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The differential geometry of spaces with analytic distances

By

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We consider here an analytic n-manifold V with a distance function $d(p, q)$ defined for all points p, q of V, satisfying the following two axioms:

1. $d(p, q)=0$ for $p=q$,

2. $d(p, q)=d(q, p)$;

at present we do not assume the so-called triangle axiom.

Further we suppose that there exists a coordinate neighborhood U containing any pair of points p, q and that the square of $d(p, q)$ is analytically expressed by the coordinates (x) and (y) of p and q respectively. Such a manifold V will be called the space with analytic distance and denoted by S^n. The function $g(x, y)$ defined by

$$g(x, y)=-\frac{1}{2}[d(x, y)]^2$$

will be called *fundamental function* of the space S^n and we shall investigate the properties of S^n from the standpoint of the differential geometry by means of the analyticity of the fundamental function.

The space with analytic distance was formerly investigated by one of my senior *Tsutomu Ôtake*, who died about ten years ago without publishing his note. In this paper we shall introduce and develope his discussions.

§ 1. Tensors in the space S^n.

It is natural by means of the geometrical meaning that the fundamental function $g(x, y)$ of S^n is invariant under a transformation of the local coordinates

$$(1\cdot1)\qquad x^i=x^i(\bar{x}),\quad y^i=y^i(\bar{y}).$$

Such a transformation of coordinates is thought of as one in the product $S^n\times S^n$. A function of (x) and (y), which is invariant under $(1\cdot1)$, is called a *scalar*. If we put

$$(1\cdot2)\qquad g_i=\frac{\partial}{\partial x^i}g(x, y),\quad g_{(i)}=\frac{\partial}{\partial y^i}g(x, y),$$

these are subjected to the transformations

$$\bar{g}_i=g_a\frac{\partial x^a}{\partial\bar{x}^i},\quad \bar{g}_{(i)}=g_{(a)}\frac{\partial y^a}{\partial\bar{y}^i}.$$

If n functions u_i of variables (x, y) are subjected to the transformation

$$\bar{u}_i=u_a\frac{\partial x^a}{\partial\bar{x}^i},$$

then u_i are called the components of the covariant vector (u) with respect to (x). We define similarly a covariant vector with respect to (y). In particular, the functions g_i defined by $(1\cdot2)$ are components of the covariant vector, which will be called the *slope vector* with respect to (x) and $g_{(i)}$ is called the slope vector with respect to (y).

The definition of a tensor of any degree is immediately given. Thus, if quantities $T^{i(k)}_{j(l)}$ are subjected to the transformation

$$\bar{T}^{i(k)}_{j(l)}=T^{a(c)}_{b(d)}\frac{\partial\bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial\bar{x}^j}\frac{\partial\bar{y}^k}{\partial y^c}\frac{\partial y^d}{\partial\bar{y}^l},$$

we call $T^{i(k)}_{j(l)}$ components of the tensor (T), which is covariant of the first degree and is contravariant of the first degree with respect to (x) and also to (y).

If we set

$$g_{i(j)}=\frac{\partial^2}{\partial x^i\partial y^j}g(x, y)=\frac{\partial g_i}{\partial y^j}=\frac{\partial g_{(j)}}{\partial x^i},$$

then we see easily that the functions $g_{i(j)}$ are components of the tensor, which is covariant of the first degree with respect to (x) and (y). This tensor is called the *relative metric tensor* at the pair of the points $p(x)$ and $q(y)$. If there exists a region R of the space S^n, such that the determinet $|g_{i(j)}(x, y)|$ does not vanish at any points (x) and (y) in R, then we shall say that the region R is *proper in* S^n. An euclidean space is proper in itself. Throughout the paper it is understood that we consider a proper region alone. Then we construct the inverse matrix $(g^{i(j)})$ of $(g_{i(j)})$, and it is easily seen that $g^{i(j)}$ are components of the contravariant tensor, which is also called the relative metric tensor.

In Riemannian geometry, we obtain from a given tensor, making use of the covariant or contravariant components of the metric tensor, tensors of the same degree but different character. This process is usually called as rising the subscripts and lowing the superscripts. Similarly in our case, we can obtain from a given tensor, making use of the relative metric tensor, new tensors, which shall be called to be *conjugate* to the given tensor. For an example, we take a tensor $T^{i\cdot(k)}_{\cdot j\cdot(l)}$ and then we have various types of tensors as follows:

$$T_{(h)j\cdot(l)}^{\quad(k)} = g_{i(h)} T^{i\cdot(k)}_{\cdot j\cdot(l)},$$

$$T^{i\cdot(k)h}_{\cdot j} = g^{h(l)} T^{i\cdot(k)}_{\cdot j\cdot(l)},$$

and so on. Especially, from covariant components of the slope vectors g_i and $g_{(i)}$, we have contravariant vectors conjugate to them, that is,

$$g^i = g^{i(j)} g_{(j)}, \qquad g^{(i)} = g^{j(i)} g_j,$$

these be called the contravariant components of the slope vectors.

If we contract a tensor by the slope vectors, then we obtain new tensors of the lower degree. As an example, for a tensor $T^{i\cdot(k)}_{\cdot j\cdot(l)}$, we have

$$T^{0\cdot(k)}_{\cdot j\cdot(l)} = g_i T^{i\cdot(k)}_{\cdot j\cdot(l)}, \qquad T^{0\cdot(0)}_{\cdot j\cdot(l)} = g_{(k)} T^{0\cdot(k)}_{\cdot j\cdot(l)},$$

$$T^{i\cdot(k)}_{\cdot j\cdot(0)} = g^{(l)} T^{i\cdot(k)}_{\cdot j\cdot(l)},$$

and so on. This process will be called the *degeneration* of a tensor and may be repeated till we have the scaler. This scalar will be called the *slope function* of the tensor. Thus, for the above tensor $T^{i\cdot(k)}_{\cdot j\cdot(l)}$, the slope function is given by

$$T = T^{i\cdot(k)}_{\cdot j\cdot(l)} g_i g^j g_{(k)} g^{(l)}.$$

It is clear that a slope function of a tensor is equal to one of tensors conjugate to it. We see easily that the slope function of the relative metric tensor is equal to the slope function of the slope vectors.

§ 2. Relative covariant differentiations.

The relative metric tensor $g_{i(j)}$ of S^n is transformed under (1·1) as follows:

$$g_{i(j)}=\bar{g}_{a(b)}\frac{\partial\bar{x}^a}{\partial x^i}\frac{\partial\bar{y}^b}{\partial y^j}.$$

Differentiation of the equation with respect to x^k and contraction by $g^{h(j)}$ give

$$\frac{\partial g_{i(j)}}{\partial x^k}g^{k(j)}=\frac{\partial\bar{g}_{a(b)}}{\partial\bar{x}^c}\bar{g}^{d(b)}\frac{\partial x^h}{\partial\bar{x}^d}\frac{\partial\bar{x}^a}{\partial x^i}\frac{\partial\bar{x}^c}{\partial x^k}+\frac{\partial^2\bar{x}^a}{\partial x^i\partial x^k}\frac{\partial x^h}{\partial\bar{x}^a}.$$

Hence, if we put

$$(2\cdot1)^{(1)}\qquad \Gamma^h_{ik}(x,y)=\frac{\partial g_{i(j)}}{\partial x^k}g^{h(j)},$$

then it follows

$$(2\cdot1)\qquad \frac{\partial^2\bar{x}^a}{\partial x^i\partial x^j}=\frac{\partial\bar{x}^a}{\partial x^k}\Gamma^k_{ij}-\frac{\partial\bar{x}^b}{\partial x^i}\frac{\partial\bar{x}^c}{\partial x^j}\bar{\Gamma}^a_{bc}.$$

Similarly, if we put

$$(2\cdot3)\qquad \Gamma^{(h)}_{(i)(k)}(x,y)=\frac{\partial g_{j(i)}}{\partial y^k}g^{j(h)},$$

then we have

$$\frac{\partial^2\bar{y}^a}{\partial y^i\partial y^j}=\frac{\partial\bar{y}^a}{\partial y^k}\Gamma^{(k)}_{(i)(j)}-\frac{\partial\bar{y}^b}{\partial y^i}\frac{\partial\bar{y}^c}{\partial y^j}\bar{\Gamma}^{(a)}_{(b)(c)}.$$

These Γ^h_{ik} and $\Gamma^{(h)}_{(i)(k)}$ are evidently symmetric with respect to subscripts, and we call them the *coefficients of linear connection* of S^n, Making use of these quantities, we shall give a process, by which we obtain from a tensor new tensors of higher degree.

We take first a scalar $T(x, y)$ and it is easily verified that the quantities

$$(2\cdot4)\qquad T_{;i}=\frac{\partial T}{\partial x^i},\qquad T_{;(i)}=\frac{\partial T}{\partial y^i},$$

define respectively components of the vector. Next, let $T_i(x, y)$ be a covariant vector with respect to (x), then the components are transformed under (1·1) into $\overline{T}_i$, which are given by

$$T_i=\overline{T}_a\frac{\partial\overline{x}^a}{\partial x^i}.$$

Differentiating with respect to x^j and substituting from (2·2), we see that the quantities

$$T_{i\,;\,j}=\frac{\partial T_i}{\partial x^j}-T_k\Gamma^k_{ij}$$

are components of covariant tensor of the second degree with respect to (x). The tensor as thus obtained will be called the *derived tensor* by means of the process of the *relative covariant differentiation* with respect to (x). On the other hand, we see that the quantities

$$T_{i\,;\,(j)}=\frac{\partial T_i}{\partial y^j}$$

are components of a tensor, which is covariant of the first degree with respect to (x) and also to (y). This tensor is said to be obtained from the original tensor by the relative covariant differentiation with respect to (y).

The above processes may be clearly generalized to the cases of tensors of any degree, and thus we have the derived tensors of the given tensor. For instance, we take a tensor $T^{i(k)}_{j(l)}$ and its derived tensors are given by the following equations:

$$T^{i(k)}_{j(l)\,;\,h}=\frac{\partial T^{i(k)}_{j(l)}}{\partial x^h}+T^{a(k)}_{j(l)}\Gamma^i_{ah}-T^{i(k)}_{a(l)}\Gamma^a_{jh},$$

$$T^{i(k)}_{j(l)\,;\,(h)}=\frac{\partial T^{i(k)}_{j(l)}}{\partial y^h}+T^{i(a)}_{j(l)}\Gamma^{(k)}_{(a)(h)}-T^{i(k)}_{j(a)}\Gamma^{(a)}_{(l)(h)}.$$

Especially, for a scalar $T(x, y)$, its derived tensors are defined by (2·4).

The derived tensors of the slope vectors are given by

$$(2\cdot5)\qquad \begin{aligned} g_{i\,;\,j}&=\frac{\partial^2 g}{\partial x^i\partial x^j}-\frac{\partial g}{\partial x^k}\Gamma^k_{ij}, & g_{i\,;\,(j)}&=g_{i(j)},\\ g_{(i)\,;\,(j)}&=\frac{\partial^2 g}{\partial y^i\partial y^j}-\frac{\partial g}{\partial y^k}\Gamma^{(k)}_{(i)(j)}, & g_{(i)\,;\,j}&=g_{j(i)}, \end{aligned}$$

these satisfy the relations

$$g_{i;j}=g_{j;i}, \quad g_{(i);(j)}=g_{(j);(i)}, \quad g_{i;(j)}=g_{(j);i}.$$

For the relative metric tensor, we have as a consequence of (2·1) and (2·3)

$$(2\cdot 6) \qquad g_{i(j);k}=0, \quad g_{i(j);(k)}=0,$$

from which we have

$$(2\cdot 7) \qquad g^{i(j)}{}_{;k}=0, \quad g^{i(j)}{}_{;(k)}=0.$$

Therefore the relative metric tensor is relatively covariant constant. It follows that the processes of the covariant differentiation and construction of tensors conjugate to a tensor are commutative. Finally we see, for the contravariant components of the slope vectors,

$$(2\cdot 8) \qquad g^{i}{}_{;j}=\delta^i_j, \quad g^{(i)}{}_{;(j)}=\delta^i_j$$

by means of (2·5), where δ's are the Kronecker's deltas.

§ 3. Relative curvature tensors.

We consider first a contravariant vector $u^i(x, y)$ with respect to (x). The derived tensor with respect to (x) is given by

$$u^i{}_{;j}=\frac{\partial u^i}{\partial x^j}+u^a\Gamma'^i_{aj},$$

and hence we have

$$u^i{}_{;j;k}=\frac{\partial^2 u^i}{\partial x^j\partial x^k}+\frac{\partial u^a}{\partial x^j}\Gamma'^i_{ak}+\frac{\partial u^a}{\partial x^k}\Gamma'^i_{aj}-u^i{}_{;a}\Gamma'^a_{jk}$$
$$+u^a\left(\frac{\partial\Gamma'^i_{aj}}{\partial x^k}+\Gamma'^b_{aj}\Gamma'^i_{bk}\right).$$

Accordingly, if we put

$$(3\cdot 1) \qquad \Gamma'^i_{a\cdot jk}(x, y)=\frac{\partial\Gamma'^i_{aj}}{\partial x^k}-\frac{\partial\Gamma'^i_{ak}}{\partial x^j}+\Gamma'^b_{aj}\Gamma'^i_{bk}-\Gamma'^b_{ak}\Gamma'^i_{bj},$$

then we have the following equation:

$$(3\cdot 2) \qquad u^i{}_{;j;k}-u^i{}_{;k;j}=u^a\Gamma'^i_{a\cdot jk}.$$

However it is shown that the quantities as above defined by (3·1) are identically equal to zero, and the same is true for $\Gamma'^{(i)}_{(a)\cdot(j)(k)}(x, y)$, which are constructed from $\Gamma'^{(i)}_{(j)(k)}$ similarly to (3·1). This facts are verified by direct calculation, but we shall here prove them by applying the rule (3·2) to the relative metric tensor. The rule can be applied equally well to tensor of any degree if only it is contravariant of the first degree with respect to (x). Thus we have the equation

$$g^{i(b)}{}_{;j;k}-g^{i(b)}{}_{;k;j}=g^{a(b)}\Gamma'^{i}_{a\cdot jk},$$

the left hand member be equal to zero as a result of (2·7). Hence we have $g^{a(b)}\Gamma'^{i}_{a\cdot jk}=0$, from which the above statemens are immediately proved. Consequently we have for any tensor (T) the identity

$$T^{\cdot\cdot\cdot}_{\cdot\cdot\cdot;j;k}-T^{\cdot\cdot\cdot}_{\cdot\cdot\cdot;k;j}=0, \tag{3·3}$$

and also we obtain

$$T^{\cdot\cdot\cdot}_{\cdot\cdot\cdot;(j);(k)}-T^{\cdot\cdot\cdot}_{\cdot\cdot\cdot;(k);(j)}=0.$$

Next, differentiating a vector $u^i(x, y)$ covariantly with respect to x^j and then y^k, we have

$$u^i_{;j;(k)}=\frac{\partial^2 u^i}{\partial x^j\partial y^k}+\frac{\partial u^a}{\partial y^k}\Gamma'^{i}_{aj}+u^a\frac{\partial\Gamma'^{i}_{aj}}{\partial y^k}.$$

On the other hand, by interchanging the order of the above differentiations,

$$u^i_{;(k);j}=\frac{\partial^2 u^i}{\partial x^j\partial y^k}+\frac{\partial u^a}{\partial y^k}\Gamma'^{i}_{aj}.$$

Hence we put

$$S^{\ i}_{j\cdot k(l)}(x, y)=\frac{\partial}{\partial y^l}\Gamma'^{i}_{jk}(x, y), \tag{3·4}$$

and it follows that

$$u^i_{;j(k)}-u^i_{;(k);j}=u^a S^{\ i}_{a\cdot j(k)}.$$

Thus we can easily establish the general formula for interchanging the order of differentiations with respect to (x) and (y). The formula is illustrated, for a tensor $T^{i(k)}_{j(l)}$, by the following equations:

$$\begin{aligned} &T^{i(k)}_{j(l);h;(m)}-T^{i(k)}_{j(l);(m);h}\\ &=T^{a(k)}_{j(l)}S^{\ i}_{a\cdot h(m)}-T^{i(k)}_{a(l)}S^{\ a}_{j\cdot h(m)}-T^{i(a)}_{j(l)}S^{\ \ (k)}_{(a)\cdot(m)h}\\ &\quad+T^{i(k)}_{j(a)}S^{\ \ (a)}_{(l)\cdot(m)h}, \end{aligned} \tag{3·5}$$

where $S^{\ \ (i)}_{(j)\cdot(k)l}$ is defined by the similar equation to (3·4). The quantities $S^{\ i}_{j\cdot k(l)}$ and $S^{\ \ (i)}_{(j)\cdot(k)l}$ are clearly components of tensors, which are called the ***relative curvature tensors***, the former be (x)-component and the latter (y)-component.

Let us find some identities satisfied by the components of the relative curvature tensors. From the definition, we have

$$S_{j\cdot k(l)}^{\;i} = S_{k\cdot j(l)}^{\;i} . \tag{3·6}$$

Next, if we apply (3·5) to the relative metric tensor, we have

$$g_{i(j);k;(l)} - g_{i(j);(l);k} = -g_{a(j)} S_{i\cdot k(l)}^{\;a} + g_{i(a)} S_{(j)\cdot(l)k}^{\;(a)} = 0 ,$$

$$g^{i(j)}{}_{;k;(l)} - g^{i(j)}{}_{;(l);k} = g^{a(j)} S_{a\cdot k(l)}^{\;i} - g^{i(a)} S_{(a)\cdot(l)k}^{\;(j)} = 0 .$$

These equations and (3·6) are expressed in terms of the tensors conjugate to the relative curvature tensors as follows:

$$S_{i(j)k(l)} = S_{(j)i(l)k} , \qquad S^{(j)i}{}_{\cdot\cdot k(l)} = S^{i(j)}{}_{\cdot\cdot(l)k} , \qquad S_{j(i)k(l)} = S_{k(i)j(l)} ,$$

from which we have

$$S_{i(j)k(l)} = S_{i(l)k(j)} . \tag{3·7}$$

Next, we apply (3·5) to the slope vectors and then we get

$$g_{i;j;(k)} - g_{i;(k);j} = -g_a S_{i\cdot j(k)}^{\;a} ,$$

$$g_{(i);(j);k} - g_{(i);k;(j)} = -g_{(a)} S_{(i)\cdot(j)k}^{\;(a)} .$$

Making use of (2·5) and (2·6), these equations are written in the forms

$$g_{i;j;(k)} + g_a S_{i\cdot j(k)}^{\;a} = 0 ,$$

$$g_{(i);(j);k} + g_{(a)} S_{(i)\cdot(j)k}^{\;(a)} = 0 .$$

The second terms of the above are tensors obtained from the relative curvature tensors by degeneration by the slope vectors, which are written as $S_{ij(k)}$ and $S_{(i)(j)k}$ respectively. Thus we have

$$\frac{\partial g_{i;j}}{\partial y^k} + S_{ij(k)} = 0 , \qquad \frac{\partial g_{(i);(j)}}{\partial x^k} + S_{(i)(j)k} = 0 . \tag{3·8}$$

If we apply the similar process to the contravariant components of the slope vectors, then we have in virtue of (2.8)

$$g^{i}{}_{;(k);j} + S^{i}{}_{\cdot j(k)} = 0 , \qquad g^{(i)}{}_{;k;(j)} + S^{(i)}{}_{\cdot(j)k} = 0 , \tag{3·9}$$

where $S^{i}_{\cdot j(k)}$ and $S^{\cdot(i)}_{\cdot(j)k}$ are defined by

$$S^{i}_{\cdot j(k)}=g^{a}S_{a\cdot j(k)}^{\;\;i}\,,\qquad S^{\cdot(i)}_{\cdot(j)k}=g^{(a)}S_{(a)\cdot(j)k}^{\;\;(i)}\,.$$

§ 4. The Bianchi's identities and the space of constant relative curvature.

We shall give some identities satisfied by the derived tensors of the relative curvature tensors. We have first from (3·4)

$$S^{\;i}_{j\cdot k(l)\,;\,(h)}=\frac{\partial^2 \Gamma^{\,i}_{jk}}{\partial y^l\partial y^h}-S^{\;i}_{j\cdot k(a)}\Gamma^{(a)}_{(l)(h)}\,.$$

It implies that the equation

$$S^{\;i}_{j\cdot k(h)\,;\,(l)}=S^{\;i}_{j\cdot k(l)\,;\,(h)} \tag{4·1}$$

is satisfied. Next, covariant differentiation of the equation

$$u_{i\,;\,j\,;\,(k)}-u_{i\,;\,(k)\,;\,j}=-u_a S^{\;a}_{i\cdot j(k)}$$

with respect to x^l gives

$$u_{i\,;\,j\,(k)\,;\,l}-u_{i\,;\,(k)\,;\,j\,;\,l}=-u_{a\,;\,l}S^{\;a}_{i\cdot j(k)}-u_a S^{\;a}_{i\cdot j(k)\,;\,l}\,.$$

We subtract from the above the equation obtained by interchanging indices j and l, and then the following equation is got as a consequence of (3·3) :

$$\begin{aligned}&u_{i\,;\,j\,;\,(k)\,;\,l}-u_{i\,;\,l\,;\,(k)\,;\,j}\\&\quad=u_{a\,;\,j}S^{\;a}_{i\cdot l(k)}-u_{a\,;\,l}S^{\;a}_{i\cdot j(k)}-u_a(S^{\;a}_{i\cdot j(k)\,;\,l}-S^{\;a}_{i\cdot l(k)\,;\,j})\,.\end{aligned}$$

The left hand member is expressed by means of (3·5) as

$$\begin{aligned}&=(u_{i\,;\,j\,;\,l\,;\,(k)}+u_{a\,;\,j}S^{\;a}_{i\cdot l(k)}+u_{i\,;\,a}S^{\;a}_{j\cdot l(k)})\\&\quad-(u_{i\,;\,l\,;\,j\,;\,(k)}+u_{a\,;\,l}S^{\;a}_{i\cdot j(k)}+u_{i\,;\,a}S^{\;a}_{l\cdot j(k)})\,,\end{aligned}$$

which is written from (3·3) and (3·6) in the form

$$=u_{a\,;\,j}S^{\;a}_{i\cdot l(k)}-u_{a\,;\,l}S^{\;a}_{i\cdot j(k)}\,.$$

Consequently we obtain

$$S^{\;a}_{i\cdot j(k)\,;\,l}=S^{\;a}_{i\cdot l(k)\,;\,j}\,. \tag{4·2}$$

It is clear that we have the equations analogous to (4·1) and (4·2) for (y)-components of the relative curvature tensors. These equations are thought of as the generalizations of the Bianchi's identities satisfied by the curvature tensor of Riemannian geometry.

The equations $(4\cdot 1)$ and $(4\cdot 2)$ give the following identities satisfied by the tensors conjugate to the derived tensors, that is,

$$(4\cdot 3)\qquad \begin{aligned} &S_{h(i)j(k);(l)}=S_{h(i)j(l);(k)}\,,\\ &S^{(h)i}_{\cdot\ \cdot j(k);(l)}=S^{(h)i}_{\cdot\ \cdot j(l);(k)}\,,\\ &S_{h(i)j(k);l}=S_{h(i)l(k);j}\,,\\ &S^{(h)i}_{\cdot\ \cdot j(k);l}=S^{(h)i}_{\cdot\ \cdot l(k);j}\,. \end{aligned}$$

Next, covariant differentiation of $(3\cdot 8)$ with respect to y^l gives

$$S_{ij(k);(l)}=S_{ij(l);(k)}\,.$$

We also obtain from $(3\cdot 9)$

$$S^{i}_{\cdot j(k);l}=S^{i}_{\cdot l(k);j}\,.$$

Now, we consider such a space C^n that the tensor $S_{i(j)k(l)}$ are expressed in the form

$$S_{i(j)k(l)}=g_{i(j)}p_{k(l)}+g_{k(j)}p_{i(l)}\,,$$

where $p_{i(j)}$ is a tensor. If follows immediately from $(3\cdot 7)$ that $p_{i(j)}=\rho g_{i(j)}$, ρ be a function of (x, y). Hence we have

$$(4\cdot 4)\qquad S_{i(j)k(l)}=\rho(g_{i(j)}g_{k(l)}+g_{k(j)}g_{i(l)})\,.$$

Covariantly differentiating $(4\cdot 4)$ with respect to x^m and making use of $(4\cdot 3)$, we have $\rho_{;m}=0$. The similar process and $(4\cdot 3)$ give also $\rho_{;(m)}=0$. Hence a scalar ρ is constant. This result is similar to the well-known theorem of Schur in Riemannian geometry. The space C^n as now considered will be called the *space of constant relative curvature* and ρ in $(4\cdot 4)$ the relative curvature.

§ 5. Riemannian spaces associating with S^n.

It follows from the definition of the fundamental function of S^n

$$(5\cdot 1)\qquad g(x, x)=0\,.$$

We shall prove the identities

$$(5\cdot 2)\qquad g_i(x, x)=0\,,\qquad g_{(i)}(x, x)=0$$

In fact, we see from $(5\cdot 1)$

$$\frac{\partial g(x, x)}{\partial x^i}=0=\left(\frac{\partial g(x, y)}{\partial x^i}\right)_{(x, x)}+\left(\frac{\partial g(x, y)}{\partial y^i}\right)_{(x, x)}\,,$$

which is written in the form $g_i(x, x)+g_{(i)}(x, x)=0$. Since $g(x, y)$ is symmetric function of (x) and (y), we have $g_i(x, x)=g_{(i)}(x, x)$, from which it follows (5·2).

If (x) and $(x+dx)$ are neighboring points, then the fundamental function $g(x+dx, x)$ is expanded as follows:

$$g(x+dx, x)=\frac{1}{2}\left(\frac{\partial^2 g(x, y)}{\partial x^i \partial x^j}\right)_{(x,x)} dx^i dx^j+\cdots .$$

Hence, if we put

$$g_{ij}(x)=-\left(\frac{\partial^2 g(x, y)}{\partial x^i \partial x^j}\right)_{(x,x)}, \tag{5·3}$$

then we have

$$[d(x+dx, x)]^2=g_{ij}(x)\,dx^i dx^j+\cdots , \tag{5·4}$$

where the function $d(x, y)$ is the distance between (x) and (y). We obtain from (2·5) and (5·2) $g_{ij}(x)=-g_{i\,;\,j}(x, x)$. On the other hand, we get

$$\frac{\partial g_i(x, x)}{\partial x^j}=\left(\frac{\partial g_i(x, y)}{\partial x^j}\right)_{(x,x)}+\left(\frac{\partial g_i(x, y)}{\partial y^j}\right)_{(x,x)},$$

from which we obtain by means of (2·5), (5·2) and (5·3) $-g_{ij}(x)+g_{i(j)}(x, x)=0$. Therefore we have

$$g_{ij}(x)=g_{i(j)}(x, x)=-g_{i\,;\,j}(x, x). \tag{5·5}$$

It is clear that the functions $g_{ij}(x)$ are components of a covariant tensor of the second degree with respect to (x), and the quadratic form $g_{ij}p^ip^j$ is positive definite by means of (5·4). Therefore we can define a Riemannian metric $ds^2=g_{ij}dx^idx^j$ in an neighborhood of a point (x) and thus we have a Riemannian space V^n, the underlying manifold be the same as S^n. We call V^n the Riemannian space *associating with* S^n. Any tensor of V^n is thought of as one of S^n, whose components are functions of variables (x) alone, and hence the tensor $g_{ij}(x)$ as above defined is called the *metric tensor* of S^n.

Let us find the relation between the coefficients of the linear connection in V^n, that is to say, the Christoffel's symbols $\left\{ {i \atop jk} \right\}$ constructed by the metric tensor g_{ij} and coefficients of the linear connection in S^n. We obtain immediately by means of (2·1) and (2·3) $\Gamma^{*i}_{jk}(x, x)=\Gamma^{*(i)}_{(j)(k)}(x, x)$. Also, from (5·5), we get

$$\frac{\partial g_{ij}(x)}{\partial x^k}=\left(\frac{\partial g_{i(j)}(x, y)}{\partial x^k}\right)_{(x,x)}+\left(\frac{\partial g_{i(j)}(x, y)}{\partial y^k}\right)_{(x,x)}.$$

The right hand menber are written from (2·6) in the form

$$=g_{a(j)}(x, x)\,\Gamma'^{a}_{ik}(x, x)+g_{i(a)}(x, x)\,\Gamma'^{(a)}_{(j)(k)}(x, x).$$

Hence we have as a consequence of (5·5)

$$\frac{\partial g_{ij}(x)}{\partial x^k}=g_{aj}(x)\,\Gamma'^{a}_{ik}(x, x)+g_{ia}(x)\,\Gamma'^{a}_{jk}(x, x).$$

Since, in V^n associating with S^n, the equation

$$\frac{\partial g_{ij}(x)}{\partial x^k}=g_{aj}(x)\left\{{a \atop ik}\right\}(x)+g_{ia}(x)\left\{{a \atop jk}\right\}(x),$$

is satisfied and both of Γ'^{i}_{jk} and $\left\{{i \atop jk}\right\}$ are symmetric with respect to subscripts, then we conclude

$$\left\{{i \atop jk}\right\}(x)=\Gamma'^{i}_{jk}(x, x)=\Gamma'^{(i)}_{(j)(k)}(x, x). \tag{5·6}$$

Further, we shall express the curvature tensor of V^n in terms of the relative curvature tensors of S^n. For this purpose we observe first

$$\frac{\partial\left\{{i \atop jk}\right\}(x)}{\partial x^l}=\left(\frac{\partial \Gamma'^{i}_{jk}(x, y)}{\partial x^l}\right)_{(x,x)}+\left(\frac{\partial \Gamma'^{i}_{jk}(x, y)}{\partial y^l}\right)_{(x,x)},$$

as a consequence of (5·6). The last term is equal to $S_{j\cdot k(l)}^{i}(x, x)$ from (3·4). Hence, from the above equation and $\Gamma'^{i}_{j\cdot kl}=0$, it follows that the curvature tensor $R_{j\cdot kl}^{i}$ of V^n

$$R_{j\cdot kl}^{i}=\frac{\partial\left\{{i \atop jk}\right\}}{\partial x^l}-\frac{\partial\left\{{i \atop jl}\right\}}{\partial x^k}+\left\{{a \atop jk}\right\}\left\{{i \atop al}\right\}-\left\{{a \atop jl}\right\}\left\{{i \atop ak}\right\}$$

is given by

$$R_{j\cdot kl}^{i}(x)=S_{j\cdot k(l)}^{i}(x, x)-S_{j\cdot l(k)}^{i}(x, x). \tag{5·7}$$

We consider a connected, complete, analytic Riemannian space V^n. In this space, there is at least one geodesic arc joining any pair of points, which is the shortest arc among arcs joining the points[(2)]. Hence we can define a distance between the points as the arc-length of the shortest arc, and so we can define the fundamental function, which is analytic as a consequence of analyticity of the space. Therefore the Riemannian space has the space S^n, with which the former associates.

Conversely, if we consider a space S^n, then a Riemannian space V^n associating with S^n may be defined as above shown. Hence, any objects in V^n are regarded as one in S^n. Thus, we may define in S^n the length of a curve, the magnitude of a vector, the angle between two vectors at the same point and so on. We shall use these notions hereafter in order to study S^n.

Finally, we give the following theorem which is easily shown by (4·4), (5·5) and (5·7):

Theorem. *A Riemannian space associating with a space C^n of constant relative curvature is of constant curvature, the curvature be equal to the relative curvature of C^n.*

§ 6. Parallel displacement.

We shall introduce the notion of parallel displacements in our space S^n. Let $u^{(i)}(x_0)$ be a contravariant vector at a fixed point (x_0) in S^n, and then we construct covariant vector $v(x)$ at any point (x), which is conjugate to u and so the components v_j of v are given by

$$v_j(x) = u^{(a)}(x_0)\, g_{j(a)}(x, x_0).$$

Hence, by means of the contravariant components $g^{ij}(x)$ of the metric tensor at the point (x), the contravariant components $v^i(x)$ of v are given by

$$v^i(x) = u^{(a)} g_{j(a)}(x, x_0)\, g^{ij}(x). \tag{6·1}$$

In the first place, if we take $(x)=(x_0)$, it follows that the vector v coincides with the original vector u, making use of (5·5). Next, if (x) is an neighboring point of (x_0) and we put $dx^i = x^i - x_0^i$, then the functions $g_{j(a)}(x, x_0)$ and $g^{ij}(x)$ are expanded in the forms

$$g_{j(a)}(x, x_0) = g_{ja}(x_0) + g_{ka}(x_0) \left\{ {k \atop j\,l} \right\} (x_0)\, dx^l + \cdots,$$

$$g^{ij}(x) = g^{ij}(x_0) - g^{aj}(x_0) \left\{ {i \atop a\,l} \right\} (x_0)\, dx^l - g^{ia}(x_0) \left\{ {j \atop a\,l} \right\} (x_0)\, dx^l + \cdots,$$

and hence we see that du^i, the differential of the vector u, is given by $du^i = -u^a \left\{ {i \atop a\,j} \right\} dx^j$, from which it follows that the vector at

(x_0) is displaced in parallel with itself from (x_0) to (x) in the sense of Levi-Civita. Hence we may say that the vector as defined by (6·1) is obtained from the original vector u by *parallel displacement* from (x_0) to (x).

The notion of parallel displacement is formally generalized to tensors of any degree. As an example, components $U^i_j(x)$ of a tensor (U) obtained from a tensor $T^{(i)}_{(j)}(x_0)$ at a point (x_0) by parallel displacement to a point (x) are given by definition as follows:

$$U^i_j(x)=T^{(a)}_{(b)}g_{h(a)}(x, x_0)g^{hi}(x)g^{k(b)}(x, x_0)g_{kj}(x).$$

If the original vector $u(x_0)$ is non-zero and if the vector $v(x)$ as given by (6·1) vanishes, then the determinant $|g_{i(j)}(x, x_0)|$ must be equal to zero and hence (x) is not included in the proper region of (x_0). Therefore we may construct a field of parallel vectors in a proper region by means of parallel displacement of a vector at a point fixed in the region. As a consequence of the well-known theorem proved by H. Hopf[(3)], we have the

Theorem. *The Euler-Poincaré characteristic of a closed, proper space S^n is equal to zero.*

We shall examine a variation of length of a vector owing to a parallel displacement. The length $|u|$ of the original vector u is $\sqrt{g_{(a)(b)}u^{(a)}u^{(b)}}$, where $g_{(a)(b)}$ is the metric tensor at the original point (x_0), and length $|v|$ of the vector v defined by (6·1) is given by

$$|v|^2=g_{ij}v^i v^j=g_{ij}u^{(a)}g_{h(a)}g^{hi}u^{(c)}g_{d(c)}g^{dj},$$

from which we have

$$|v|^2=g_{i(a)}g_{j(b)}g^{ij}u^{(a)}u^{(b)}. \tag{6·2}$$

Thus we may say that a parallel displacement changes generally a length of a vector. The condition that the length of any vector at (x_0) is invariant under parallel displacement to (x) is given by the following equation:

$$g_{(a)(b)}=g_{i(a)}g_{j(b)}g^{ij}, \tag{6·3}$$

which is easily seen from (6·2).

Next, suppose that the vector $v(x)$ is obtained from $u(x_0)$ by parallel displacement from (x_0) to (x) and then the vector $\bar{u}(x_0)$ is obtained from the above $v(x)$ by parallel displacement from (x)

to the original point (x_0). The vector v is given by $(6\cdot 1)$ and $\bar{u}$ is given by means of $(6\cdot 1)$ as follows:

$$\bar{u}^{(i)}=u^{(j)}g_{k(j)}g^{ka}g_{a(h)}g^{(h)(i)}.$$

If the vector $\bar{u}$ coincides with u and latter is any vector at (x_0), then the condition $(6\cdot 3)$ is necessary and sufficient, as will be immediately shown. Therefore the condition that, if a vector $v(x)$ is parallel to $u(x_0)$, then the latter is conversely parallel to the former, is given by $(6\cdot 3)$.

Now, by the definition, when we construct a vector $v(x)$ at a point (x) parallel to a vector $u(x_0)$ at a point (x_0), we have first a vector v_i conjugate to $u^{(i)}$ and then have the contravariant components v^i of v by rising subscript. On the other hand, we may proceed in the following manner. First we construct covariant components of the given vector by lowing superscript and then the covariant components by parallel displacement of the covariant vector as above found. This process gives us the vector $\bar{v}$, its covariant components be as follows:

$$\bar{v}_i(x)=u_{(a)}g^{b(a)}g_{bi}=u^{(c)}g_{(c)(a)}g^{b(a)}g_{bi}.$$

Since the covariant and contravariant components of the vector should be thought of as different representations of the same object, it seems to be natural that $\bar{v}$ as above given coincides with covariant components of v, namely $v_i=u^{(c)}g_{i(c)}$. We have immediately the equation $(6\cdot 3)$ as the condition, under which the above requirement is satisfied.

Consequently, we know that the equation $(6\cdot 3)$ must be required in order that the notion of parallel displacement has various desirable properties. The space S^n, such that $(6\cdot 3)$ holds throughout the space, will be called the space *admitting a parallelism.* These spaces will be treated in detail in the Section 8.

§ 7. Geodesics with the center.

Let C be a curve in S^n, the equation be given by the form $x^i=x^i(s)$; where s is arc-length. The unit vector ξ tangent to C is given by $\xi^i=dx^i/ds$. We take any point (x_0) fixed in S^n and construct a vector $\lambda(x_0)$ from ξ by parallel displacement from (x) to (x_0). Then the components of λ are given by

$$\lambda^{(i)}(x_0)=\xi^a(x)g_{a(h)}(x,\ x_0)g^{(h)(i)}(x_0). \tag{7·1}$$

When the original point (x) of ξ displaces along the curve C, the final point of the vector λ will describe a curve γ around (x_0). If we denote by σ arc-length of γ, we obtain

$$d\sigma^2 = g_{(i)(j)}(x_0)\, d\lambda^{(i)}\, d\lambda^{(j)}. \tag{7·2}$$

From (7·1) we have

$$\frac{d\lambda^{(i)}}{ds} = \eta^a g_{a(h)} g^{(h)(i)}, \tag{7·3}$$

where we put

$$\eta^a = \frac{d\xi^a}{ds} + \xi^b \xi^c \Gamma^a_{bc}(x, x_0) = \xi^a{}_{;b}\, \xi'^b. \tag{7·4}$$

It follows from (7·3) that the vector $\mu^{(i)} = d\lambda^{(i)}/ds$ coincides with the vector obtained from η by parallel displacement to (x_0). The vector η is called the *first normal vector* of the curve C with reference to the point (x_0). From (7·2) and (7·3) it follows that the length $|\mu|$ of μ is given by $|d\sigma/ds|$.

The first normal η of C may be depend upon the choice of a reference point (x_0), and the condition that the vector be uniquely determined is given by the equation

$$S^i_{j \cdot k(l)}(x, x_0) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{7·5}$$

which is immediate result of (7·4). Further, the vector η may not be orthogonal to the tangent vector in general. We shall show in the next section that the first normal vector is orthogonal to the tangent vector in spaces admitting a parallelism.

A curve C, such that the curve γ as above defined is reduced to a point,i s called the *geodesic with the center* (x_0). For such a curve, each vector obtained from its tangent vector by parallel displacement to the center (x_0) is constant, as the origin of tangent vector displaces along C. From (7·4) it follows that the differential equation

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}(x, x_0) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \tag{7·6}$$

defines the geodisic with the center (x_0). The equation (7·6) is analogous to the equation of geodesic in Riemannian space. It is clear that the condition for a geodesic, such that any point of S^n

may be its center, is given by (7·5) and (7·6).

We consider a space C^n of constant relative curvature. According to (4·4) and (7·5) we have easily $\rho=0$, so that the relative curvature vanishes. Hence we have the

Theorem. *The first normal vector of a curve in a space of constant relative curvature is independent of choice of the reference point, if and only if the relative curvature vanishes.*

Corollary. *A geodesic with the center in a space of constant relative curvature has any point as its center, if and only if the relative curvature vanishes.*

§ 8. Some properties of S^n admitting a parallelism I.

We defined spaces admitting a parallelism in the Section 5. Such spaces may have some interesting geometrical properties, and hence we consider such spaces in this section. We shall denote hereafter by P^n a space admitting a parallelism.

In the first place, we have from the definition (6·4)

P 1. Each of the space P^n is proper in itself.

As a consequence of the theorem in the Section 6, we have

P 2. The Euler-Poincaré characteristic of closed P^n is equal to zero.

The discussions in the Section 6 give the following three properties of P^n.

P 3. The Parallel displacements in P^n preserve a length of a vector.

P 4. The Parallel displacements of vectors in P^n satisfy the condition of reflexivity. That is to say, if a vector $v(y)$ is parallel to a vector $u(x)$, then the latter is also parallel to the former.

P 5. In P^n, a vector obtained from a vector by parallel displacement is uniquely determined by the latter. Strictly speaking, the vector obtained from the contravariant components of the given vector u by the parallel displacement coincides with the vector obtained from the covariant components of u by the parallel displacement.

The equation (6·4) is written in the convenient form

$$(8\cdot1)\qquad \begin{aligned} g_{ia}g^{a(j)} &= g_{i(a)}g^{(a)(j)}, \\ g_{(i)(a)}g^{j(a)} &= g_{a(i)}g^{ja}, \end{aligned}$$

from which we see

$$g_{ab}g^{a(c)}g_{(c)(i)}g^{b(d)}g_{(d)(j)}=g_{(i)(j)}\,.$$

This result may be stated as follows:

P 6. The metric tensor at any point (x) of P^n is obtained from the metric tensor at a point (x_0) by the parallel displacement from (x_0) to (x).

Since P^n has the property *P 3*, we can define an angle between a vector u at a point (x) and a vector v at a point (y). We suppose first that both of the vectors u and v are of length unit. If the vector $\bar{u}$ at (y) is obtained from u by parallel displacement from (x) to (y), this vector $\bar{u}$ is of length unit as well. The angle α between $\bar{u}$ and v is given by

$$\cos\alpha=g_{i(j)}u^i v^{(j)},$$

which is easily verified by means of $(7\cdot1)$ and $(8\cdot1)$. On the other hand, if the vector $\bar{v}$ at (x) is obtained from v by the parallel displacement from (y) to (x), then the angle between u and $\bar{v}$ is also given by the above equation, which is immediately seen. Therefore we define the angle between unit vectors u at (x) and v at (y) as the quantity α given by the above equation. Thus the angle between $u(x)$ and $v(y)$, whose lengths are not necessarily unit, is defined as follows:

$$|u|\cdot|v|\cos(u\ v)=g_{i(j)}u^i v^{(j)}.$$

The definition and $(8\cdot1)$ gives us the following two properties of P^n.

P 7. If a vector v at a point (y) is parallel to a vector u at a point (x), then the angle between them is equal to zero or π.

P 8. Let u and v be two vectors at a point (x), and $\bar{u}$ and $\bar{v}$ be the vectors obtained from u and v respectively by the parallel displacement to a point (y). Then the angle between $\bar{u}$ and $\bar{v}$ is equal to the angle between u and v.

From $(9\cdot1)$ we have relative covariant differentiation with respect to x^k, $g_{ia\,;\,k}g^{a(i)}=0$, which gives

$$(8\cdot2)\qquad g_{ij\,;\,k}=0\,.$$

Thus we have

P 9. The metric tensor in P^n is relative covariant constant.

From this it follows that

$$\Gamma'^{i}_{jk}(x, y) = \left\{ {i \atop j\,k} \right\}_{(x)} .$$

Hence coefficients Γ'^{i}_{jk} of the linear connection of P^n are independent of the second variables (y) and so we see by means of the definition of the relative curvature tensor the following property:

P 10. The relative curvature tensor of P^n is identically equal to zero.

From (5·7) we see

P 11. The Riemannian space associating with P^n is locally euclidean.

We consider the inverse of *P 10.* Let S^n be such that the relative curvature tensors vanish. We call such a space to be *relatively flat.* Coefficients $\Gamma'^{i}_{jk}(x, y)$ of the linear connection of the space is independent of (y), and so they are equal to the Christoffel's symbols $\left\{ {i \atop j\,k} \right\}_{(x)}$ as a consequence of (5·6), so that the metric tensor is relative covariant constant. Conversely, if the metric tensor is relative covariant constant, then the space is evidently relatively flat. Thus we have an important class of relatively flat spaces, and the class inclndes all of spaces admitting a parallelism. It is clear that, if S^n is relatively flat, the first normal vector of a curve is equal to the vector which is the first normal in the sense of the Riemannian geometry, and hence the vector is orthogonal to the curve.

§ 9. Some properties of S^n admitting a parallelism II.

We shall see some properties of a closed P^n in the large. Let $u(x)$ be a field of differentiable vectors of length unit. That there exists such a field is already known, by means of *P 2* in the last section. From the assumption we have $g_{ij}(x)u^i(x)u^j(x)=1$. Relative covariant differentiation of this equation with respect to x^k gives

$$g_{ij}(x)u^i(x)u^j{}_{;k}(x)=0 , \tag{9·1}$$

in consequence of (8·2). This implies that the rank of the matrix $(u^j{}_{;k}(x))$ is less than n throughout the space. Further we take a unit vector $v^{(i)}$ at a fixed point (x_0) and then the angle $\alpha(x)$ between u and v is given by

$$\cos \alpha(x) = g_{j(i)}(x, x_0)v^{(i)}u^j(x) .$$

This is a differentiable function of (x) defined throughout the space. Hence the function has to receive its minimum and maximum in P^n, since P^n is closed. Therefore there exist at least two points (x_p) $(p=1, 2)$ where all of the partial derivatives of $\cos\alpha(x)$ are equal to zero. Thus we have

$$g_{j(i)}(x, x_0)\, v^{(i)} u^j{}_{;k}(x) = 0 \qquad (x=x_p). \tag{9·2}$$

Now the equations (9·1) and (9·2) are expressible in the forms

$$\begin{aligned} u^j{}_{;k}(x)\, u_j(x) &= 0 \\ u^j{}_{;k}(x)\, v_j(x) &= 0 \end{aligned} \qquad (x=x_p), \tag{9·3}$$

where $u_j(x)$ are the values of covariant components of u at the point (x_p) and $v_j(x_p)$ are covariant components of the vector obtained from v at the point (x_0) by parallel displacement to the point (x_p). As a consequence of (9·3), if $u_j(x_p)$ and $v_j(x_p)$ are linearly independent, then the matrix $(u^j{}_{;k}(x_p))$ is of rank less than $(n-1)$. If that is not the case, then we have easily $u_j(x_p) = \pm v_j(x_p)$, from which it follows that $u^i(x_p)$ is equal to a vector obtained from the given v at the point (x_0) by the parallel displacement to the point (x_p) to within algebraic sign.

Gathering the foregoing results we have the

Theorem. *Let u be a field of differentiable vectors of length unit in a closed P^n. Then the rank of the matrix $(u^j{}_{;k}(x))$ is less than n throughout the space. Further, if the rank is equal to $(n-1)$ throughout the space, then there exist at least two points, where u is equal to a vector obtained from the given vector at the point fixed in P^n by parallel displacement to within algebraic sign.*

It is to be remarked that the first part of the theorem is also satisfied in a closed Riemannian space whose Euler-Poincaré characteristic vanishes. The proof is quite similar to the above shown.

§ 10. Squares of spaces S^n.

We shall return to the consideration of a general space S^n. A product space $(S^n)^2$ of S^n by itself will be called the *square* of S^n, in which coordinates (z) of a point are given by $z^i = x^i$, $z^{n+i} = y^i$, $(i=1, \cdots, n)$. We consider a transformation of coordinates only of the type (1·1) in the square $(S^n)^2$ and then tensors of

$(S^n)^2$ can be defined. As an example, we denote by $T_{\alpha\beta}$ (*) components of a tensor of the second degree in $(S^n)^2$. These quantities are subjected under (1·1) to the following transformations.

$$\bar{T}_{ij}=T_{ab}\frac{\partial x^a}{\partial \bar{x}^i}\frac{\partial x^b}{\partial \bar{x}^j}\,,\qquad \bar{T}_{i(j)}=T_{a(b)}\frac{\partial x^a}{\partial \bar{x}^i}\frac{\partial y^b}{\partial \bar{y}^j}\,,$$

$$\bar{T}_{(i)j}=T_{(a)b}\frac{\partial y^a}{\partial \bar{y}^i}\frac{\partial x^b}{\partial \bar{x}^j}\,,\qquad \bar{T}_{(i)(j)}=T_{(a)(b)}\frac{\partial y^a}{\partial \bar{y}^i}\frac{\partial y^b}{\partial \bar{y}^j}\,.$$

Now, we consider a covariant vector $u_i(x, y)$ with respect to (x) in S^n and then $u_{(i)}=u_a g^{a(b)}g_{(b)(i)}$ are covariant components of a vector conjugate to the given vector. If the space is P^n admitting a parallelism, then we have the given vector u_i from $u_{(i)}$ by the converse process. Therefore we have a vector $(u_\alpha) =(u_i, u_{(i)})$ of the square $(P^n)^2$.

The above method applies equally well for any tensors of P^n. Thus we consider a covariant tensor $u_{ij}(x, y)$ of the second degree with respect to (x), from which we define $u_{i(j)}$, $u_{(i)j}$ and $u_{(i)(j)}$ as follows:

$$u_{i(j)}=u_{ia}g^{a(b)}g_{(b)(j)}\,,\qquad u_{(i)j}=u_{aj}g^{a(b)}g_{(b)(i)}\,,$$

$$u_{(i)(j)}=u_{ac}g^{c(d)}g_{(d)(j)}g^{a(b)}g_{(b)(i)}=u_{a(j)}g^{a(b)}g_{(b)(i)}\,.$$

However, we see easily by means of (8·1) $u_{(a)j}g^{b(a)}g_{bi}=u_{ij}$. Hence we obtain a covariant tensor $(u_{\alpha\beta})=(u_{ij}, u_{i(j)}, u_{(i)j}, u_{(i)(j)})$. Each of u_{ij}, $u_{i(j)}$, $u_{(i)j}$ and $u_{(i)(j)}$ is called a *factor* of $(u_{\alpha\beta})$, and the tensor $(u_{\alpha\beta})$ the *extension* of each factor. It has above shown that the extension is uniquely determined by each factor in the case of a space admitting a parallelism.

From (8·1) we obtain a tensor

$$(g_{\alpha\beta})=(g_{ij}, g_{i(j)}, g_{(i)j}=g_{j(i)}, g_{(i)(j)}),$$

which is the extension of the relative metric tensor as well as the metric tensor. This $(g_{\alpha\beta})$ is called the fundamental tensor of $(P^n)^2$.

Let (T) be a tensor in S^n, then we have its derived tensors $(T^{\cdot\cdot\cdot}_{\cdot\cdot\cdot;j})$ and $(T^{\cdot\cdot\cdot}_{\cdot\cdot\cdot;(j)})$ with respect to (x) and also to (y). Hence it is clear that the quantities

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(*) In the following it is understood that latin indices teak the values $1,\cdots, 2n$.

$$T^{\cdots}_{\cdots/j}=T^{\cdots}_{\cdots;j}+T^{\cdots}_{\cdots;(a)}g^{b(a)}g_{bj},$$

$$T^{\cdots}_{\cdots/(j)}=T^{\cdots}_{\cdots;(j)}+T^{\cdots}_{\cdots;a}g^{a(b)}g_{(b)(j)},$$

are respectively components of tensors. These tensors $(T^{\cdots}_{\cdots/j})$ and $(T^{\cdots}_{\cdots/(j)})$ shall be called the *derived tensors of compounded type* with respect to (x) and to (y). We see easily for the relative metric tensor

$$g_{i(j)/k}=0, \quad g_{i(j)/(k)}=0.$$

We consider a square $(S^n)^2$ of S^n and a tensor $T^{\alpha}_{\cdot\beta}$, which is the extension of the tensor $T^i_{\cdot j}$ in S^n, and whose factors are $T^i_{\cdot j}$, $T^i_{\cdot(j)}$, $T^{(i)}_{\cdot j}$ and $T^{(i)}_{\cdot(j)}$. But we see immediately that the quantities $T^{\alpha}_{\cdot\beta;j}$ and $T^{\alpha}_{\cdot\beta;(j)}$ are not always an extension of one of its factors. Because, for instance, the equation

$$T^i_{\cdot j;(k)}=T^i_{\cdot j;a}g^{a(b)}g_{(b)(k)},$$

will not hold. However, in $(P^n)^2$, we shall prove that the derived tensor $(T^{\alpha}_{\cdot\beta/j}, T^{\alpha}_{\cdot\beta/(j)})$ of compounded type is defined the extension of one of its factors. In fact, factors of $T^{\alpha}_{\cdot\beta}$ are given from $T^i_{\cdot j}$ by the following equations:

$$(10\cdot1)\qquad \begin{aligned} &T^i_{\cdot(j)}=T^i_{\cdot a}g^{a(b)}g_{(b)(j)}, \quad T^{(i)}_{\cdot j}=T^a_{\cdot j}g_{a(b)}g^{(b)(i)},\\ &T^{(i)}_{\cdot(j)}=T^a_{\cdot b}g_{a(c)}g^{(c)(i)}g^{b(d)}g_{(d)(j)}. \end{aligned}$$

Since the equations

$$g_{ij/k}=0, \quad g_{ij/(k)}=0$$

are evidently statisfied, we have from $(10\cdot1)$

$$T^i_{\cdot(j)/k}=T^i_{\cdot a/k}g^{a(b)}g_{(b)(j)},$$

and so on. Hence $T^i_{\cdot(j)/k}$, $T^{(i)}_{\cdot j/k}$ and $T^{(i)}_{\cdot(j)/k}$ are conjugate to $T^i_{\cdot j/k}$. Next we have

$$T^i_{\cdot j/a}g^{a(b)}g_{(b)(k)}=(T^i_{\cdot j;a}+T^i_{\cdot j;(c)}g^{d(c)}g_{da})g^{a(b)}g_{(b)(k)},$$

from which we have as a consequence of $(8\cdot1)$

$$=T^i_{\cdot j;(c)}g^{d(c)}g_{da}g^{ab}g_{b(k)}+T^i_{\cdot j;a}g^{a(b)}g_{(b)(k)}$$

$$=T^i_{\cdot j;(k)}+T^i_{\cdot j;a}g^{a(b)}g_{(b)(k)}=T^i_{\cdot j/(k)}.$$

Hence $T^i_{\cdot j/(k)}$ is conjugate to $T^i_{\cdot j/k}$, from which $T^i_{\cdot(j)/(k)}$, $T^{(i)}_{\cdot j/(k)}$ and $T^{(i)}_{\cdot(j)/(k)}$ are also conjugate to $T^i_{\cdot j/k}$. Consequently we have proved the above statement.

It is at once shown that, for P^n, covariant differentiation of compounded type are not depend upon the order of differentiations; while, for general space S^n, the circumstances are complicated, because we have terms containing the derived tensor of the metric tensor, which does not always vanish. But we obtain the following terms:

$$B_{i\cdot kl}^{\cdot j}=g^{a(b)}(S_{i\cdot k(b)}^{\cdot j}g_{al}-S_{i\cdot l(b)}^{\cdot j}g_{ak}),$$

$$B_{i\cdot k(l)}^{\cdot j}=S_{i\cdot k(l)}^{\cdot j}-S_{i\cdot a(b)}^{\cdot j}g^{a(c)}g_{(c)(l)}g^{d(b)}g_{dk},$$

$$B_{i\cdot(k)(l)}^{\cdot j}=-g^{a(b)}(S_{i\cdot a(k)}^{\cdot j}g_{(b)(l)}-S_{i\cdot a(l)}^{\cdot j}g_{(b)(k)}),$$

and the similar terms $B_{(i)\cdot kl}^{\cdot(j)}$, $B_{(i)\cdot k(l)}^{\cdot(j)}$ and $B_{(i)\cdot(k)(l)}^{\cdot(j)}$. It is a little interesting for us that the curvature tensor of the Riemannian space associating with S^n coincides with the quantities $B_{i\cdot kl}^{\cdot j}(x, x)$ $B_{i\cdot k(l)}^{\cdot j}(x, x)$, $B_{i\cdot(k)(l)}^{\cdot j}(x, x)$, $\cdots$.

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