Addition and corrections to my paper “A treatise on the 14-th problem of Hilbert”

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Concerning the 14-th problem of Hilbert, Zariski [3] conjectured the following:

Conjecture of Zariski. Let $D$ be a positive divisor on a normal projective variety $V$ defined over a field $k$ and let $R[D]$ be the set of functions $f$ on $V$ defined over $k$ such that $(f)^{-1} + nD > 0$ for some natural number $n$. Then $R[D]$ will be an affine ring over $k$.

He proved there that if the answer of this conjecture is affirmative, then the answer of the following problem is affirmative:

The generalized 14-th problem of Hilbert: Let $\mathfrak{o}$ be a normal affine ring over a field $k$ and let $L'$ be a function field contained in the function field of $\mathfrak{o}$. Is then $\mathfrak{o} \cap L'$ an affine ring?

In the present paper, we shall show at first that the generalized 14-th problem of Hilbert is equivalent to the conjecture of Zariski and then we shall give some corrections to my paper [2].

§ 1. The proof of the equivalence.

Since Zariski [3] proved that the affirmative answer of the conjecture of Zariski implies the affirmative answer of the generalized 14-th problem of Hilbert, we have only to prove the converse. The writer proved in [2] that the generalized 14-th problem is equivalent to

Problem A. Let $\mathfrak{a}$ be an ideal of a normal affine ring $\mathfrak{o}$ over a field $k$. Is then the $\mathfrak{a}$-transform of $\mathfrak{o}$ an affine ring?

Therefore we have only to prove that:

The affirmative answer of Problem A implies the affirmative answer of the conjecture of Zariski.

Now we shall use the notations as in the conjecture of Zariski. Let $L$ be the field of quotients of $R[D]$ and let $\mathfrak{o}$ be a normal
affine ring of $L$ contained in $R[D]$. We denote by $v$ in general
groups which corresponds to $k$-prime divisors on $V$ which are not
components of $D$. Then obviously $R[D]$ is the intersection of all
of $v$'s, hence $R[D] = \cap_{v} (L \cap v)$, which shows that $R[D]$ is a Krull
ring (see [2, p. 60]) and if $q$ is a prime ideal of rank 1 in $R[D]$,
then there exists one $v$ such that $R[D]_{q} = L \cap v$. Furthermore,
since each $L \cap v$ is a spot (see [2, footnote 3]), $R[D]_{v}$ is a spot.
Let $\Omega$ be the set of prime ideals $q$ of rank 1 in $R[D]$ such that
$q \cap v$ is not of rank 1. Since $\nu(q \cap v)$ is dominated by one $v$, $q \in \Omega$
means that the spot $\nu(q \cap v)$ is an isolated fundamental spot with
respect to $V$, hence $\Omega$ is a finite set. Since $R[D]_{q}$ is a spot, we
can reduce easily to the case where $\Omega$ is empty (see [2, Proposition A]). Thus we assume that $\Omega$ is empty. Next, let $\Psi$ be the
set of prime ideals $p$ of rank 1 in $\nu$ such that there exists no prime
ideal $q$ of rank 1 in $R[D]$ which lies over $p$. Then $\nu_{p}$ ($p \in \Psi$)
is dominated by none of $v$, which shows that $\nu_{\Psi}$ corresponds to only
components of $D$, which shows that $\Psi$ is a finite set. Let $\alpha$
be the intersection of members of $\Psi$. Then $R[D]$ is the $\alpha$-transform
of $\nu$. Therefore the equivalence is proved.

§ 2. Corrections.

In [2, Theorem 4] we asserted that if $D$ is a closed set of an
affine model $A$ of dimension 2 ($A \neq D$), then $A - D$ has an associated
affine model. This is correct under the additional assumption that
$A$ is normal and in the non-normal case the assertion is not true
as will be shown by an example in § 3. One error in the proof
exists in 1. 4, p. 67 of the paper.) Namely, we stated that from
$\nu_{q} = \tilde{s}_{q'}$ it follows that $q'' = q \theta_{m'}$: $q \theta_{m'}$ is a primary ideal belonging
to $m' \tilde{s}_{m'}$: But we needed really the normality in that conclusion.
In fact, the example which will be shown in § 3 shows the non-
validity of this conclusion in the non-normal case. Since, even in
the normal case, that conclusion may not be obvious, we shall give
a detailed proof of that conclusion in § 4.

By this reason, in that Theorem 4, we must assume that $A$
is normal. Under the assumption of normality, the proof of
Theorem 4 is valid and there remains no difficulty (except the
fact which we shall prove in § 4).

On the other hand, Proposition 5 (p. 69) should be asserted
also under the additional assumption that $\nu$ is a normal ring.
§ 3. An example.

Let $x$, $y$ and $z$ be indeterminates and let $k$ be a field. Let $f$ be an element of $k[x, y, z]$ such that

1. $f$ is irreducible, and
2. $f = y(z + yt) + x(u, y^2 + uz + u, z^2)$ with $t \in k[x, y]$ and $u, u_2, u_3 \in k[x, y, z]$.

Set $\mathfrak{o} = k[x, y, z]/\langle f \rangle$. Then $x, y$ generate a prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{o}$; $y, z$ generate a prime ideal $\mathfrak{q}$ of rank 1 in $\mathfrak{o}$. $\mathfrak{o}$ is not normal. Let $\hat{\mathfrak{o}}$ be the $\mathfrak{p}$-transform of $\mathfrak{o}$. We first consider $\mathfrak{p}^{-1}$. It is obviously generated by 1 and $z_i = (z + yt)/x$. Therefore $\mathfrak{o} [\mathfrak{p}^{-1}]$ is generated by $x, y, z_i$ satisfying a relation similar to $f$ stated in (2) as is easily seen. Thus $\hat{\mathfrak{o}}$ is obtained by successive adjunction of elements $z_1, z_2, ..., z_n$, such that $z_i = (z_{i-1} + yt_{i-1})/x$ with $t_{i-1} \in k[x, y]$. Though we have already seen in essential that $\hat{\mathfrak{o}}$ is not an affine ring, we shall see a little more. Since $xz_i = z_{i-1} + yt_{i-1} (z_i = z)$, we see that $z_i \in \mathfrak{q}$. Thus $x$ and $y$ generate a maximal ideal $\mathfrak{m}$ of $\hat{\mathfrak{o}}$. Therefore if $\hat{\mathfrak{o}}_m$ is Noetherian, $\hat{\mathfrak{o}}_m$ must be a regular local ring. Let $\mathfrak{q}'$ be the uniquely determined prime ideal of rank 1 in $\hat{\mathfrak{o}}$ such that $\hat{\mathfrak{o}}_{\mathfrak{q}'} = \mathfrak{o}_q$. Since $y, z \in \mathfrak{q}$ and $x \notin \mathfrak{q}$, $z_i = (z + yt)/x$ must be in $\mathfrak{q}'$. By the same reason, we have $z_i \in \mathfrak{q}'$ for every $i$. Therefore $\mathfrak{q}'$ is generated by $z, z_1, z_2, ..., z_n$. Therefore $\mathfrak{q}'$ is contained in $\mathfrak{m}$. Since $\mathfrak{o}_q = \hat{\mathfrak{o}}_{\mathfrak{q}'}$, we see that $\hat{\mathfrak{o}}_m$ is not a normal ring and $\hat{\mathfrak{o}}_m$ cannot be a regular local ring and $\hat{\mathfrak{o}}_m$ cannot be a Noetherian ring. Now, if $q \hat{\mathfrak{o}}_m : q' \hat{\mathfrak{o}}_m$ is a primary ideal belonging to $m \hat{\mathfrak{o}}_m$, then the treatment in [2, p. 69] shows that $q'$ is generated by a finite number of elements. But we see now easily that $q'$ cannot be generated by any finite number of the $z_i$'s. Thus $q \hat{\mathfrak{o}}_m : q' \hat{\mathfrak{o}}_m$ is not a primary ideal belonging to $m \hat{\mathfrak{o}}_m$ but is contained in $q' \hat{\mathfrak{o}}_m$.

§ 4. A lemma on Krull ring.

In order to verify the statement in [2, p. 67, l. 4] in the normal case, it will be sufficient to prove the following lemma. 3)

Lemma. Let $q$ be a prime ideal of rank 1 in a Krull ring $\mathfrak{a}$.
If $a$ is an ideal contained in $q$ such that $a \mathfrak{a}_q = q \mathfrak{a}_q$, then $a : q$ is not contained in $q$.

Proof. Since $\mathfrak{a}$ is a Krull ring, $\mathfrak{a}_q$ is a discrete valuation ring. Therefore there exists an element $a \in a$ such that $a \mathfrak{a}_q = q \mathfrak{a}_q$ (because...
\[ a \mathfrak{a}_\mathfrak{a} = q \mathfrak{a}_\mathfrak{q} \]. Since \( \mathfrak{a} \) is a Krull ring, \( a \mathfrak{a} \) is the intersection of a finite number of primary ideals and we see easily that \( a \mathfrak{a} : q \) is not contained in \( q \).

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Notes

1) There is one more error concerning non-normal case in p. 67. Namely, we constructed the ring \( \mathfrak{a}^* \); then \( \mathfrak{a}^* \) may have a maximal ideal \( \mathfrak{m}^* \) of rank 1. This is the reason why proposition 5 should be asserted under an additional condition (see the end of this section).
2) There is a case where \( q^\prime \mathfrak{a} = \mathfrak{m}^\prime \mathfrak{a} \). In such a case, we have obviously \( q \mathfrak{a}^\prime = q \mathfrak{a} \mathfrak{m}^\prime \) and \( q \mathfrak{a}^\prime \mathfrak{m}^\prime \) has a finite base. Therefore we disregarded such a simple case.
3) This lemma was used in the first step of the proof of [1, Theorem 3].