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A property of an ample linear system on a non-singular variety

By

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We shall treat here the same subject as is stated in the preceding paper[1] using the dual map into the Grassmann variety. The contents of this paper are almost as the same as the contents of § 3 of my paper " On the characteristic linear systems of algebraic families " (will appear in Illinois' Journal), but I would like to present here again as a memory of Prof. Zariski following the advice of Prof. Akizuki. Before to state the complete form of the final result we must introduce some auxiliary notions.

Let V^r be an irreducible variety and E be an ample linear system of divisors on V without fixed component. Then E defines an everywhere biregular birational transformation of V onto a projective variety V_E. Let $n=\dim E$, and k a common field of definition for V and E. Then the variety V_E is defined over k, and belongs to a projective space L^n (i. e. not contained in any hyperplane of L). Let P, $\bar{P}$ be the corresponding generic points of V, V_E over k and $T_{\bar{P}}$ the tangent linear variety to V_E at $\bar{P}$. Then the Plücker coordinates $c(T_{\bar{P}})$ is rational over $k(P)$, and the point $c(T_{\bar{P}})$ has a locus $V_E{}^*$ over k. We shall call this variety *the dual variety of V with respect to the linear system E*, and the map φ_E of V onto $V_E{}^*$ defined over k by $\varphi_E(P)=c(T_{\bar{P}})$ will be called the *dual map* of V onto V_E^*. The map φ_E is defined at every simple point of V.

Now our theorem is as follows.

Theorem 1. *Let E be an ample linear system on a non-singular variety V defined over k and assume that the dual map φ_E of V*

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1) Y. Akizuki and H. Matsumura, On the dimensions of algebraic system of curves with nodes on a surface, in the same number of this Memoirs.

with respect to the linear system E is everywhere 1 to 1.[2] *Let* $\mathfrak{M}_d$ *be the set of divisors which has at least d multiple points, together with their specializations over k, then* $\mathfrak{M}_d$ *form a finite number of algebraic subsystems of E. Let B be a component of* $\mathfrak{M}_d$ *such that the generic member of B has only a finite number of multiple points, then the dimension of B is not less than* $n-d$, *where n is the dimension of E.*

We shall divide the proof in several steps to make the roles of the assumptions clear.

Let V be a non-singular variety in a projective n-space L^n, not contained in any hyperplane, ank k a field of definition for V. Let L_1 be the linear system on V which are composed of the hyperplane sections of V, and V^* be the dual variety of V with respect to the linear system L_1 (which will be simply be called a dual variety of V), and φ be the dual map of V onto V^*. Then V^* is a subvariety of the Grassmann variety $\mathfrak{G}(r, n)$ which consists of the set of r dimensional linear varieties in L^n. Let L' be the dual space of L and T be the correspondence between L' and $\mathfrak{G}$ such that for any point x of L', $T(x)$ is the set of r dimensional linear varieties contained in the hyperplane x. Since $T(x)$ is also a Grassmann variety $\mathfrak{G}(r, n-1)$, T is an irreducible correspondence between L' and $\mathfrak{G}$. We shall say that a hyperplane x has *d contacts* with V if the intersection $(x\times V^*)\cap T$ contains d points. Now we have the

Lemma 1. *Let* $\mathfrak{M}_d$ *be the set of points of* L' *which has at least d contacts with V, together with their specializations over k, then* $\mathfrak{M}_d$ *form a bunch of subvarieties of* L', *normally algebraic over k.*

Proof. Let $\mathfrak{G}^{(d)}=\underbrace{\mathfrak{G}\times\cdots\times\mathfrak{G}}_{d}$ and $T^{(d)}$ be the correspondence between L' and $\mathfrak{G}^{(d)}$ such that for any point x of L' we have $T^{(d)}(x)=\underbrace{T(x)\times\cdots\times T(x)}_{d}$. We shall consider the intersection product $(L'\times V^*\times\cdots\times V^*)\cap T^{(d)}$, and $\mathfrak{B}_i(i=1, \cdots, s)$ be the components of the intersection. Let $\mathrm{proj}_{L'}\mathfrak{B}_i=B_i$, and select among B_i's such one that the generic point x of B_i has at least d contacts with V. Let $B_i(i=1, \cdots, t)$ $(t\leqq s)$ be such ones. We shall show

2) As an example of the linear system E satisfying these conditions we can give the linear system on V composed of the sections of V with the hypersurfaces of order $m(\geqq 2)$.

that $\bigcup_{i=1}^{t} B_i = \mathfrak{M}_d$. It is clear by definition that $\bigcup_{i=1}^{t} B_i$ is contained in $\mathfrak{M}_d$. Let x be a point of $\mathfrak{M}_d$. It is sufficient to show that x is contained in $\bigcup_{i=1}^{t} B_i$ under the assunption that x has at least d contacts with V. Let $P_i^*(i=1, \cdots, d)$ be the points of V^* contained in $T(x)$, then the point $x \times P_1^* \times \cdots \times P_d^*$ is contained in $(L' \times V^* \times \cdots \times V^*) \cap T^{(d)}$. Let $\mathfrak{B}$ be the component of this intersection containing the point $x \times P_1^* \times \cdots P_d^*$, and $\bar{x} \times \bar{P}_1^* \times \cdots \times \bar{P}_d^*$ be a generic point of $\mathfrak{B}$ over $\bar{k}$ (it is clear that $\mathfrak{B}$ is algebraic over k). Then since $x \times P_1^* \cdots \times P_d^*$ is a specialization of $\bar{x} \times \bar{P}_1^* \times \cdots \times \bar{P}_d^*$ over $\bar{k}$, and $P_i^* \neq P_j^*$ for $i \neq j$, we see that the hyperplane $\bar{x}$ has at least d contacts with V. Hence $B = \text{proj}_{L'} \mathfrak{B}$ must be one of $B^i (1 \leqq i \leqq t)$. This prove that $\mathfrak{M}_d = \bigcup_{i=1}^{t} B_i$.

It is immediate to see that the conjugate of B_i $(1 \leqq i \leqq t)$ over k is also one of $B_i (1 \leqq i \leqq t)$ and $\mathfrak{M}_d$ is seen to be normally algebraic over k. q. e. d.

We shall recall here that the Grassmann variety $\mathfrak{G}(r, n)$ is an irreducible variety of dimension $(r+1)(n-r)$, defined over the field of definition for the ambiant space L^n. Now we shall show the

Lemma 2. *$\mathfrak{G}(r, n)$ is a non-singular variety.*

Proof. Let P^* be an arbitrary point of $\mathfrak{G}$ and $\bar{P}^*$ the generic point of $\mathfrak{G}$ over a field of definition k for $\mathfrak{G}$. Let H, $\bar{H}$ be the r dimensional linear varieties corresponding to P^* and $\bar{P}^*$ respectively. Let σ be the proper projective transformation of L^n onto itself such that $\sigma(\bar{H}) = H$. Then σ induces an everywhere biregular birational transformation of $\mathfrak{G}$ onto itself, transforming the point $\bar{P}^*$ onto P^*. Since $\bar{P}^*$ is a simple point of $\mathfrak{G}$, P^* is also a simple point of $\mathfrak{G}$. q. e. d.

Lemma 3. *Assume that the dual variety V^* of V has the dimension $r(=\dim V)$, then the component of $\mathfrak{M}_d$ whose generic member has at most a finite number of contacts with V has the dimension $\geqq n-d$.*

Proof. We shall now count the dimensions of the components $\mathfrak{B}_i$ (appeared in the proof of Lemma 1). Since $L' \times \mathfrak{G}^{(d)}$ is a non-singular variety and $\dim \mathfrak{G}^{(d)} = (r+1)(n-r)d$, $\dim T^{(d)} = n + (r+1)(n-1-r)d$ we see that the dimensions of $\mathfrak{B}_i$ are all of $\geqq n-d$. Let $\mathfrak{B}$ be one of $\mathfrak{B}_i$'s $(i=1, \cdots, t)$ and $\text{proj}_{L'} \mathfrak{B} = B$, and assume that the generic point x of B over $\bar{k}$ has at most a finite number of contacts with V. This means that the components of $(x\times V^*)\cap T$ are all of dimension 0, hence the point in $(x\times V^*)\cap T$ are all algebraic over $k(x)$. Let $x\times P_1^*\times\cdots\times P_d^*$ be a generic point of $\mathfrak{B}$ over $\bar{k}$, then $\dim_k(x, P_1^*, \cdots, P_d^*)=\dim_k(x)$ and we see that $\dim B=\dim\mathfrak{B}\geqq n-d$. q. e. d.

Let L_1 be as before the linear system of hyperplane sections of V, and C_x be a member of L_1 which is the intersection product of the hyperplane x and V. Then a point P of V is a multiple point of C_x if and only if x touches with V at P, i. e. $\varphi(P)$ is contained in $T(x)$.

Lemma 4. *Assume that the dual map φ of V onto the dual variety V^* is everywhere 1 to 1. Then the set of divisors of* L_1 *which has at least d multiple points, together with their specializations form a finite number of algebraic subsystems of* L_1.

Proof. In this case, a hyperplane x has d contacts with V if and only if the divisor C_x has d multiple points, on acount of the 1 to 1 correspondence of φ. Moreover the correspondence between the point x of L' and the member C_x of L_1 is also 1 to 1, since V is not contained in any hyperplane and V has no singular subvarieties of codimension 1. The rest follows from the preceding Lemmas. q. e. d.

Now the proof of the Theorem 1 is immediate. In fact, since the linear system E is ample, E defines an everywhere biregular birational map of V onto V_E which is contained in a projective $n(=\dim E)$ space, not contained in any hyperplane. Moreover any member of E corresponds, in biregularly, to a hyperplane section of V_E. Thus the Theorem 1 is reduced to the case of the linear system L_1.

It is not difficult to generalize the Theorem 1 to the case when V has some singular subvarieties whose codimensions are $\geqq 2$. In this case we say that a member C of E has d *variable multiple points* $P_i(i=1, \cdots, d)$, if P_i's are all simple points of V. Then if we assume that the dual map φ of V onto V_E^* is everywhere 1 to 1 except the multiple points of V and if we denote by $\mathfrak{M}_d$ the set of divisors which has at lest d variable multiple points, together with their specializations, then the Theorem 1 holds in this generalized form.

Theorem. 2. *Let E be an ample linear system on an irreducible* *variety V^r which has no singularity of codimension 1, and assume that $n = \dim E$ is greater than r.*[3] *Let $\mathfrak{M}$ be the set of divisors in E which has at least one variable singular point. Then $\mathfrak{M}$ exists and it is an irreducible algebraic subsystem of E defined over a field k which is a common field of definition for V and E. Moreover if the dual variety V_E^* of V with respect to the linear system E has the same dimension r as V, then* $\dim \mathfrak{M} = n-1$.

Proof. By the same process as before, we can reduce the problem to the case where E is a linear system L_1, hence we can assume that V is contained in a projective n-space L^n, not contained in any hyperplane. Let L', $\mathfrak{G}(r, n)$ and the correspondence T between L' and $\mathfrak{G}$ be as before, and we shall show that the intersection product $(L' \times V^*) \cap T$ is an irreducible variety defined over k. Let P^* be an arbitrary point of $\mathfrak{G}$, then the intersection product $(L' \times P^*) \cdot T = T^{-1}(P^*) \times P^*$ is defined and it is an irreducible variety defined over $k(P^*)$, and whose dimension is equal to $n - r - 1$ ($\geqq 0$, by the assumption). Let P^* be a generic point of V^* over k and x be a generic point of $T^{-1}(P^*)$ over $k(P^*)$. Then, since $k(x, P^*)$ is a regular extension of k, the point $x \times P^*$ has a locus S over k. We shall show that $(L' \times V^*) \cap T = S$. Counting the dimension, we see that S is a proper component of this intersection. Let $x' \times Q^*$ be an arbitrary point of $(L' \times V^*) \cap T$. Then since Q^* is a point of V^*, Q^* is a specialization of P^* over k. Let y be an isolated specialization of x over $P^* \to Q^*$ with reference to k. Then we have $\dim_{k(P^*)}(x) \leqq \dim_{k(Q^*)}(y)$. Hence y must be a generic point of $T^{-1}(Q^*)$ over $k(Q^*)$ and the equality holds. Since x' is a point of $T^{-1}(Q^*)$ (x', Q^*) is a specialization of (y, Q^*) over k. Thus we see that (x', Q^*) is a specialization of (x, P^*) over k and the point $x' \times Q^*$ is contained in S. Thus the algebraic family $\mathfrak{M}$ is parameterized by an irreducible variety $\mathrm{proj}_{L'} S$, which is defined over k. Since E is an ample linear system, the generic member of E cannot have variable singularities, and the dimension of $\mathfrak{M}$ cannot be n. Thus, under the assumptions of the theorem, we must have $\dim \mathfrak{M} = n-1$. q. e. d.

At the end of the paper we shall propose here some questions which seems to me very interesting. To avoid the confusion we

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3) This assumption $n > r$ is essential. In fact if $n = r$ we can find the following counter example: Let V be a projective r space L, and E be the linear system of hyperplanes of L, then there cannot exist such $\mathfrak{M}$.

shall restrict ourselves to the consideration of a non-singular variety. Let E be an ample linear system on a non-singular variety which satisfies the conditions of Theorem 1. We shall call a component B of $\mathfrak{M}_d$ a *proper component* of $\mathfrak{M}_d$, if the generic member of B has exactly d multiple points. Then if the linear system is given, we ask the upper bound for d such that $\mathfrak{M}_d$ containes a proper component. Is it also possible to decide the exact dimension of the proper component of $\mathfrak{M}_d$? The author has some reason to imagine that the excess of the dimension of the proper component of $\mathfrak{M}_d$ from the integer $n-d(n=\dim E)$ has a close connection with the geometric genus if the ambient variety V is a non singular surface. In the case of plane algebraic curves, these problems are treated by F. Severi.[4)]

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4) Cf. F. Severi, Vorlesungen über Algebraische Geometrie, Teubner, Berlin (1921), Anfang F, or O. Zariski, Algebraic surfaces, Ergebnisse der Mathematik (1935), Chap. VIII.