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On the algebraic theory of sheets of an algebraic variety

By

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The motivation of the present note is stated at the begining of [1]. In order to determine the dimension of the algebraic sub-system of curves with prescribed number of nodes in a linear system on a surface, Severi used the " method of analytic branches " successfully. ([9] Anhang F, [10] p. 85 and p. 167.) The preceding two papers [1], [8] have shown how one can avoid explicit use of the analytic method, but it seems nevertheless to be of some interest to translate the method of analytic branches (or sheets) into the language of local algebra, which we will do in this note.

In § 1 we shall define the *decomposition number* $\mathrm{dn}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of a ring $\mathfrak{o}$, which is an algebraic translation of the geometric notion of the number of sheets of a variety at a point. From a property of the decomposition number we deduce the following proposition: *Let V be an r-dimensional algebraic variety. Then those points of V which decompose into d or more points on the derived normal variety $\widetilde{V}$ of V, together with their specializations, make up an algebraic subset $\mathfrak{B}_d$ of V. If V is embedded in an $(r+1)$-dimensional non-singular variety U, then each component of $\mathfrak{B}_d$ has dimension $\geq r+1-d$.* Here one may replace " $P(\in V)$ decomposes into d points on $\widetilde{V}$" by " V has d sheets at P" by virtue of Zariski's theorem of analytic normality, as is well known.

In § 2 we shall apply the above proposition to the geometric problem treated in [1] and [8], and shall show how well does Severi's reasoning work also in the abstract case.

In § 3 we shall show, by means of the theory of Henselian rings, that a component of the intersection of several sheets of an algebraic variety (or varieties) is a sheet of some algebraic subvariety.

As for the notations we follow [4]. "Variety" will always mean as usual an absolutely irreducible algebraic variety, and "ring" will always imply a commutative ring with a unit element. If $\mathfrak{o}$ is a ring, we denote by $\{\mathfrak{o}\}$ the total ring of quotients of $\mathfrak{o}$. An integrally closed integral domain is called a normal ring. Following Abhyankar, we shall mean by saying "$(\mathfrak{o}, \mathfrak{m})$ is a local ring" that $\mathfrak{o}$ is a local ring and $\mathfrak{m}$ is its maximal ideal.

1. Let $\mathfrak{o}$ be a ring, $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$, and $\tilde{\mathfrak{o}}$ be the integral closure of $\mathfrak{o}$ in $\{\mathfrak{o}\}$. We call $\sum[\tilde{\mathfrak{o}}/\tilde{\mathfrak{p}} : \mathfrak{o}/\mathfrak{p}]_s$ (where the sum is taken over all prime ideals $\tilde{\mathfrak{p}}$ of $\tilde{\mathfrak{o}}$ such that $\tilde{\mathfrak{p}} \cap \mathfrak{o} = \mathfrak{p}$) the *decomposition number* of $\mathfrak{p}$ and denote it by $\mathrm{dn}(\mathfrak{p})$. It is equal to the number of the homomorphisms of $\tilde{\mathfrak{o}}$ into the algebraic closure of the field $\{\mathfrak{o}/\mathfrak{p}\}$ which extend the natural homomorphism $\mathfrak{o} \to \mathfrak{o}/\mathfrak{p}$. If $(\mathfrak{o}, \mathfrak{m})$ is the spot (=the local ring) of a point P on a variety V (over a ground field k), $\mathrm{dn}(\mathfrak{m})$ is nothing but the number of the points on the derived normal model $\tilde{V}$ (with respect to k) of V which correspond to P. In this case we denote the number also by $\mathrm{dn}(P)$.

Assume that $\mathfrak{o}$ is a Noetherian ring without nilpotent elements, and let $\mathfrak{n}_1, \cdots, \mathfrak{n}_s$ be the prime ideals of (0). Set $\mathfrak{o}_i = \mathfrak{o}/\mathfrak{n}_i$. Then $\{\mathfrak{o}\}$ is the direct sum of $\{\mathfrak{o}_1\}, \cdots, \{\mathfrak{o}_s\}$ and $\tilde{\mathfrak{o}}$ is the direct sum of $\tilde{\mathfrak{o}}_1, \cdots, \tilde{\mathfrak{o}}_s$. Therefore it is easy to see that if $\mathfrak{p}$ is a prime ideal containing just s' of the prime divisors of (0), say $\mathfrak{n}_1, \cdots, \mathfrak{n}_{s'}$, then $\mathrm{dn}(\mathfrak{p}) = \sum_1^{s'} \mathrm{dn}(\mathfrak{p}/\mathfrak{n}_i)$. In particular, $\mathrm{dn}(\mathfrak{p}) \geq s'$.

Assume, moreover, that $\mathfrak{o}$ is a spot. We denote by $\tilde{\mathfrak{o}}$ and $\mathfrak{o}^*$ the derived normal ring and the completion of $\mathfrak{o}$ respectively. Let $\tilde{\mathfrak{o}}_1, \cdots, \tilde{\mathfrak{o}}_r$ be the quotient rings of $\tilde{\mathfrak{o}}$ by its maximal ideals, and let $\mathfrak{n}_1^*, \cdots, \mathfrak{n}_s^*$ be the prime ideals of (0) in $\mathfrak{o}^*$. On one hand, the completion $\tilde{\mathfrak{o}}^*$ of the semilocal ring $\tilde{\mathfrak{o}}$ is the direct sum of the completions $\tilde{\mathfrak{o}}_i^*$ of $\tilde{\mathfrak{o}}_i$, which are normal rings by the analytical normality of normal spots. Since every element of $\mathfrak{o}$ other than 0 is not a zero-divisor in $\mathfrak{o}^*$, $\tilde{\mathfrak{o}}^*$ (which contains $\mathfrak{o}^*$ since $\mathfrak{o}$ is a subspace of $\tilde{\mathfrak{o}}$) is contained in $\{\mathfrak{o}^*\}$, hence it coincides with the integral closure of $\mathfrak{o}^*$ in $\{\mathfrak{o}^*\}$. On the other hand, this last ring is the direct sum of the derived normal rings of $\mathfrak{o}^*/\mathfrak{n}_i^*$. Hence we have $r=s$, as is well known. It is known that $\tilde{\mathfrak{o}}^*=\tilde{\mathfrak{o}}\otimes_{\mathfrak{o}}\mathfrak{o}^*$. If $\mathfrak{o}$ is the spot of a point P on a variety V with respect to an algebraically closed ground field k over which P is rational, the prime divisors $\mathfrak{n}_1^*, \cdots, \mathfrak{n}_s^*$ of (0) in $\mathfrak{o}^*$ correspond, by definition, to the sheets of V at P. Thus " V has d sheets at P" is equivalent to "$\mathrm{dn}(P)=d$."

Proposition 1. *Let* $\mathfrak{p}^*$ *be a prime ideal of the completion* $\mathfrak{o}^*$ *of a spot* $\mathfrak{o}^*$, *and let* $\mathfrak{p}=\mathfrak{p}^*\cap\mathfrak{o}$. *Then we have* $\mathrm{dn}(\mathfrak{p}^*)=\mathrm{dn}(\mathfrak{p})$.

Proof. Using the above notations, $\mathrm{dn}(\mathfrak{p}^*)$ is equal to the number of homomorphisms of $\tilde{\mathfrak{o}}^*$ into the algebraic closure $\bar{K}$ of $\{\mathfrak{o}^*/\mathfrak{p}^*\}$ which extend the natural homomorphism $\mathfrak{o}^*\to\mathfrak{o}^*/\mathfrak{p}^*$. Such a homomorphism determines a homomorphism of $\tilde{\mathfrak{o}}$ into the algebraic closure $\bar{k}$ of $\{\mathfrak{o}/\mathfrak{p}\}$ (in $\bar{K}$) extending the natural homomorphism $\mathfrak{o}\to\mathfrak{o}/\mathfrak{p}$. Conversely, any homomorphism $\tilde{\mathfrak{o}}\to\bar{k}\subseteq\bar{K}$ which extends the natural homomorphism $\mathfrak{o}\to\mathfrak{o}/\mathfrak{p}$ is induced by a homomorphism $\tilde{\mathfrak{o}}^*=\tilde{\mathfrak{o}}\otimes_{\mathfrak{o}}\mathfrak{o}^*\to\bar{K}$ which extends the natural homomorphism $\mathfrak{o}^*\to\mathfrak{o}^*/\mathfrak{p}^*$, by the universal mapping property of tensor product. This proves our proposition.

Corollary. *Let P be a point of a variety V. Let* $\mathfrak{U}$ *be an algebroid subvariety of V passing through P, and let U be the smallest* (*algebraic*) *variety containing* $\mathfrak{U}$. *Then, if* $\mathfrak{U}$ *is contained in d of the sheets of V at P, V has at least d sheets at the generic points of U.*

Remark. If the d sheets in the Corollary are normal at P, i. e. if $\mathfrak{o}^*/\mathfrak{n}_i^*$ $(1\leq i\leq d)$ are normal rings, then V has exactly d sheets at the generic points of U.

Proposition 2. *Let V be an r-dimensional variety defined over a field k, and let $\tilde{V}$ be the derived normal variety* (*with respect to k*) *of V. Then those points of V which decompose into d or more points on $\tilde{V}$, together with their specializations, make up an algebraic subset* $\mathfrak{B}_d$ *of V, normally algebraic over k. If V is embedded in an $(r+1)$-dimensional non-singular variety U, then each component of* $\mathfrak{B}_d$ *has dimension* $\geq r+1-d$.[1]

Proof. We fix k as the ground field. The first half of the

1) If we abandon the assumption $\dim U=r+1$, and if we set $\dim U=u$, then we have only to replace $r+1-d$ by $rd-u(d-1)$ in the conclusion and in the proof, by virtue of the dimension theory of algebroid varieties developed in [3.]

proposition follows readily from the following lemma. (Cf. [4] Chap. 2, Prop. 6.)

Lemma. *Let W be a k-subvariety of V and let P be its generic point. If* $\mathrm{dn}(P)=t$ $(<d)$, *then* $\mathrm{dn}(Q)=t$ *holds for points Q of W almost everywhere.*

Let $\widetilde{W}_1, \cdots, \widetilde{W}_g$ be the subvarieties of $\widetilde{V}$ which correspond to W. By a property of the derived normal model, a point of $\widetilde{V}$ which corresponds to a point of W is in some $\widetilde{W}_i$. The points of W which correspond to points belonging to the intersections $\widetilde{W}_i \cap \widetilde{W}_j (i \neq j)$ form an algebraic subset of W, which we omit from W. We may assume also that W is an affine variety, and that the affine rings A_i of W_i are of the form $A_i = A[a_i]$, where A is the affine ring of W and a_i is a root of an irreducible separable monic polynomial $f_i(x) \in A[x]$ (since we can disregard purely inseparable extensions). The points of W at which at least one of the discriminants of the f_i vanishes form an algebraic subset, which we omit from W also. The remaining points satisfy $\mathrm{dn}(Q)=t$. Thus the lemma is proved.

Let us turn to the second half of the Proposition. We have to show that a point P of V with $\mathrm{dn}(P) \geq d$ is contained in a subvariety, of dimension $\geq r+1-d$, such that $\mathrm{dn}(P') \geq d$ at its generic points P'. Let $K(\supseteq k)$ be an algebraically closed field over which P is rational, and let $\mathfrak{o}$ be the local ring of P on U (with respect to K). Let (f) be the prime ideal of V in $\mathfrak{o}$, and let $(f_1^*), \cdots, (f_\delta^*)$ be the (minimal) prime divisors of $f\mathfrak{o}^*$ in the completion $\mathfrak{o}^*$ of $\mathfrak{o}$. Then $\delta = \mathrm{dn}(P) \geq d$. Let $\mathfrak{p}^*$ be a minimal prime divisor of the ideal $(f_1^*, \cdots, f_d^*)$, and set $\mathfrak{p} = \mathfrak{p}^* \cap \mathfrak{o}$. Then

$$\mathrm{rank}\ \mathfrak{o} = \mathrm{rank}\ \mathfrak{p}\mathfrak{o}^* \leq \mathrm{rank}\ \mathfrak{p}^* \leq d,$$

and $\dim \mathfrak{p} \geq r+1-d$. On the other hand, $\mathrm{dn}(\mathfrak{p}) = \mathrm{dn}(\mathfrak{p}^*) \geq d$. Let P' be a generic point of the subvariety corresponding to $\mathfrak{p}$ over K. Then the locus of P' over k satisfies our requirement.

2. Now we apply Proposition 2 to the problem stated at the begining of [1]. We use the notations of [1], and use also the results proved in the Lemmas 4, 5 of [1]. Let λ be a point in $\mathfrak{C}_a$. Then there correspond on $\overline{\mathfrak{M}}$ $d'(\geq d)$ points $\lambda \times Q_j$, which are all simple on $\overline{\mathfrak{M}}$ by Lemma 4 of [1]. We assume that $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ are birational. (This assumption is not very restrictive by Lemma 5 of [1].) In our proposition 2 we set $V = \mathfrak{M}$. Then we see that

$\mathrm{dn}(\lambda)=d'$. From the algebraic set constructed in Proposition 2, we take up only those components which contain some points of $\mathfrak{E}_d$. Let $\bar{\lambda}$ be a generic point of such a component. Then the points on $\mathfrak{M}$ corresponding to $\bar{\lambda}$ must be all simple, since they have simple points (of the form $\lambda\times Q$, $\lambda\in\mathfrak{E}_d$) as specializations. Therefore the number of nodes on $D(\bar{\lambda})$ must be equal to $\mathrm{dn}(\bar{\lambda})$, which is not less than d.

3. In this section we want to make the meaning of the Corollary of Proposition 1 clearer by showing that the components of the intersection of several sheets of a variety are sheets of some (algebraic) varieties. For that purpose we use basic results of the theory of Henselian rings, which are contained in Chap. 1 of [5] and Chap. 1 of [6].[2] Let us recall: A Henselian ring is, by definition, a quasi-local ring for which Hensel's lemma holds with respect to the maximal ideal. If $(\mathfrak{o}, \mathfrak{m})$ is a quasi-local normal ring, the Henselization of $\mathfrak{o}$ is defined as follows. Let $\bar{\mathfrak{o}}$ be the integral closure of $\mathfrak{o}$ in the maximal separable extension of the field $\{\mathfrak{o}\}$, and let $\bar{\mathfrak{m}}$ be a maximal ideal of $\bar{\mathfrak{o}}$. Denote by $\tilde{\mathfrak{o}}$ the decomposition ring of $\bar{\mathfrak{m}}$, and let $\tilde{\mathfrak{m}}=\bar{\mathfrak{m}}\cap\tilde{\mathfrak{p}}$. Then we call $\mathfrak{o}'=\tilde{\mathfrak{o}}_{\tilde{\mathfrak{m}}}$ the Henselization of $\mathfrak{o}$. It is a Henselian normal ring and is determined uniquely up to isomorphisms over $\mathfrak{o}$. If $\mathfrak{o}$ is a local ring (i.e. $\cap\,\mathfrak{m}^n=0$), then $\mathfrak{o}'$ is a local ring containing $\mathfrak{o}$ as a dense subspace. If $\mathfrak{o}$ is Noetherian, so is $\mathfrak{o}'$. If $\mathfrak{o}$ is a regular local ring, then also $\mathfrak{o}'$ is regular (since it is Noetherian and since its completion is regular).

Proposition 3. *Let* $(\mathfrak{o}, \mathfrak{m})$ *be a normal spot, and let* $\mathfrak{p}$ *be a prime ideal of* $\mathfrak{o}$. *Denote by* $(\mathfrak{o}', \mathfrak{m}')$ *the Henselization of* $\mathfrak{o}$, *and let* $\mathfrak{p}'$ *be a prime divisor of* $\mathfrak{p}\mathfrak{o}'$. *Then the local domain* $\mathfrak{o}'/\mathfrak{p}'$ *is analytically irreducible.*

This proposition is essentially contained in Th. 7 of [6], according to which the derived normal ring of $\mathfrak{o}'/\mathfrak{p}'$ is the Henzelization of a normal spot (the quotient ring of the derived normal ring of $\mathfrak{o}/\mathfrak{p}$ by one of its maximal ideal), hence analytically ir-

2) We want to make a remark on the Chap. 2 of [6] on this occasion. Its § 4 seems to contain some errors, so the proof of Th. 4 must be amended. The proofs of Th. 5 and 6 are independent of Th. 4. In reading the proof of Th. 6 attention should be paid to the fact that, if $\bar{\mathfrak{q}}\neq 0$, $\bar{\mathfrak{o}}/\bar{\mathfrak{q}}$ is integrally closed in the algebraic closure of its quotient field. This is because every monic polynomial in $(\bar{\mathfrak{o}}/\bar{\mathfrak{q}})[x]$ is the residue of a *separable* monic polynomial in $\bar{\mathfrak{o}}[x]$.

reducible. From this we can easily derive our proposition. (For example, we can proceed as follows: since $\mathfrak{o}'/\mathfrak{p}'$ is analytically unramified, the derived normal ring is finite $(\mathfrak{o}'/\mathfrak{p}')$-module[3] and hence contains $\mathfrak{o}'/\mathfrak{p}'$ as a subspace. Therefore the completion $(\mathfrak{o}'/\mathfrak{p}')^*$ is an integral domain.) Again, we can prove the analytical irreducibility of the derived normal ring of $\mathfrak{o}'/\mathfrak{p}'$ without using the cited Theorem 7, by a theorem of Zariski which was used in his proof of analytical normality of normal spot (see e. g. Samuel, Méthodes d'Algèbre Abstraite en Géométrie Algébrique, p. 66). We shall give here another proof.

Lemma. *Let* $(\mathfrak{o}, \mathfrak{m})$ *be a Henselian integrity domain and let* R *be an integral extension of* $\mathfrak{o}$. *Then* R *is quasi-local.* (Azumaya [2] Th. 22.)

Proof. Assume the contrary and let $\mathfrak{m}_1$, $\mathfrak{m}_2$ be two maximal ideals of R. Take an element a of R such that $a \in \mathfrak{m}_1$, $a \notin \mathfrak{m}_2$. Then $\mathfrak{o}[a]$ is not quasilocal. Let $f(x)$ be an irreducible monic polynomial in $\mathfrak{o}[x]$ such that $f(a)=0$. (It may be reducible in $\{\mathfrak{o}\}[x]$.) Then $\mathfrak{o}[a]$ is a homomorphic image of $\mathfrak{o}[x]/(f)$. The maximal ideals of $\mathfrak{o}[a]$ (which lie necessarily over $\mathfrak{m}$) are images of maximal ideals of $\mathfrak{o}[x]/(\mathfrak{m}, f) = (\mathfrak{o}/m)[x]/(\bar{f}(x))$, where $\bar{f}(x) = f(x) \bmod \mathfrak{m}[x]$. Since $\mathfrak{o}$ is Henselian, $\bar{f}(x)$ cannot have two distinct irreducible factors. Therefore $(\mathfrak{o}/\mathfrak{m})[x]/(\bar{f}(x))$ has only one prime ideal, and $\mathfrak{o}[a]$ must be quasi-local. Contradiction.

We shall use also the following facts on Noetherian local rings. Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let E be a finite A-module. We topologize E by taking $\{\mathfrak{m}^n E;\ n=1, 2, \cdots\}$ as a system of neighborhoods of 0. Then E is a Hausdorf topological group and its completion E^* coincides with $E \otimes_A A^*$. A submodule of E is also a subspace of E. Let $(B, \mathfrak{n})$ be another Noetherian local ring containing A as a dense subspace. Then $\otimes_A B$ is an exact functor: if $E \to F \to G$ is an exact sequence of A-modules (not necessarily finite), then also $E \otimes B \to G \otimes B \to F \otimes B$ is exact. Moreover, the natural mapping $E \to E \otimes B$ is injective.[4]

Proof of Proposition 3. Let $\mathfrak{o}^*$ be the completion of $\mathfrak{o}$ (and of $\mathfrak{o}'$). We set $R = \mathfrak{o}/\mathfrak{p}$, $R' = \mathfrak{o}'/\mathfrak{p}\mathfrak{o}'$, $R^* = \mathfrak{o}^*/\mathfrak{p}\,\mathfrak{o}^*$. Then R' is a

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3) Cf. Nagata 7.

4) We learned these theorems from E. Artin and J. P. Serre in their stay in Kyoto.

Henselian Noetherian ring containing R as a dense subspace, and R^* is the completion of R and of R'. Let $\widetilde{R}$ be the derived normal ring of R, and let $\widetilde{R}_i (1 \leq i \leq s)$ be the quotient rings of R by its maximal ideals. In the completion $\widetilde{R}^*$ of $\widetilde{R}$, we consider the composite ring S of $\widetilde{R}$ and R'. Since $\widetilde{R}^* = \widetilde{R} \otimes R^* = (\widetilde{R} \otimes_R R') \otimes_{R'} R^*$, and since the last tensor product contains $\widetilde{R} \otimes R'$ by the cited theorem of Serre, we have $S = \widetilde{R} \otimes R'$, and $S^* = S \otimes_{R'} R^* = \widetilde{R}^*$. Let $S_1, \cdots, S_h$ be the quotient rings of the semi-local ring S by its maximal ideals. Then, since $\sum_1^h \oplus S_j^* = \widetilde{R}^* = \sum_1^s \oplus \widetilde{R}_i^*$, we see that $h = s$ and that the S_j are normal rings (as they are analytically normal). Now, we denote by $\mathfrak{n}_1, \cdots, \mathfrak{n}_t$ and by $\mathfrak{n}_1^*, \cdots, \mathfrak{n}_s^*$ the minimal prime divisors of (0) in R' and in R^*, respectively. By the analytical unramifiedness of spots, R^* (hence also R') has no nilpotent elements. The integral closure $\widetilde{R}'$ of R' in $\{R'\}$ is the direct sum of t normal rings (the derived normal rings of $R'/\mathfrak{n}_j$). By the lemma above, each direct summand is a local ring, therefore can be identified with the quotient ring of $\widetilde{R}'$ by one of its maximal ideals. Thus we see that $\widetilde{R}' = \sum S_j$ and that $t = s$. Moreover, the completion $(R'/n_j)^*$ is contained in $S_j^* = \widetilde{R}_j^*$ and hence an integral domain, q. e. d.

Proposition 4. *Let $\mathfrak{o}$ be a normal spot and let $\mathfrak{o}'$ be its Henselization. Then any prime ideal $\mathfrak{q}'$ of $\mathfrak{o}'$ is analytically irreducible.*

Proof. We adjoin a basis of $\mathfrak{q}'$ to $\mathfrak{o}$, take the derived normal ring $\mathfrak{o}_1$ of the extension ring and construct the quotient ring $\mathfrak{o}_2$ of $\mathfrak{o}_1$ by $\mathfrak{m}' \cap \mathfrak{o}_1$. Then $\mathfrak{o}_2$ is a normal spot, hence analytically irreducible, and has the same rank as $\mathfrak{o}'$. Therefore $\mathfrak{o}'$ contains $\mathfrak{o}_2$ as a dense subspace. From this, and from our proof of Proposition 3, the assertion of Proposition 4 follows immediately.

Proposition 5. *Let $\mathfrak{o}$ be a normal spot and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ be prime ideals of $\mathfrak{o}$. Let $\mathfrak{o}^*$ be the completion of $\mathfrak{o}$, and let $\mathfrak{p}_1^*, \cdots, \mathfrak{p}_g^*$ be prime ideals of $\mathfrak{o}^*$ such that each $\mathfrak{p}_i^*$ is a (minimal) prime divisor of $\mathfrak{p}_j \mathfrak{o}^*$ for some j. Let $\mathfrak{p}^*$ be a minimal prime divisor of $(\mathfrak{p}_1^*, \cdots, \mathfrak{p}_g^*)\mathfrak{o}^*$, and set $\mathfrak{p} = \mathfrak{p}^* \cap \mathfrak{o}$. Then we have* rank $\mathfrak{p}$ = rank $\mathfrak{p}^*$ *(and hence $\mathfrak{p}^*$ is a minimal prime divisor of $\mathfrak{p}\mathfrak{o}^*$).*

Proof. By Proposition 3, $(\mathfrak{p}_1^*, \cdots, \mathfrak{p}_g^*)\mathfrak{o}^*$ is the extended ideal of an ideal of the Henselization $\mathfrak{o}'$ of $\mathfrak{o}$. Therefore by Proposition 4, $\mathfrak{p}^*$ is the extended ideal of $\mathfrak{p}' = \mathfrak{p}^* \cap \mathfrak{o}'$. Since $\mathfrak{o}'$ is a quotient ring of an integral extension of $\mathfrak{o}$, we have rank $\mathfrak{p}^*$ = rank $\mathfrak{p}'$ = rank $\mathfrak{p}$.

Corollary. *Each component of the intersection of several sheets of one or more algebraic varieties, is a sheet of an algebraic variety.*

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