen suitablly. They are also the differential equations of the path $C_p^{(+)}$, and by the transformation of the second parameter-group $\mathfrak{G}_r^{(-)}$ with parameters p^α's any point $a(t)$ on C_{a_0} is transformed to a point $a(t)$ on $C_p^{(+)}$ which corresponds to the same value of t. Since $\vec{A}_b(a)$'s are independent, we can put

$$\bar{A}_a^\alpha(a)=\rho_a^b(a)\,A_b^\alpha(a), \tag{3·2}$$

where $\rho_a^b(a)$'s are functions of a. As $\mathfrak{R}(a)$ and $\bar{\mathfrak{R}}(a)$ coincide at the points a_0 and p,

$$\rho_a^b(a_0)=\rho_a^b(p)=\delta_a^b.$$

It is known that these functions satisfy

$$\frac{\partial \rho_a^b}{\partial a^\alpha} = c_{ef}^{\ b} \cdot \rho_a^e(a) A_\alpha^f(a),$$

where c_{ef}^{b}'s are the constants of structure of G_r ([4], cf. p. 30). At any point $a(t)$ on C_{a_0} these $\rho_a^b(a)$ are expanded formally to the series

$$(3\cdot 3) \qquad \rho_a^b(a(t)) = \delta_a^b + c_{af_1}^{\ b} u^{f_1} + \frac{1}{2!} c_{e_1 f_1}^{\ b} c_{af_2}^{\ e_1} u^{f_1} u^{f_2}$$

$$+ c_{e_1 f_1}^{\ b} c_{e_2 f_2}^{\ e_1} c_{af_3}^{\ e_2} u^{f_1} u^{f_2} u^{f_3} + \cdots,$$

where $u^a = e^a t$ (which are canonical parameters). As $\rho_a^b(p) = \delta_a^b$ and $C_p^{(+)}$ is also represented by (3·1), the equations (3·3) are that of $\rho_a^b(a)$ at a point on $C_p^{(+)}$, too.

If $|c_{ef}^b| < c$ for all e, f, b from 1 to r, and if we put $u = \sum_\alpha |u^\alpha|$, we have that the series on the right in (3·3) is less than the series

$$1 + cu + \frac{1}{2!} rc^2 u^2 + \frac{1}{3!} rc^3 u^3 + \cdots,$$

whose sum is $(\exp(rcu) - 1)/r + 1$, and consequently the series (3·3) have the meaning as the values of $\rho_a^b(a)$ so far as canonical parameters u^α's exist on C_{a_0} (or $C_p^{(+)}$). Since

$$(3\cdot 4) \qquad e^a A_a^\alpha = \bar{e}^a \bar{A}_a^\alpha$$

for suitable constants $\bar{e}^a$'s ([4], cf. p. 200), (3·1) are also represented by

$$(3\cdot 5) \qquad \frac{da^\alpha}{dt} = \bar{e}^a \bar{A}_a^\alpha.$$

They are the differential equations of C_{a_0} in the case where it is regarded as a trajectory of one-parameter sub-group of the second parameter-group $\mathfrak{G}_r^{(-)}$. From (3·4) we have on C_{a_0},

$$(3\cdot 6) \qquad e^b = \bar{e}^a \rho_a^b(a),$$

and ρ_a^b's have same values at corresponding points on C_{a_0} and $C_p^{(+)}$, hence substituting e^a in (3·1) as the equations of $C_p^{(+)}$ by (3·6) we can obtain (3·5) again. This means that, when $C_p^{(+)}$ is regarded as a trajectory of one-parameter sub-group of $\mathfrak{G}_r^{(+)}$, $C_p^{(+)}$ is also $(-)$-parallel to C_{a_0}, that is, $C_p^{(+)}$ and $C_p^{(-)}$ coincide as curves. In this case a point $a(t)$ on $C_p^{(-)}$ corresponds to a point on C_a

having the same value of t.

Moreover, making use of (3·1) and (3·5) C_{a_0} is expressed by the series

$$a^\alpha = a_0^\alpha + te^a A_a a_0^\alpha + \cdots + \frac{t^m}{m!} e^{a_1}\cdots e^{a_m} A_{a_1}\cdots A_{a_m} a_0^\alpha + \cdots,$$

where

$$A_a(a) = A_a^\alpha(a) \frac{\partial f}{\partial a^\alpha},$$

and

$$a^\alpha = a_0^\alpha + t\bar{e}^a \bar{A}_a a_0^\alpha + \cdots + \frac{t^m}{m!} \bar{e}^{a_1}\cdots \bar{e}^{a_m} \bar{A}_{a_1}\cdots \bar{A}_{a_m} a_0^\alpha + \cdots;$$

where

$$\bar{A}_a(a) = \bar{A}_a^\alpha(a) \frac{\partial f}{\partial a^\alpha}$$

respectively. From (3·4) and these two series we know that a point $a(t)$ on C_{a_0} as a trajectory of a sub-group of $\mathfrak{G}_r^{(+)}$ and a pointed $a(t)$ on C_{a_0} as a trajectory of a sub-group of $\mathfrak{G}_r^{(-)}$ are coincident for every value of t. Accordingly $C_p^{(+)}$ and $C_p^{(-)}$ coincide not only as curves but also point-wisely.

Now, let a_1 be a point on C_{a_0} and b_1 corresponding one on $C_p^{(+)}$ $(\equiv C_p^{(-)})$. When b_1 is regarded as a point on $C_p^{(+)}$, we have

(3·7) $$T_{b_1} = T_{a_1} T_p,$$

on the other hand when it is regarded as a point on $C_p^{(-)}$, we have

(3·8) $$T_{b_1} = T_p T_{a_1}.$$

Since similar relations are consistent for every such pair of a_2 and b_2 as the pair of a_1 and b_1, where a_2 may not be on C_{a_0}, we have

(3·9) $$T_{b_2} = T_{a_2} T_p,$$

(3·10) $$T_{b_2} = T_p T_{a_2}.$$

From (3·7) and (3·9), it follows that

$$T_{b_2} T_{b_1}^{-1} = T_{a_2} T_{a_1}^{-1}.$$

Moreover the relations of (3·8) and (3·10) are similar to those of (1·1) and (1·2), hence we have:

THEOREM 3. *If $\mathfrak{R}(a)$ and $\bar{\mathfrak{R}}(a)$ coincide at a point p other than the origin, the group-space has a cyclic point p with respect to the origin.*

From Theorems 2, 3 we have:

THEOREM 4. *A necessary and sufficient condition that there exists a cyclic point p with respect to the origin is that the two kinds of repères coincide at the origin and at p.*

When there exists a cyclic point p with respect to the origin, the relative position of the two kinds of repères at a is coincident with the one at b which is transformed from a by the transformation of $\mathfrak{G}_r^{(-)}$ with parameters p^α's. This is evident since the functions ρ^b_a's at a are equal to those at b. Let us call b a *cyclic point with respect to a.* When there exists a cyclic point with respect to the origin, there exists necessarily a cyclic point for every point in S. Hence we say merely in this case that the group-space has a *cycle.*

We can choose any point in S as an origin and determine arbitrarily relative position of two kinds of repères at only one point, hence we have:

THEOREM 5. *A necessary and sufficient condition that the group-space has a cycle is that the relative position of the two repères at a point coincides with the one at another certain point.*

4. For an example let us consider the group of motions in the euclidean plane defined by the equations

$$x'^1 = a^1 + x^1 \cos a^3 - x^2 \sin a^3,$$
$$x'^2 = a^2 + x^1 \sin a^3 + x^2 \cos a^3.$$

The coordinates of the origin a_0^α in the group-space are given by $(0, 0, 0)$. The vectors $\vec{A}_b$ and $\vec{\bar{A}}_b$ are defined as follows:

$$\left\| \begin{matrix} \vec{A}_1 \\ \vec{A}_2 \\ \vec{A}_3 \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a^2 & -a^1 & -1 \end{matrix} \right\|$$

and

$$\left\| \begin{matrix} \vec{\bar{A}}_1 \\ \vec{\bar{A}}_2 \\ \vec{\bar{A}}_3 \end{matrix} \right\| = \left\| \begin{matrix} \cos a^3 & -\sin a^3 & 0 \\ \sin a^3 & \cos a^3 & 0 \\ 0 & 0 & -1 \end{matrix} \right\|$$

In the above matrices we have determined the vectors so that the relation $\mathfrak{R}(a_0) = \bar{\mathfrak{R}}(a_0)$ follows. $\mathfrak{R}(a)$ and $\bar{\mathfrak{R}}(a)$ coincide at $(a^1, a^2,$

$a^3) = (0, 0, 2n\pi)$ where $n=0, \pm 1, \pm 2, \cdots$. Thus all points of the type $(0, 0, 2n\pi)$ where $n=\pm 1, \pm 2, \cdots$ are cyclic points with respect to the origin. Hence this group has a cycle.

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