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On the derived normal rings of Noetherian integral domains

By

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It was communicated to the writer that Y. Mori [4, II][1] proved that the derived normal ring of a Noetherian integral domain of rank 2 is also Noetherian.[2] In the present paper we want to give a new proof of the result. In the way, we shall show a detailed and a little clearer proof of the result due to Y. Mori [4] that the derived normal ring of a Noetherian integral domain is a Krull ring.[3]

Terminology and results stated in Nagata [7] will be used freely. Further, some basic results on local rings (see, for example [8, § 1]) will be used freely.

§ 1. Krull-Akizuki's theorem.

PROPOSITION 1 (Krull-Akizuki's theorem).[4] *Let $\mathfrak{o}$ be a Noetherian integral domain of rank* 1 *and let K be the field of quotients of $\mathfrak{o}$. Let L be a finite algebraic extension of K. If $\mathfrak{o}'$ is a subring of L containing $\mathfrak{o}$, then for every ideal $\mathfrak{a}'(\neq 0)$ of $\mathfrak{o}'$, $\mathfrak{o}'/\mathfrak{a}'$ is a finite $\mathfrak{o}/(\mathfrak{a}'\cap\mathfrak{o})$-module. Consequently, $\mathfrak{o}'$ is Noetherian.*

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1) The result was anounced by him at the Autumn meeting of the Mathematical Society of Japan in 1953.

2) As was shown by Nagata [6], i) there exists an example of Noetherian integral domain $\mathfrak{o}$ of rank 2 which has an integral extension $\mathfrak{o}'$ contained in the derived normal ring of $\mathfrak{o}$ such that $\mathfrak{o}'$ is not Noetherian and ii) there exists an example of Noetherian integral domain of rank 3 such that the derived normal ring is not Noetherian. On the other hand, as is well known as Krull-Akizuki's theorem, when $\mathfrak{o}$ is a Noetherian integral domain of rank 1, then every almost finite integral extension of $\mathfrak{o}$ is Noetherian (see § 1).

3) For the definition, see § 2.

4) The writer owes the present formulation of the theorem to Cohen [2].

PROOF. (1) Consider a finite integral extension $\mathfrak{o}''$ of $\mathfrak{o}$ such that i) $\mathfrak{o}''$ is contained in $\mathfrak{o}'$ and ii) $\mathfrak{o}''$ and $\mathfrak{o}'$ have the same field of quotients. Then obviously $\mathfrak{o}''$ is Noetherian and of rank 1. Further for every ideal $\mathfrak{a}''(\neq 0)$ of $\mathfrak{o}''$, $\mathfrak{o}''/\mathfrak{a}''$ is a finite $\mathfrak{o}/(\mathfrak{a}''\cap\mathfrak{o})$-module. Therefore we may assume that $\mathfrak{o}=\mathfrak{o}''$, i.e., $L=K$.

(2) Set $\mathfrak{a}=\mathfrak{a}'\cap\mathfrak{o}$ and let $\mathfrak{p}_1,\cdots\mathfrak{p}_n$ be prime divisors of $\mathfrak{a}$. Further let S be the intersection of complementary sets of $\mathfrak{p}_i$'s with respect to $\mathfrak{o}$. Then $\mathfrak{o}'/\mathfrak{a}'=\mathfrak{o}'_S/\mathfrak{a}'\mathfrak{o}'_S$ and $\mathfrak{o}/\mathfrak{a}=\mathfrak{o}_S/\mathfrak{a}\mathfrak{o}_S$ (because every element of S is a unit modulo $\mathfrak{a}$). Therefore we may assume that $\mathfrak{p}_1,\cdots,\mathfrak{p}_n$ are all of maximal ideals in $\mathfrak{o}$, that is, $\mathfrak{a}$ is contained in the J-radical of $\mathfrak{o}$.

(3)[5] Now we shall prove the assertion under the assumption that $\mathfrak{o}'$ is integral over $\mathfrak{o}$. Let x be an element of $\mathfrak{a}$ different from zero. Let $y_1,\cdots,y_s$ be arbitrary elements of $\mathfrak{o}'$. Set $\mathfrak{o}^*=\mathfrak{o}[y_1,\cdots,y_s]$. Since x is an element of the J-radical of $\mathfrak{o}$ and since $\mathfrak{o}$ is of rank 1, there exists an integer a such that $x^a\mathfrak{o}^*\subseteq\mathfrak{o}$. On the other hand, set $h=l(\mathfrak{o}/x\mathfrak{o})$ and $h^*=l(\mathfrak{o}^*/x\mathfrak{o}^*;\mathfrak{o})$.[6] Under the isomorphism from $\mathfrak{o}$ onto $x\mathfrak{o}$ which mapps z to xz, $x^r\mathfrak{o}$ is mapped onto $x^{r+1}\mathfrak{o}$ and therefore $l(x^r\mathfrak{o}/x^{r+1}\mathfrak{o})=h$ for every r. Similarly $l(x^r\mathfrak{o}^*/x^{r+1}\mathfrak{o}^*;\mathfrak{o})=h^*$. From $x^a\mathfrak{o}^*\subseteq\mathfrak{o}$, we have $l(x^a\mathfrak{o}^*/x^{r+a}\mathfrak{o}^*;\mathfrak{o})\leq l(\mathfrak{o}/x^{r+a}\mathfrak{o})$ and $rh^*\leq(r+a)h$ for every r. Therefore we have $h^*\leq h$.[7] Since this holds for every finite integral extension of $\mathfrak{o}$ contained in $\mathfrak{o}'$, we have $l(\mathfrak{o}'/x\mathfrak{o}';\mathfrak{o})\leq l(\mathfrak{o}/x\mathfrak{o})$ and the proof is completed in this case.

(4) Now we shall prove the general case. Let $\mathfrak{o}''$ be the integral closure of $\mathfrak{o}$ in $\mathfrak{o}'$. Then since $\mathfrak{o}''$ is Noetherian (of rank 1) and since $\mathfrak{o}''/(\mathfrak{a}'\cap\mathfrak{o}'')$ is a finite $\mathfrak{o}/\mathfrak{a}$-module by (3), we may assume that $\mathfrak{o}=\mathfrak{o}''$ (and we repeat the same reduction in (2) above). Then the assertion follows from

PROPOSITION 2. *Let $\mathfrak{o}$ be a Noetherian integral domain (of an arbitrary rank) and let $\mathfrak{o}'$ be a subring of the field of quotients of $\mathfrak{o}$ which contains $\mathfrak{o}$. If $\mathfrak{o}$ is integrally closed in $\mathfrak{o}'$ and if, for every prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, there exists a prime ideal of $\mathfrak{o}'$ which lies over $\mathfrak{p}$, then we have $\mathfrak{o}=\mathfrak{o}'$.*

PROOF. Assuming the contrary, let a be an element of $\mathfrak{o}'$

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5) The writer owes an important idea of the present proof to Chevalley [1].

6) The symbol $l(\ )$ denotes the length of the primary ring in the parentheses and the symbol $l(\ ;\mathfrak{o})$ denotes the length of the module in the parentheses as an $\mathfrak{o}$-module.

7) We can prove that $h^*=h$ by the same way as in the theory of multiplicity (see [8]).

which is not in $\mathfrak{o}$ and let $\mathfrak{a}$ be the set of elements of $\mathfrak{o}$ whose product with a is in $\mathfrak{o}$; $\mathfrak{a}$ is an ideal of $\mathfrak{o}$. Let $\mathfrak{p}$ be a prime divisor of $\mathfrak{a}$ and set $\mathfrak{b}=\mathfrak{a}:\mathfrak{p}$. Since $\mathfrak{b}\neq\mathfrak{a}$, $a\mathfrak{b}\not\subseteq\mathfrak{o}$ and therefore there exists an element b of $a\mathfrak{b}$ which is not in $\mathfrak{o}$. Since $\mathfrak{p}\mathfrak{o}'\cap\mathfrak{o}=\mathfrak{p}$ and since $a\mathfrak{b}\mathfrak{p}\subseteq\mathfrak{o}$, we have $b\mathfrak{p}\subseteq\mathfrak{p}$. Therefore we have easily $b^n\mathfrak{p}\subseteq\mathfrak{p}$ for every integer $n(>0)$ by induction on n, which shows that b is integral over $\mathfrak{o}$ and we have a contradiction. Thus we have $\mathfrak{o}'=\mathfrak{o}$.

COROLLARY. *Le $\mathfrak{o}$ be a Noetherian integral domain of rank* 1 *and let $\mathfrak{o}'$ be an almost finite integral extension of $\mathfrak{o}$. Then $\mathfrak{o}'$ is a Noetherian integral domain of rank* 1. *Furthermore, if $\mathfrak{p}$ is a maximal ideal of $\mathfrak{o}$, then there exist only a finite number of maximal ideals in $\mathfrak{o}'$ which lie over $\mathfrak{p}$; if a maximal ideal ideal $\mathfrak{p}'$ of $\mathfrak{o}'$ lies over $\mathfrak{p}$, then $\mathfrak{o}'/\mathfrak{p}'$ is a finite algebraic extension of $\mathfrak{o}/\mathfrak{p}$.*

(The last half of this corollary will be generalized in § 3.)

§ 2. Krull rings.

We say that an integral domain $\mathfrak{o}$ is a *Krull ring* if the following conditions are satisfied:

(1) *If $\mathfrak{p}$ is a prime ideal of rank* 1 *in $\mathfrak{o}$, then $\mathfrak{o}_{\mathfrak{p}}$ is a discrete valuation ring.*

(2) *Every principal ideal in $\mathfrak{o}$ is the intersection of a finite number of primary ideals of rank* 1.

REMARK 1. The above condition (2) is equivalent to the following two conditions:

(2–1) *Every principal ideal in $\mathfrak{o}$ has only a finite number of minimal prime divisors.*

(2–2) *$\mathfrak{o}$ is the intersection of all $\mathfrak{o}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of rank* 1.

PROOF. Assume that (2) holds good. Then (2–1) holds obviously. Let d be an element of $\mathfrak{d}=\cap\mathfrak{o}_{\mathfrak{p}}$ ($\mathfrak{p}$ runs over all prime ideals of rank 1). d can be expressed as a/b $(a, b\in\mathfrak{o})$. Then $d\in\mathfrak{d}$ shows that $a\mathfrak{o}_{\mathfrak{p}}\subseteq b\mathfrak{o}_{\mathfrak{p}}$ for every $\mathfrak{o}_{\mathfrak{p}}$ and $a\mathfrak{o}\subseteq b\mathfrak{o}$ by (2), which shows that $\mathfrak{d}=\mathfrak{o}$. Conversely, assume that (2–1) and (2–2) hold. For an element a of $\mathfrak{o}$, let $\mathfrak{a}$ be the intersection of primary components of $a\mathfrak{o}$ belonging to prime divisors of rank 1. If $b\in\mathfrak{a}$, then b/a is in all $\mathfrak{o}_{\mathfrak{p}}$, $\mathfrak{p}$ being a prime ideal of rank 1. Therefore $b/a\in\mathfrak{o}$ by (2–2). Hence $\mathfrak{a}=a\mathfrak{o}$ and (2) holds good.

REMARK 2. By the conditions (1) and (2–2), we see that a

Krull ring is a normal ring. As is well known, a Noetherian normal ring is a Krull ring. In the non-Noetherian case, there are normol rings which are not Krull rings (for instance, non-discrete valuation rings are normal but are not Krull rings).

REMARK 3. Let $\mathfrak{o}$ be an integral domain and let P be a set of prime ideals of $\mathfrak{o}$ such that if $\mathfrak{p}$ is a prime divisor of a principal ideal in $\mathfrak{o}$ then there exists a member of P which contains $\mathfrak{p}$. Then we have $\mathfrak{o}=\cap_{\mathfrak{p}\in P}\mathfrak{o}_{\mathfrak{p}}$, as is easily seen by the same way as in the proof of Remark 1.

PROPOSITION 3. *An integral domain $\mathfrak{o}$ is a Krull ring if and only if there exists a family F of discrete valution rings of the field of quotients K of $\mathfrak{o}$ such that* 1) *$\mathfrak{o}$ is the intersection of all rings in F and* 2) *every non-zero element of $\mathfrak{o}$ is a unit in all, but a finite number, of rings in F.* (Krull [3])

PROOF. If $\mathfrak{o}$ is a Krull ring, then the family F of all $\mathfrak{o}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of rank 1 in $\mathfrak{o}$, satisfies the above conditions 1) and 2) by virtue of Remark 1. In order to prove the converse, we shall prove a lemma:

We shall say for a moment that a ring is a K-ring if it satisfies the conditions stated in our proposition. Then

LEMMA 1. *If $\mathfrak{o}$ is a* K-*ring, then every ring of quotients of $\mathfrak{o}$ is also a* K-*ring.*

PROOF. Let F be the family of discrete valuation rings in the condition and let S be a multiplicatively closed subset of $\mathfrak{o}$ which does not contain zero. Let F' be the set of rings in F in which every element of S is a unit. Set $\mathfrak{d}=\cap_{\mathfrak{v}\in F'}\mathfrak{v}$. We have only to show that $\mathfrak{o}_S=\mathfrak{d}$. Obviously $\mathfrak{d}$ contains $\mathfrak{o}_S$ and we shall show the converse inclusion. Let a/b $(a, b\in\mathfrak{o})$ be an element of $\mathfrak{d}$ and let $\mathfrak{v}_1,\cdots,\mathfrak{v}_n$ be all rings in F in which b is a non-unit; we renumber them so that $\mathfrak{v}_i\in F'$ if and only if $i\leq r$. For each $i>r$, there exists an element $s_i\in S$ which is a non-unit in $\mathfrak{v}_i$. Then choosing a power s of the product of s_i's, we have $sa/b\in\mathfrak{v}_i$ for every $i>r$. Since a/b is already in $\mathfrak{v}_i$ for $i\leq r$, we have $sa/b\in\mathfrak{v}_i$ for every i. Since b is a unit in every ring in F other than $\mathfrak{v}_i$'s, we have $sa/b\in\cap_{\mathfrak{v}\in F}\mathfrak{v}=\mathfrak{o}$. Since S is multiplicatively closed, s is in S and a/b is in $\mathfrak{o}_S$. Thus the lemma is settled.

REMARK. By our proof and by our treatment in 1) below, we see the following fact:

If $\mathfrak{o}$ is Krull ring, if F is a set of discrete valuation rings as in

Proposition 3 *and if* $\mathfrak{p}$ *is a prime ideal of rank* 1 *in* $\mathfrak{o}$, *then* $\mathfrak{o}_{\mathfrak{p}}$ *must be in* F.

Now we shall return to the proof of the proposition.

1) If $\mathfrak{p}$ is a prime ideal of rank 1 in $\mathfrak{o}$, then $\mathfrak{o}_{\mathfrak{p}}$ is a K-ring by Lemma 1. Then a discrete valuation ring (of the field of of quotients K of $\mathfrak{o}_{\mathfrak{p}}$ which is different from K itself) can contain $\mathfrak{o}_{\mathfrak{p}}$ if and only if it dominates $\mathfrak{o}_{\mathfrak{p}}$. Therefore by the second condition in Proposition 3, $\mathfrak{o}_{\mathfrak{p}}$ is the intersection of a finite number of discrete valuation rings. Since the intersection of n valuation rings has n maximal ideals, provided that there is no inclusion relation among the valuation rings, we see that $\mathfrak{o}_{\mathfrak{p}}$ must be a discrete valuation ring, which proves the condition (1).

(2) Let a be an element of $\mathfrak{o}$ different from zero and let $\mathfrak{v}_1, \cdots, \mathfrak{v}_n$ be all rings in F (which is in the proposition) in which a is a non-unit. Then for an element b of $\mathfrak{o}$, b/a is in $\mathfrak{o}$ if and only if $b/a \in \mathfrak{v}_i$ for every i. Therefore $a\mathfrak{o}=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ with $\mathfrak{q}_i=\mathfrak{o} \cap a\mathfrak{v}_i$. Let $\mathfrak{p}'_i$ be the maximal ideal of $\mathfrak{v}_i$ and set $\mathfrak{p}_i=\mathfrak{p}'_i \cap \mathfrak{o}$. Then, since $\mathfrak{v}_i$ is a discrete valuation ring, we see that $\mathfrak{q}_i$ is a strongly primary ideal belonging to $\mathfrak{p}_i$. Thus we have

(*) Any principal ideal of $\mathfrak{o}$ is the intersection of a finite number of strongly primary ideals.

This property shows (2–1) as a special case. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ of rank greater than 1. Then we have only to show that $\mathfrak{p}$ is not a prime divisor of any principal ideal of $\mathfrak{o}$. Assume the contrary, i.e., assume that $\mathfrak{p}$ is a prime divisor of a principal ideal $a\mathfrak{o}$. Then by the property (*) we see that $\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$ is a prime divisor of $a\mathfrak{o}_{\mathfrak{p}}$. On the other hand, $\mathfrak{o}_{\mathfrak{p}}$ is a K-ring by Lemma 1. Therefore we may assume that $\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{o}$. Since $\mathfrak{o}$ is not a discrete valuation ring (because rank $\mathfrak{p} \geq 2$), $\mathfrak{o}$ is not the intersection of a finite number of discrete valuation rings. Therefore there exists a discrete valuation ring $\mathfrak{v} \in F$ such that $a\mathfrak{v}=\mathfrak{v}$. Let $\mathfrak{q}'$ be the maximal ideal of $\mathfrak{v}$ and set $\mathfrak{q}=\mathfrak{q}' \cap \mathfrak{o}$. Let h be an element of $\mathfrak{q}$. By (*), we have $a\mathfrak{o} : \mathfrak{p} \neq a\mathfrak{o}$. Let b be an element of $a\mathfrak{o} : \mathfrak{p}$ which is not in $a\mathfrak{o}$. Then since $h \in \mathfrak{q} \subseteq \mathfrak{p}$, we have $bh \in a\mathfrak{o}$ and $z=(b/a)h$ is in $\mathfrak{o}$. Since a is a unit in $\mathfrak{v}$ and since h is a non-unit in $\mathfrak{v}$, z is a non-unit in $\mathfrak{v}$ and z is in $\mathfrak{q}$. Therefore $(b/a)^n h$ is in $\mathfrak{q}$ for every integer $n(>0)$, as is easily seen by induction on n. Let w be a valuation whose ring is in F. Then $(b/a)^n h \in \mathfrak{o}$ shows that $n \cdot w(b/a)+w(h) \geq 0$. Since $w(h)$ is fixed, since n is arbitrary and since w is discrete, we have $w(b/a) \geq 0$. Thus we have $b/a \in \mathfrak{o}$, which is a contradiction. Thus (2) is proved. Therefore every K-ring is a Krull ring, and the proposition is proved completely.

§ 3. The derived normal rings of Noetherian integral domains (I).

We want to prove here the following

THEREM 1. *The derived normal ring $\mathfrak{o}'$ of a Noetherian local integral domain $\mathfrak{o}$ is a Krull ring. Further, $\mathfrak{o}'$ has only a finite numcer of maximal ideals and each residue class field of $\mathfrak{o}'$ is a finite algebraic extension of that of $\mathfrak{o}$.*[8)]

PROOF. Let $\mathfrak{o}^*$ be the completion of $\mathfrak{o}$, let $\mathfrak{n}^*$ be the radical of $\mathfrak{o}^*$ and let K be the field of quotients of $\mathfrak{o}$. Set $\mathfrak{r}=\mathfrak{o}^*/\mathfrak{n}^*$. Since $\mathfrak{n}^* \cap \mathfrak{o}=0$, we may regard that $\mathfrak{o}$ is a subring of $\mathfrak{r}$. Every element $a(\neq 0)$ of $\mathfrak{o}$ is not a zero-divisor in $\mathfrak{o}^*$, hence a is not in any prime divisor of $\mathfrak{n}^*$, which shows that a is not a zero-divisor in $\mathfrak{r}$. Hence K is a subfield of the total quotient ring R of $\mathfrak{r}$. Let $\mathfrak{r}'$ be the integral closure of $\mathfrak{r}$ in R. Then we have

(*) $\mathfrak{r}' \cap K=\mathfrak{o}'$.

Indeed, it is obvious that $\mathfrak{o}'$ is contained in $\mathfrak{r}' \cap K$. Let a/b $(a, b \in \mathfrak{o})$ be an element of $\mathfrak{r}' \cap K$. Since $a/b \in \mathfrak{r}'$, there exist $c_1, \cdots, c_n \in \mathfrak{r}$ such that $(a/b)^n+c_1(a/b)^{n-1}+\cdots\cdots+c_n=0$. Let $d_1, \cdots, d_n$ be representatives of $c_1, \cdots, c_n$ in $\mathfrak{o}^*$. Then we have $a^n+d_1a^{n-1}b+\cdots+d_n b^n \in \mathfrak{n}^*$. Since every element of $\mathfrak{n}^*$ is nilpotent, we see that there exist elements $e_1, \cdots, e_m$ of $\mathfrak{o}^*$ such that $a^m+e_1a^{m-1}b+\cdots+e_mb^m=0$, which shows that $a^m \in (\sum_1^m a^{m-i}b^i\mathfrak{o}^*) \cap \mathfrak{o}=\sum_1^m a^{m-i}b^i \mathfrak{o}$. Therefore the above e_i's can be choosen from $\mathfrak{o}$. Hence a/b is integral over $\mathfrak{o}$ and $a/b \in \mathfrak{o}'$. Thus (*) is proved.

Since the derived normal ring of a complete Noetherian local integral domain $\mathfrak{s}$ is a finite $\mathfrak{s}$-module (for the proof, see [5]) and since $\mathfrak{r}$ is a subdirect sum of complete local integral domains, we see that $\mathfrak{r}'$ is a finite $\mathfrak{r}$-module and $\mathfrak{r}'$ is the direct sum of normal Noetherian rings, say $\mathfrak{r}_1, \cdots, \mathfrak{r}_t$. For every prime ideal $\mathfrak{p}^*_i$ of rank 1 in $\mathfrak{r}_i$, we set $v(\mathfrak{p}^*_i)=K_1+\cdots K_{i-1}+(\mathfrak{r}_i)_{\mathfrak{p}^*_i}+K_{i+1}+\cdots+K_t$, where K_j is the field of quotients of $\mathfrak{r}_j$. Since each $\mathfrak{r}_i$ is a Noetherian

8) The first half of this theorem was proved by Mori [4], while the finiteness of maximal ideals was proved by Chevalley [1].

normal ring, $\mathfrak{r}'$ is the intersection of all the $\mathfrak{v}(\mathfrak{p}^*_i)$'s. On the other hand, $\mathfrak{v}(\mathfrak{p}^*_i) \cap K$ is a discrete valuation ring of K (may be K itself). We have $\mathfrak{o}'=K\cap\mathfrak{r}'=\cap\,(K\cap\mathfrak{v}(\mathfrak{p}^*_i))$. Since that an element $a(\neq 0)$ is a non-unit in $K\cap\mathfrak{v}(\mathfrak{p}^*_i)$ means that a is in the rank 1 prime ideal $\mathfrak{r}_1+\cdots+\mathfrak{r}_{i-1}+\mathfrak{p}^*_i+\mathfrak{r}_{i+1}+\cdots+\mathfrak{r}_t$, we see that the set of valuation rings $K\cap\mathfrak{v}(\mathfrak{p}^*_i)$ satisfies the conditions in Proposition 3 and $\mathfrak{o}'$ is a Krull ring. In order to prove the last half of the theorem, we consider an arbitrary finite integral extension $\mathfrak{o}''$ of $\mathfrak{o}$ contained in $\mathfrak{o}'$. Then the completion of $\mathfrak{o}''$ is the ring $\mathfrak{o}^*[\mathfrak{o}'']$. Therefore the number of maximal ideals of $\mathfrak{o}''$ is not greater than the number of prime divisors of zero of $\mathfrak{o}^*$ and each residue class field of $\mathfrak{o}''$ is a subfield of one of residue class fields of $\mathfrak{r}'$. Therefore the same is true of $\mathfrak{o}'$ and the theorem is proved completely.

COROLLARY. *Let $\mathfrak{o}'$ be the derived normal ring of a Noetherian integral domain $\mathfrak{o}$. If $\mathfrak{p}'$ is a prime ideal of $\mathfrak{o}'$, then $\mathfrak{o}'/\mathfrak{p}'$ is an almost finite integral extension of $\mathfrak{o}/(\mathfrak{p}'\cap\mathfrak{o})$. On the other hand, if $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$, there exist only a finite number of prime ideals of $\mathfrak{o}'$ which lies over $\mathfrak{p}$.*

PROOF. We may assume that $\mathfrak{p}=\mathfrak{p}'\cap\mathfrak{o}$. Let S be the complementary set of $\mathfrak{p}$ with resect to $\mathfrak{o}$. Then $\mathfrak{o}'_S$ is the derived normal ring of $\mathfrak{o}_\mathfrak{p}$. Applying the theorem to $\mathfrak{o}_\mathfrak{p}$ and observing that $\mathfrak{p}'\mathfrak{o}'_S$ is maximal, we see the assertion easily.

§ 4. The derived normal rings of Noetherian integral domains (II).

In the present section, we want to prove the following

THEOREM 2. *If $\mathfrak{o}'$ is the derived normal ring of a Noetherian integral domain $\mathfrak{o}$, then $\mathfrak{o}'$ is a Krull ring.*

In order to prove this theorem, we first state a remark on imbedded prime divisors, namely

LEMMA 2. *Let $\mathfrak{o}$ be a Noetherian local integral domain of rank >1. Then a maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$ is an imbedded prime divisor of a principrl ideal $a\mathfrak{o}\,(a \in \mathfrak{o})$ if and only if there exists an element $b \in \mathfrak{o}$ such that $b\notin a\mathfrak{o}$ and $\mathfrak{m}$ is the conductor of $\mathfrak{o}[b/a]$ over $\mathfrak{o}$.*[9]

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9) This can be generalized by the same way to general Noetherian rings, assuming that a is not a zero-divisor. Therefore, the corollary can be generalized, if we talk about principal ideals generated by nonzero-divisors, to general Noetherian rings.

This follows immeditely from a proposition in [7, § 9].

COROLLARY *If a prime ideal* $\mathfrak{p}$ *of a Noetherian integral domain* $\mathfrak{o}$ *is a prime divisor of a principal ideal of* $\mathfrak{o}$, *then for every element* $a(\neq 0)$ *of* $\mathfrak{p}$, $\mathfrak{p}$ *is a prime divisor of* $a\mathfrak{o}$.

PROOF. When $\mathfrak{p}$ is of rank 1, the assertion is obvious. When $\mathfrak{p}$ is not of rank 1, then Lemma 2 can be applied to $\mathfrak{o}_{\mathfrak{p}}$ and the corollary is proved.

Next we prove

LEMMA 3. *Let* $\mathfrak{o}'$ *be the derived normal ring of a Noetherian integral domain* $\mathfrak{o}$. *If* $\mathfrak{p}'$ *is a minimal prime divisor of a principal ideal* $a\mathfrak{o}'$ $(a \in \mathfrak{o}, a\neq 0,\ a\mathfrak{o}\neq\mathfrak{o})$, *then* $\mathfrak{p}=\mathfrak{p}'\cap\mathfrak{o}$ *is a prime divisor of* $a\mathfrak{o}$.

PROOF. Let S be the complementary set of $\mathfrak{p}$ with respect to $\mathfrak{o}$. Then $\mathfrak{o}'_S$ is the derived normal ring of $\mathfrak{o}_{\mathfrak{p}}$ and $\mathfrak{p}'\mathfrak{o}'_S$ is a minimal prime divisor of $a\mathfrak{o}'_S$. Therefore we may assume that $\mathfrak{o}$ is a local ring with maximal ideal $\mathfrak{p}$. Let x be an element of $\mathfrak{p}'$ which is not in any other maximal ideal of $\mathfrak{o}'$ and set $\mathfrak{o}''=\mathfrak{o}[x]$, $\mathfrak{p}''=\mathfrak{p}'\cap\mathfrak{o}''$. Then $\mathfrak{p}'$ is the unique prime ideal which lies over $\mathfrak{p}''$. Since $\mathfrak{p}'$ is of rank 1 (because of the validity of Theorem 1), we have rank $\mathfrak{p}''=1$. Since $\mathfrak{o}''/x\mathfrak{o}''$ is a homomorphic image of the local ring $\mathfrak{o}$, we have $x\mathfrak{o}''$ is a primary ideal belonging to $\mathfrak{p}''$. Since $\mathfrak{p}''$ is of rank 1 and since $\mathfrak{p}''$ lies over $\mathfrak{p}$, $\mathfrak{p}''$ is a minimal prime divisor of $c\mathfrak{o}''$ for every element $c(\neq 0)$ of $\mathfrak{p}$. Since the property that $\mathfrak{p}$ is a prime divisor of a principal ideal of $\mathfrak{o}$ does not depend on the choice of principal ideals by the corollary to Lemma 2, we may assume that $a\mathfrak{o}''$ is contained in $\mathfrak{o}$. Since $\mathfrak{p}''$ is maximal, we can choose an element y of $\mathfrak{p}''$ which is not in any other maximal ideal of $\mathfrak{o}''$ such that $y(1-y) \in a\mathfrak{o}''$ (consider the ring $\mathfrak{o}''/a\mathfrak{o}''$). Set $b=y(1-y)$. Since $a\mathfrak{o}''\subseteq\mathfrak{o}$, b is in $\mathfrak{o}$. Now we may replace x by y and $\mathfrak{o}''$ will denote $\mathfrak{o}[y]$. Since $y\mathfrak{o}''$ is a primary ideal belonging to $\mathfrak{p}''$, $y\mathfrak{o}''\cap\mathfrak{o}$ contains a power of $\mathfrak{p}$, say $\mathfrak{p}^n$. Then for every element d of $\mathfrak{p}$, there exists a relation such that $d^n=y(u+vy)$ $(u, v \in \mathfrak{o})$, because y satisifes a monic equation of degree 2. Since $vy^2=vy-vb$, we have $d^n=(u+v)y-vb$ and $(u+v)y=d^n+vb$. We consider the set of all paires (f, g) of elements of $\mathfrak{o}$ such that $fy=g$; let $\mathfrak{n}$ be the set of all such f and $\mathfrak{n}'$ that of g $(\mathfrak{n}'=\mathfrak{n}y)$. Since y is not in any prime ideal which lies over some prime ideal of $\mathfrak{o}$ other than $\mathfrak{p}$, we see that the set of prime ideals of $\mathfrak{o}$ which contains f coincides with that of g for each pair (f, g); the same can be applied to a and the element $y(1-y)=b$ and we see that b is in every prime divisor of $a\mathfrak{o}$. Now since d is an arbitrary element of $\mathfrak{p}$, we have $\mathfrak{n}'+a\mathfrak{o}$ is a primary ideal belonging to $\mathfrak{p}$ and therefore $\mathfrak{n}+a\mathfrak{o}$ is a primary ideal belonging to $\mathfrak{p}$. Since $\mathfrak{n}$ contains $a\mathfrak{o}$, we have $\mathfrak{n}$ is a primary ideal belonging to $\mathfrak{p}$, which shows that there exists an integral extension $\mathfrak{o}^*(\neq\mathfrak{o})$ such that $\mathfrak{p}\mathfrak{o}^*\subseteq\mathfrak{o}$ and $\mathfrak{p}$ is a prime divisor of $a\mathfrak{o}$ by Lemma 2.

Now we shall return to the proof of Theorem 2. The validity of our assertion in local case (Theorem 1) shows in particular the validity of the conditions (1) and (2-2) in § 2 (definition of Krull ring). The validity of (2-1) follows from Lemma 3 and the corollary to Theorem 1. Thus the theorem is proved completely.

§ 5. The derived normal ring of a Noetherian integrial domain of rank 2.

LEMMA 4. *A ring $\mathfrak{o}$ is Noetherian if (and only if) every prime ideal of $\mathfrak{o}$ has a finite base.* (Cohen [2])

PROOF.[10] Assuming the contrary, let P be the set of ideals of $\mathfrak{o}$ which have no finite bases. Then P is an inductive set and contains a maximal member $\mathfrak{a}$ by Zorn's lemma. Since $\mathfrak{a}\in P$, $\mathfrak{a}$ is not a prime ideal and therefore there exist ideals $\mathfrak{b}$ and $\mathfrak{c}$ which contain $\mathfrak{a}$ properly such that $\mathfrak{b}\mathfrak{c}\subseteq\mathfrak{a}$. By the maximality of $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ have finite bases and so does $\mathfrak{b}\mathfrak{c}$, too. Further $\mathfrak{o}/\mathfrak{c}$ is Noetherian. $\mathfrak{b}/\mathfrak{b}\mathfrak{c}$ can be regarded as a finite module over $\mathfrak{o}/\mathfrak{c}$ and $\mathfrak{a}/\mathfrak{b}\mathfrak{c}$ is its submodule. Since $\mathfrak{o}/\mathfrak{b}$ is Noetherian, $\mathfrak{a}/\mathfrak{b}\mathfrak{c}$ is a finite module. Since $\mathfrak{b}\mathfrak{c}$ has a finite base, we see that $\mathfrak{a}$ has a finite base, which is a contradiction to our assumption that $\mathfrak{a}\in P$. Thus P is empty and $\mathfrak{o}$ is Noetherian.

Now we want to prove

THEOREM 3. *The derived normal ring $\mathfrak{o}'$ of a Noetherian integral domain $\mathfrak{o}$ rank 2 is also Noetherian.*

PROOF. (1) We first prove that $\mathfrak{o}'$ is Noetherian under the assumption that every maximal ideal of $\mathfrak{o}'$ has a finite base.[11] By virtue of Lemma 4, we have only to show that every prime ideal $\mathfrak{p}'$ of rank 1 in $\mathfrak{o}'$ has a finite base. Set $\mathfrak{p}=\mathfrak{p}'\cap\mathfrak{o}$. If $\mathfrak{p}$ is maximal,

10) The present proof was given by Cohen [2]. Another proof was given by Nagata [9].

11) The main idea of the present step of the proof was giuen by Mr. Y. Mori.

then $\mathfrak{p}'$ is maximal and we have nothing to prove. Thus we assume that $\mathfrak{p}$ is not maximal, consequently $\mathfrak{p}$ is of rank 1. Since there are only a finite number of prime ideals in $\mathfrak{o}'$ which lie over $\mathfrak{p}$, we may assume that $\mathfrak{p}'$ is the unique prime ideal which lies over $\mathfrak{p}$. Let a be an element of $\mathfrak{p}'$ such that $a\mathfrak{o}'_{\mathfrak{p}'}=\mathfrak{p}'\mathfrak{o}'_{\mathfrak{p}'}$ (existence follows from Theorem 1 or Theorem 2). We may assume that a is in $\mathfrak{o}$. Set $\mathfrak{q}=\mathfrak{p}\mathfrak{o}':\mathfrak{p}'$. Since $\mathfrak{p}\mathfrak{o}'_{\mathfrak{p}'}=\mathfrak{p}'\mathfrak{o}'_{\mathfrak{p}'}$ by our assumption and since $\mathfrak{p}'$ is the unique prime ideal which lies over $\mathfrak{p}$, $\mathfrak{q}\cap\mathfrak{o}$ is not of rank 1. Since every prime divisor of $\mathfrak{q}$ lies over that of $\mathfrak{q}\cap\mathfrak{o}$, $\mathfrak{q}$ has only a finite number of prime divisors and they are maximal. Since every maximal ideal of $\mathfrak{o}'$ has a finite base, $\mathfrak{q}$ contains a power of the intersection $\mathfrak{n}$ of all prime divisors of $\mathfrak{q}$, say, $\mathfrak{n}^e\subseteq\mathfrak{q}$. Since $\mathfrak{n}^e$ has a finite base and since $\mathfrak{o}'/\mathfrak{n}^e$ is Noetherian by Lemma 4, we see that $\mathfrak{q}$ has a finite base. On the other hand, since $\mathfrak{o}'/\mathfrak{p}'$ is an almost finite integral extension of $\mathfrak{o}/\mathfrak{p}$ by the corollary to Theorem 1, we see that $\mathfrak{o}'/\mathfrak{p}'$ is Noetherian by Krull-Akizuki's theorem (Proposition 1). We may regard $\mathfrak{q}/\mathfrak{p}'\mathfrak{q}$ as an $\mathfrak{o}'/\mathfrak{p}'$-module; this is a finite module because $\mathfrak{q}$ has a finite base. Therefore $(\mathfrak{p}'\cap\mathfrak{q})/\mathfrak{p}'\mathfrak{q}$ is a finite module. Since $\mathfrak{p}'\cap\mathfrak{q}$ contains $\mathfrak{p}\mathfrak{o}'$, since $\mathfrak{p}\mathfrak{o}'$ contains $\mathfrak{p}'\mathfrak{q}$ and since $\mathfrak{p}\mathfrak{o}'$ has a finite base, we see that $\mathfrak{p}'\cap\mathfrak{q}$ has a finite base. Since $\mathfrak{o}'/(\mathfrak{p}'\cap\mathfrak{q})$ is a subdirect sum of Noetherian rings $\mathfrak{o}'/\mathfrak{p}'$ and $\mathfrak{o}'/\mathfrak{q}$, we see that $\mathfrak{p}'/(\mathfrak{p}'\cap\mathfrak{q})$ has a finite base. Now since $\mathfrak{p}'\cap\mathfrak{q}$ has a finite base, we see that $\mathfrak{p}'$ has also a finite base. Thus $\mathfrak{o}'$ is Noetherian.

(2) Now we have to prove that every maximal ideal $\mathfrak{m}'$ of $\mathfrak{o}'$ has a finite base. Since there exist only a finite number of maximal ideals of $\mathfrak{o}'$ which lie over $\mathfrak{m}=\mathfrak{m}'\cap\mathfrak{o}$, we may assume that $\mathfrak{m}'$ is the unique prime ideal of $\mathfrak{o}'$ which lies over $\mathfrak{m}$. Then $\mathfrak{o}'/\mathfrak{m}\mathfrak{o}'=\mathfrak{o}'_{\mathfrak{m}'}/\mathfrak{m}\mathfrak{o}'_{\mathfrak{m}'}$. Therefore, in account of the fact that $\mathfrak{m}$ has a finite base, we may assume that $\mathfrak{m}$ is the unique maximal ideal of $\mathfrak{o}$ and therefore that $\mathfrak{m}'$ is the unique maximal ideal of $\mathfrak{o}'$. By virtue of Krull-Akizuki's theorem (Proposition 1), we may assume that $\mathfrak{m}$ is of rank 2. Let a be an element of $\mathfrak{m}$ different from zero and let $b(\neq 0)$ be an element of $\mathfrak{m}$ such that $a\mathfrak{o}$ and $b\mathfrak{o}$ have no common minimal prime divisor. Let x be a transcendental element over $\mathfrak{o}$. We denote by $\mathfrak{o}(x)$ and $\mathfrak{o}'(x)$ the rings $\mathfrak{o}[x]_{\mathfrak{m}\mathfrak{o}[x]}$ and $\mathfrak{o}'[x]_{\mathfrak{m}'\mathfrak{o}'[x]}$ respectively (cf. [8]). In order to prove that $\mathfrak{m}'$ has a finite base, it is sufficient to show that $\mathfrak{m}'\mathfrak{o}'(x)$ has a finite base, as is easily seen. Since $a\mathfrak{o}'$ and $b\mathfrak{o}'$ has no common prime divisor

by our assumption and by Theorem 1, we see easily that $(ax+b)\mathfrak{o}'(x)$ is a prime ideal. Set $\mathfrak{p}^* = (ax+b)\mathfrak{o}'(x) \cap \mathfrak{o}(x)$. Then $\mathfrak{o}(x)/\mathfrak{p}^*$ is a Noetherian integral domain of rank 1 and $\mathfrak{o}'(x)/(ax+b)\mathfrak{o}'(x)$ is a subring of the field of quotients of $\mathfrak{o}(x)/\mathfrak{p}^*$ (because $\mathfrak{o}'(x)/(ax+b)\mathfrak{o}'(x)$ is a ring of quotients of $\mathfrak{o}'[b/a]$). Therefore by Krull-Akizuki's theorem (Proposition 1), $\mathfrak{o}'(x)/(ax+b)\mathfrak{o}'(x)$ is Noetherian and $\mathfrak{m}'\mathfrak{o}'(x)$ has a finite base. Thus every maximal ideal of $\mathfrak{o}'$ has a finite base. Thereby the proof of the theorem is completed.

BIBLIOGRAPHY

[1] C. Chevalley, La notion d'anneau de décomposition, Nagoya Math. J., Vol. 7 (1954), pp. 21-33.

[2] I. S. Cohen, commutative rings with restricted minimum condition, Duke Math. J., Vol. 17 (1950), pp. 27-42.

[3] W. Krull, Idealtheorie, Ergebnisse der Math., Vol. 4, No. 3(1935).

[4] Y. Mori, On the integral closure of an integral domain, Memoirs Kyoto University, ser. A, Vol. 27, No. 3 (1953), pp. 249-256; Errata, Vol. 28, No. 3 (1954) pp. 327-328.

[4, II] Y. Mori, On the integral closure of an integral domain, II, forthcoming.

[5] M. Nagata, Some remarks on local rings, II, Memoirs Kyoto University, ser. A, Vol. 28, No. 2 (1954), pp. 109-120.

[6] M. Nagata, Noet on integral closures of Noetherian integrity domains, Memoirs Kyoto University, ser. A, vol. 28, No. 2 (1954), pp. 121-124.

[7] M. Nagata, Basic theorems on general commutative rings, Memoirs Kyoto University, ser. A, Vol. 29, No. 1 (1955), pp. 59-77.

[8] M. Nagata, The theory of multiplicity in general local rings, forthcoming.

[9] M. Nagata, On the chain problem of prime ideals, forthcoming.