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On the hypersurface sections of algebraic varieties embedded in a projective space

By

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The problem to decide whether the intersection-product of two varieties is irreducible or not seems to us very important in algebraic geometry. Recently Prof. Chevalley proposed to us a problem of this character. (The final form of the problem is stated in Theorem 2 in this paper.) We heard that Chevalley has a proof already. But the proof in this paper is simple and quite different from his principle. We wish to express our hearty thanks to Prof. Akizuki for his kind encouragement.

Let $\boldsymbol{L}^N$ be a projective space of dimension N. Then as is well known every positive cycle $\boldsymbol{X}$ in $\boldsymbol{L}^N$ can be represented as a point of a suitable projective space by the coefficients of the associated form of $\boldsymbol{X}$.[1] We shall call it the Chow point of $\boldsymbol{X}$ and denote by $C(\boldsymbol{X})$. In particular the totality of Chow points of the hypersurfaces of order m in $\boldsymbol{L}^N$ constitutes a projective space $\boldsymbol{L}'^{l(N,m)}$ of dimension $l(N, m)=\binom{N+m}{N}-1$.

LEMMA 1. *Let $\boldsymbol{H}_1$ and $\boldsymbol{H}_m$ be a hyperplane and a hypersurface in $\boldsymbol{L}^N$ such that* $\dim_k C(\boldsymbol{H}_m)=l(N, m)-g$ *and that the intersection-product $\boldsymbol{H}_1\cdot\boldsymbol{H}_m$ is defined, then we have* $\dim_{k(C(H_1))} C(\boldsymbol{H}_1\cdot\boldsymbol{H}_m)\geq l(N, m)-g-N$, *where k is any field over which $\boldsymbol{L}^N$ is defined and g any posive integer.*

PROOF. As is easily shown we have

$$\dim_{k(C(H_1))} C(\boldsymbol{H}_m)\geq l(N, m)-g-N\,.$$

Hence there exist[2] on $\boldsymbol{H}_m$ at least $l(N, m)-g-N$ independent

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1) Cf. B. L. van der Waerden, "Einführung in die algebraische Geometrie." Julius Springer in Berlin, 1939.

2) Cf. Y. Nakai, "Notes on Chow points of algebraic varieties." Mem. Coll. Sci., Univ. of Kyoto, vol. XXVIII, 1953.

generic points M_i of L^N over $k(C(H_1))$, $i=1, 2, \cdots, l(N, m)-g-N$. Let M_i' $(i=1, 2, \cdots, l(N-1, m)-g-N)$ be $l(N-1, m)-g-N$ independent generic points of H_1 over $k(C(H_1))$, then we have the following specialization

$$(M_1, \cdots, M_{l(N-1,m)-g-N}, M_{l(N-1,m)-g-N+1}, \cdots, M_{l(N,m)-g-N})$$
$$\rightarrow (M_1', \cdots, M'_{l(N-1,m)-g-N}, M_{l(N-1,m)-g-N+1}, \cdots, M_{l(N,m)-g-N})$$

with reference to $k(C(H_1))$. Let H_m' be a specialization of H_m over the above specialization. Then if $H_1\cdot H_m'$ is defined, $H_1\cdot H_m$ has the uniquely determined specialization $H_1\cdot H_m'$ over the specialization $H_m\rightarrow H_m'$ with reference to $k(C(H_1))$[3], therefore it follows that

$$\dim_{k(C(H_1))} C(H_1\cdot H_m)\geq \dim_{k(C(H_1))} C(H_1\cdot H_m').$$

But now, since the points M_i' $(i=1, 2, \cdots, l(N-1, m)-g-N)$ belong to H_m', we have

$$\dim_{k(C(H_1))} C(H_1\cdot H_m')\geq l(N-1, m)-g-N.$$

Thus we have only to prove that $H_1\cdot H_m'$ is defined. As is well known $H_1\cdot H_m'$ is defined if and only if H_1 is not a component of H_m'. Assume now that H_m' is of the form H_1+H_{m-1}', then since $l(N, m-1)+1$ points $M_{l(N-1,m)-g-N+1}, \cdots, M_{l(N,m)-g-N}$ lie on H_{m-1}', we have

$$\dim_{k(C(H_1))} C(H_{m-1}')\geq l(N, m-1)+1\,.$$

and this leads to a contradiction. q.e.d.

The following lemma plays an essential role in the proof of the theorems.

LEMMA 2. *Let L^N be a projective space defined over k, and r, d, g any given positive integers such that $1\leq r\leq N-1$. Then there exists an integer $m_0(r, d, g)$, depending only on r, d, g, with the following property; let H_m be a hypersurface of order m in L^N such that $\dim_k C(H_m)\geq l(N, m)-g$ and $m\geq m_0$, then H_m contains no subvarieties of degree not greater than d.*

PROOF. We shall use the induction on the dimension N of the ambient projective space. When $N=2$, the assertion is trivial. Suppose that the lemma is proved for a projective space of dimension $\leq N-1$ for any r, d, g, and we proceed to the case L^N.

3) Cf. T. Matsusaka, "Specialization of cycles on a projective model." Mem. Coll. Sci., Univ. of Kyoto, vol. XXVI; 1950.

As is well known the totality of positive cycles of dimension r and of degree d in a projective space L^N builds up a bunch of varieties $\mathfrak{B}(r, d; N)$ in a suitable projective space. Let us denote by $e(r, d; N)$ the maximal dimension of the components of $\mathfrak{B}(r, d; N)$. Then it is clear that $e(r, d; N) > e(r, d'; N)$ for $d > d'$.

Let H_1 be any hyperplane in L^N, then since H_1 is a $(N-1)$-dimensional projective space, our lemma is valid for H_1 by induction assumptions. Then for integers r, d, and $g+N$, there exists a positive integer m' satisfying the requirements in our lemma, i.e. if H'_m is a hypersurface in H_1 such that $m \geq m'$ and $\dim_{k(C(H_1))}(C(H'_m)) \geq l(N-1, m) - g - N$, then H'_m cannot contain any subvariety of dimension r and of degree $\leq d$. Let m'' be so chosen that $Nm'' \geq g + e(r, d; N)$ and m''' an integer such that for any integer $m \geq m'''$, H_m cannot contain H_1. We put $m_0 = \max(m', m'', m''')$. Then m_0 will be shown to satisfy all our requirements in our lemma.

Assume that for some $m \geq m_0$ there exists a hypersurface H_m in L^N such that $\dim_k(C(H_m)) \geq l(N, m) - g$ and contains a subvariety Γ of dimension r and of degree $d' (\leq d)$. Let Y be the locus of $C(H_m)$ over the algebraic closure of $k(C(\Gamma))$ and Z a subvariety of $\mathfrak{B}(N-1, m; N)$ composed of m hyperplanes in L^N. Then we have

$$\begin{aligned}\dim Y + \dim Z &\geq l(N, m) - g - \dim_k(C(\Gamma)) + Nm \\ &\geq l(N, m) - g - e(r, d'; N) + Nm \\ &\geq l(N, m) - g - e(r, d; N) + Nm \\ &\geq l(N, m) + Nm'' - g - e(r, d; N) \\ &\geq l(N, m).\end{aligned}$$

Hence $Y \cap Z \neq \phi$. This shows that there exists a specialization of H_m to $\sum_{i=1}^{m} H_1^{(i)}$ over $k(C(\Gamma))$. Since the inclusion relations are preserved under specializations of cycles, Γ must be contained in some hyperplane, say, $H_1^{(1)}$, hence in $H_1^{(1)} \cap H_m$. But now since $m \geq m'''$, $H_m \cdot H_1^{(1)}$ is defined and by Lemma 1 $H'_m = H_1^{(1)} \cdot H_m$ is a hypersurface in $H_1^{(1)}$ such that $\dim_{k(C(H_1^{(1)}))} C(H'_m) \geq l(N-1, m) - g - N$. Hence from the choice of m $(\geq m')$, H'_m cannot contain any variety of dimension r and of degree $\leq d$ and this is a contradiction. Thus the proof is completed. q.e.d.

Now we can state the theorems.

THEOREM 1. *Let* V^n $(n\geq 2)$ *be a projective model in* L^N, *and* k *a field of definition for* V *and* L. *Let* g *be an arbitrary integer, then there exists a positive integer* s *such that the hypersurface section of* V *with* H_m *is defined and irreducible provided* $m\geq s$ *and* $\dim_k(C(H_m))\geq l(N, m)-g$.

PROOF. Let t be an integer such that $l(N, t)-\varphi(t)\geq g$[4] and $d=t\deg(V)$, where $\deg(V)$ is the degree of V in L^N. Then for given integers $n-1$, d, g there exists an integer m_0 satisfying the conditions of Lemma 2. We put $s=\max(m_0, t+1)$. Then s will satisfy the requirements of the theorem.

Let H_m be a hypersurface in L^N such that $m\geq s$ and $\dim_k(C(H_m))\geq l(N, m)-g$, and Y the locus of $C(H_m)$ over the algebraic closure of k. Let H_{m-t} be a hypersurface of order $m-t$ such that $V\cdot H_{m-t}$ is defined and irreducible and $\bar{H}_t$ a generic hypersurface of order t with reference to $k(C(H_{m-t}))$. Let Z be the locus of $C(H_{m-t}+\bar{H}_t)$ over the algebraic closure of $k(C(H_{m-t}))$. Then we have $\dim Z=l(N, t)\geq g+\varphi(t)$. Hence $Y\cap Z$ contains a component of dimension $\geq\varphi(t)$ and there exists a point in $Y\cap Z$ such that the corresponding hypersurface H' intersects properly with V. Now H' is of the form $H_{m-t}+H_t$ and hence $V\cdot H_m$ has the specialization of the form $V\cdot H_{m-t}+V\cdot H_t$. Since $V\cdot H_{m-t}$ is irreducible, $V\cdot H_m$ must contain a component whose degree is not less than $(m-t)\deg(V)$. Then if $V\cdot H_m$ is reducible, it must also contain a component of degree not greater than d by the theorem of Bezout. But it contradicts to the choice of number s by Lemma 2. Thus the proof is completed. q.e.d.

The following theorem follows immediately from THEOREM 1.

THEOREM 2 *Let* V^n $(n\geq 2)$ *be a projective model in* L^N *and* U *be the parameter variety of the linear systems cut out on* V *by hypersurfaces of some order* m *in* L^N. *The points of* U *such that the corresponding members of the linear systems are not irreducible build up a bunch of subvarieties* W *in* U. *Let* $s(m)$ *be the dimension of the components of* U *and* $s'(m)$ *be the maximal dimension of the components of* W. *The* $\lim_{m\to\infty}(s(m)-s'(m))=\infty$.

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4) Let $\mathfrak{a}$ be the defining homogeneous ideal of V in $k[X_0, X_1, \cdots, X_n]$, then $\varphi(t)$ denotes the maximal number of linearly independent forms of order t contained in $\mathfrak{a}$. Then we see easily that if $\dim_k(C(H_t))\geq\varphi(t)$, H_t cannot contain V, that is, $V\cdot H_t$ is defined.

As an application of the above theorems we can see easily the following: let $\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots, \boldsymbol{P}_s$ be a set of finite number of points on a projective model $\boldsymbol{V}$, then there exists a subvariety $\boldsymbol{V}'$ of $\boldsymbol{V}$ of any positive dimension, containing all the points $\boldsymbol{P}_i$ $(i=1, 2, \cdots, s)$ such that $\boldsymbol{V}'$ is the complete intersection of $\boldsymbol{V}$ with hypersurfaces of the ambient projective space.[5)]

5) Cf. H. Nishimura and Y. Nakai, "On the existence of a curve connecting given points on abstract varieties," forthcoming.