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Appendix to the paper "Note on the boundedness and the ultimate boundedness"

By

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In the foregoing paper [2] we have discussed the boundedness and the ultimate boundedness of solutions of a system of differential equations. Now we consider a system of differential equations,

$$\frac{dx}{dt}=F(t, x), \tag{1}$$

where x denotes an n-dimensional vector and $F(t, x)$ is a given vector field which is defined and continuous in the domain

$$\Delta: \quad 0 \leqq t < \infty, \quad |x| < \infty.$$

$|x|$ represents *the sum of the squares* of its components. And let

$$x=x(t;\ x_0, t_0)$$

be a solution through the initial point (t_0, x_0). Except otherwise stated, we adopt the symbols and the promises in [2].

At first, we will discuss the boundedness of solutions under perturbations. Corresponding to the differential equation (1), we consider an equation

$$\frac{dx}{dt}=F(t, x)+H(t, x), \tag{2}$$

where $H(t, x)$ is a continuous vector field defined in Δ. Here we give a definition for the boundedness which corresponds to the stability under constantly acting perturbations.

Definition. The solutions of (1) are said to be *totally bounded* (or *bounded under constantly acting perturbations*), if for any $\alpha > 0$, there exist two positive numbers β and γ such that, if $|x_0| \leqq \alpha$, then we have $|x(t; x_0, t_0)| < \beta$ for any $t \geqq t_0$ (t_0, arbitrary), where $x=x(t; x_0, t_0)$ is the solution of the equation (2) in which we have

$|H(t, x)|<\gamma$, provided $\alpha<|x|<\beta$.

Therefore if the solutions of (1) are totally bounded, they are naturally uniformly bounded.

Theorem 1. *If the solutions of a linear system,*

$$x'=A(t)\cdot x\,, \tag{3}$$

are totally bounded, they are uniformly ultimately bounded. In fact we have $|x(t)|\rightarrow 0$ *when* $t\rightarrow\infty$. $A(t)$ *is a square matrix and the dot represents matrix multiplication with the column vector* x.

Proof. Let

$$x_1=B_{i1}(t),\ x_2=B_{i2}(t),\ \cdots\cdots,\ x_n=B_{in}(t)$$

be the solution of (3) such as $x_1=\cdots=x_{i-1}=x_{i+1}=\cdots=x_n=0$, $x_i=1$ for $t=0$. For $i=1, 2, \cdots, n$, we have n solutions which are linearly independednt. Then the general solution is of the form

$$x=B(t)\cdot C,$$

where C is a column vector of constant elements and $B(t)$ is the matrix

$$\begin{pmatrix} B_{11}(t) & B_{21}(t) & \cdots & B_{n1}(t) \\ B_{12}(t) & B_{22}(t) & \cdots & B_{n2}(t) \\ \cdots & \cdots & \cdots & \cdots \\ B_{1n}(t) & B_{2n}(t) & \cdots & B_{nn}(t) \end{pmatrix}.$$

Since $B(t)$ is non-singular, there exists $B^{-1}(t)$. Hence the solution such as $x=x_0$ at $t=t_0$ is of the form

$$x=B(t)\cdot B^{-1}(t_0)\cdot x_0\,.$$

On the other hand, the solution of the equation,

$$x'=A(t)\cdot x+\delta\cdot x\ \ (\delta>0:\text{ sufficiently small constant}),$$

such as $x=x_0$ at $t=t_0$ is of the form

$$x=B(t)\cdot B^{-1}(t_0)\cdot x_0\cdot e^{\delta t-\delta t_0}\,. \tag{4}$$

Since the solutions of (3) are totally bounded, if $|x_0|\leqq\alpha$, there exists $\beta(\alpha)$ and we have

$$|B(t)\cdot B^{-1}(t_0)\cdot x_0|e^{2(\delta t-\delta t_0)}<\beta(\alpha),$$

where the absolute value means the sum of the squares of its elements. Hence we have

(5) $$|B(t)\cdot B^{-1}(t_0)\cdot x_0|<\beta(\alpha)e^{-2\delta t+2\delta t_0}.$$

And for any $\varepsilon>0$, if $t>t_0+\frac{1}{2\delta}\log\frac{\beta(\alpha)}{\varepsilon}$, we have

$$\beta(\alpha)e^{-2\delta t+2\delta t_0}<\varepsilon.$$

Therefore if $t>t_0+\frac{1}{2\delta}\log\frac{\beta(\alpha)}{\varepsilon}$, we have

$$|B(t)\cdot B^{-1}(t_0)\cdot x_0|<\varepsilon.$$

Namely the solutions of (3) are uniformly ultimately bounded, since $\frac{1}{2\delta}\log\frac{\beta(\alpha)}{\varepsilon}$ depends only on α. From (5) clearly we have $|x(t)|\to 0$ as $t\to\infty$.

Theorem 2. *Let R be a positive constant which may be sufficiently great and let Δ^* be the domain such as*

$$0\leqq t<\infty,\quad |x|\geqq R.$$

Suppose that there exists a positive continuous function $\varphi(t,x)$ satisfying the following conditions in Δ^*; namely

1° *$\varphi(t,x)$ has the property A (cf. p. 279 in [2]),*

2° *$\varphi(t,x)$ tends to infinity uniformly for $|x|\to\infty$,*

3° *$\varphi(t,x)$ satisfies locally the Lipschitz condition with regard to x, that is,*

$$|\varphi(t,x)-\varphi(t,\bar{x})|\leqq K\sqrt{|x-\bar{x}|},$$

where K is a positive constant depending only on L for $|x|\leqq L$, $|\bar{x}|\leqq L$ (L: arbitrary),

4° $\overline{\lim_{h\to 0}}\frac{1}{h}\{\varphi(t+h,x+hF(t,x))-\varphi(t,x)\}$, *i.e., $D_F\varphi$ has the property B (cf. p. 279 in [2]).*

Then the solutions of (1) are totally bounded.

Proof. It is sufficient to see that there exists γ such as $|H(t,x)|<\gamma$ and we have

$$\overline{\lim_{h\to 0}}\frac{1}{h}\{\varphi(t+h,x+hF+hH)-\varphi(t,x)\}\leqq 0.$$

By the conditions 1° and 2°, we determine β suitably for α. In $\alpha\leqq|x|\leqq\beta$, using the conditions 3° and 4°, we have

$$D_{F+H}\varphi\leqq\overline{\lim_{h\to 0}}\frac{1}{h}\{\varphi(t+h,x+hF)+Kh\sqrt{|H(t,x)|}-\varphi(t,x)\}$$

$$\leqq D_F\varphi+K\sqrt{|H(t, x)|}$$
$$\leqq -\lambda+K\sqrt{|H(t, x)|},$$

where λ is determined by the property B. Hence, if $|H(t, x)|\leqq \lambda^2/K^2$, we have

$$D_{F+H}\varphi\leqq 0.$$

Thus we can see that the solutions of (2) such as $|x_0|\leqq\alpha$ are bounded by β. Since λ and K are determined depending only on α, the solutions of (1) are totally bounded.

Theorem 3. *We assume that $F(t, x)$ in the system (1) has continuous partial derivatives of the first order with respect to x. If $F(t, x)$ is periodic of t and the solutions issuing from $t=0$ are equi-bounded and the solutions are ultimately bounded, then they are totally bounded.*

Proof. By Theorem 12 in [2], when (1) satisfies the assumptions of the theorem, there exists a positive function $\varphi(t, x)$ defined in Δ^* (R be sufficiently great) which is continuous with its partial derivatives of the first order and tends to infinity uniformly as $|x|\to\infty$ and satisfies the following conditions;

1° $\varphi(t, x)$ has the property A,

2° $\dfrac{\partial\varphi}{\partial t}+\dfrac{\partial\varphi}{\partial x}\cdot F(t, x)$ has the property B.

Moreover observing the construction of $\varphi(t, x)$, clearly we see that $\varphi(t, x)$ is periodic of t and hence $\dfrac{\partial\varphi}{\partial x}$ is so also. Therefore $\left|\dfrac{\partial\varphi}{\partial x}\right|$ is bounded, if $|x|$ is bounded. From

$$|\varphi(t, x)-\varphi(t, \bar{x})|\leqq\sum_{i=1}^{n}\left|\frac{\partial\varphi}{\partial x_i}(t, x+\theta(x-\bar{x}))\right||x_i-\bar{x}_i|$$
$$\leqq K\sqrt{|x-\bar{x}|},$$

we see that $\varphi(t, x)$ satisfies all the conditions in Theorem 2. Hence the solutions of (1) are totally bounded.

Now we assume that $F(t, x)$ and $\dfrac{\partial F}{\partial x_i}$ are bounded when $|x|$ is bounded and $\dfrac{\partial F}{\partial x_i}$ is continuous and that the solutions are uniformly bounded and uniformly ultimately bounded. Then we can obtain a similar function $\varphi(t, x)$ (*cf.* p. 290 in [2]). In this case, we can prove the boundedness of $\left|\dfrac{\partial\varphi}{\partial x}\right|$ by using the boundedness of

$\frac{\partial F}{\partial x_i}$. Namely utilizing the boundedness of $\frac{\partial F}{\partial x_i}$ and the equation of variation, we will see that

$$\left| \frac{\partial |x(t+\tau;\, x,\, t)|}{\partial x_i} \right| < a e^{\mu\tau},$$

where a and μ are constants determined depending only on $|x|$. By the symbol in the foregoing paper (p. 288 in [2]),

$$\frac{\partial \varphi}{\partial x_i} = \int_t^{\infty} G'(|x(\tau;\, x,\, t)|) \frac{\partial |x(\tau;\, x,\, t)|}{\partial x_i}\, d\tau$$

and hence considering the property of G' and taking X suitably, we have

$$\left| \frac{\partial \varphi}{\partial x_i} \right| \leqq a\, G'(X) \int_t^{t+T} e^{\mu(\tau - t)}\, d\tau = \frac{a\, G'(X)}{\mu} [e^{\mu T} - 1].$$

Since a, T, μ and $G'(X)$ are determined depending only on $|x|$, for a suitable number $A(|x|)$ depending only on $|x|$ we have

$$\left| \frac{\partial \varphi}{\partial x_i} \right| \leqq A(|x|).$$

Thus our $\varphi(t, x)$ satisfies also all the conditions in Theorem 2 and hence the solutions are totally bounded.

Next we will discuss the ultimate boundedness of solutions under perturbations which have a given order when $|x|$ is great.

Definition. For a given positive function $f(|x|)$, the solutions of (1) are said to be *ultimately bounded under constantly acting perturbations of order* $f(|x|)$, if there exist two positive constants B and α so that $|H(t, x)| < \alpha f(|x|)$ for $|x| \geqq B$, and then we have

$$\overline{\lim_{t\to\infty}}\, |x(t;\, x_0,\, t_0)| < B,$$

where $x = x(t;\, x_0,\, t_0)$ is any solution of (2).

Theorem 4. *We suppose that there exists a positive continuous function* $\varphi(t, x)$ *satisfying the following conditions in* Δ^*; *namely*

1° $\varphi(t, x)$ *has the property* A,

2° $\varphi(t, x)$ *tends to infinity uniformly as* $|x| \to \infty$,

3° $|\varphi(t, x) - \varphi(t, \bar{x})| \leqq K\sqrt{|x - \bar{x}|}$, *where* $\bar{x}$ *may be sufficiently near* x *and* K *is a constant depending only on* $|x|$,

4° $D_F \varphi \leqq -G(|x|)$, *where* $G(\eta)$ *is a positive continuous function for* $\eta \geqq R$, 5° $K^2(|x|)f(|x|)=O(G^2(|x|)) \quad (|x|\to\infty)$.

Then the solutions of (1) *are ultimately bounded under constantly acting perturbations of order* $f(|x|)$.

Proof. It is sufficient to show that α is determined in Δ^* and if $|H(t, x)|<\alpha f(|x|)$, then $D_{F+H}\varphi$ has the property B. We have

$$D_{F+H}\varphi \leqq D_F\varphi + K\sqrt{|H(t, x)|}$$
$$\leqq -G(|x|)+K(|x|)\sqrt{|H(t, x)|}.$$

On the other hand, by 5°, we can take an α such as

$$\alpha \leqq \frac{G^2(|x|)}{4K^2(|x|)f(|x|)}.$$

Since, for such an α, we have

$$D_{F+H}\varphi \leqq -G(|x|)+K(|x|)\sqrt{\alpha f(|x|)}$$
$$\leqq -G(|x|)+K(|x|)\sqrt{f(|x|)}\frac{G(|x|)}{2K(|x|)\sqrt{f(|x|)}}$$
$$\leqq -\frac{1}{2}G(|x|),$$

hence $D_{F+H}\varphi$ has the property B.

When (1) satisfies the conditions in Theorem 3, there exists $\varphi(t, x)$ satisfying the conditions 1°, 2° and 3° in Theorem 4. And in place of 4° we have

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial x}\cdot F(t, x)\leqq -G(|x|),$$

where $G(|x|)$ is monotone and $G(|x|)\to\infty$ as $|x|\to\infty$. Therefore if we have $\left|\frac{\partial\varphi}{\partial x}\right|\cdot f(|x|)=O(G^2(|x|))$, then the solutions of (2) are ultimately bounded.

Thus when x is a *2-dimensional vector* in (1) and (2), it is seen that there exists a periodic solution also, even if we give such perturbations as in the above.

This time we consider a system of differential equations of *Carathéodory's type.* We suppose that in a system,

$$\frac{dx}{dt}=F(t, x), \tag{6}$$

$F(t, x)$ is finite and it is a measurable function of t when x is fixed and it is a continuous function of x when t is fixed, and that, if $|x| \leqq L$ (L: arbitrary), we have

(7) $$\sqrt{|F(t, x)|} \leqq M(t),$$

where $M(t)$ is summable for $0 \leqq t \leqq \tau$ (τ: arbitrary) and of course, it may depend on L. Remark again that $|F(t, x)| = \sum_{i=1}^{n} F_i(t, x)^2$. Then we can obtain similar results as those in the foregoing paper [2].

In the paper [1] we have proved a theorem analogous to the following one. Namely *if there exists a function $\varphi(t, x)$ satisfying the following conditions in Δ^*;*

1° *$\varphi(t, x)$ is positive continuous and it has the property A,*
2° *$\varphi(t, x)$ tends to infinity uniformly as $|x| \to \infty$,*
3° *$\varphi(t, x)$ has the property a.c.u. (cf. p.* 253 *in* [1]),
4° *$\varphi(t, x)$ satisfies locally the Lipschitz condition with regard to x,*
5° *$D_r \varphi \leqq -\kappa$ a.e. (for $|x| \leqq L$, $\kappa > 0$ is a constant and it may depend on L),*

then the solutions of (6) *are uniformly ultimately bounded.*

Now we will show that its reciprocal problem holds good for a certain equation.

Theorem 5. *We suppose that $F(t, x)$ in the system* (6) *satisfies locally the generalized Lipschitz condition with regard to x, i.e., for $|x| \leqq L$, $|\bar{x}| \leqq L$ (L: arbitrary)*

$$\sqrt{|F(t, x) - F(t, \bar{x})|} \leqq K(t) \sqrt{|x - \bar{x}|},$$

where $K(t)$ is a summable function and it may depend on L. If $F(t, x)$ is periodic of t and the solutions issuing from $t=0$ to the right are equi-bounded and the solutions are ultimately bounded, then there exists such a function $\varphi(t, x)$ as the above-mentioned.

Proof. Without the loss of generality, we can assume that the period of $F(t, x)$ is 1. We can see that now the solutions are uniformly bounded and uniformly ultimately bounded just as in the case where $F(t, x)$ is continuous. Hence we assume that the solutions are uniformly ultimately bounded for the bound B. Namely if $|x_0| \leqq \alpha$, there is T depending only on α such as if $t > t_0 + T$, then $|x(t; x_0, t_0)| < B$. Such T is infinite in number; so we take the infimum of such T, denoted by $T(\alpha)$. Then $T(\alpha)$ is non-decreasing with respect to α. At first we put

$$\sup_{\substack{|x_0|\leqq\eta \\ t_0\leqq t<\infty \\ 0\leqq t_0<\infty}} |x(t;\, x_0,\, t_0)| = f(\eta).$$

$f(\eta)$ is uniquely determined and it is non-decreasing with respect to η. If we put

$$\sup_{|x_0|\leqq f(\eta)} \sqrt{|F(t,\, x_0)|} = g(t,\, \eta),$$

then $g(t, \eta)$ is a measurable function of t and it is a non-decreasing function of η and clearly it is periodic of t. Moreover $g(t, \eta)$ is summable with respect to t, since it is bounded by a summable function $M(t)$. Let $h(t, \eta)$ and $k(t, \eta)$ be

$$\int_{\eta}^{\eta+1} g(t,\, s)\,ds = h(t,\, \eta)$$

and

$$\int_{\eta}^{\eta+1} h(t,\, s)\,ds = k(t,\, \eta)$$

respectively. Then we have clearly $g(t, \eta) \leqq h(t, \eta) \leqq k(t, \eta)$. For fixed η, $k(t, \eta)$ is a measurable function of t, for fixed t it is continuous with its derivative with respect to η and it is non-decreasing with respect to η and moreover it is periodic of t. If we put

$$\tilde{k}(t,\, \eta) = k(t,\, \eta) + \eta,$$

then $\tilde{k}(t, \eta)$ has the same properties as those of $k(t, \eta)$. Now we consider a function of (t, η) for $0 \leqq t < \infty$, $\eta \geqq 0$ as follows; namely

$$G(t,\, \eta) = \begin{cases} [\tilde{k}(t,\, 4\eta) - \tilde{k}(t,\, 4B)](\eta - B) & (\eta \geqq B) \\ 0 & (0 \leqq \eta < B). \end{cases}$$

Then for fixed t, this function has the same properties as those of $G(\eta)$ which we have used in the foregoing paper (p. 288 in [2]). Clearly this is a measurable function of t for fixed η and this is periodic of t. Since we have

$$G(t,\, \eta) \leqq \tilde{k}(t,\, 4\eta)\eta\,,$$

we have for $\eta \leqq L$

$$G(t,\, \eta) \leqq L\tilde{k}(t,\, 4L),$$

while $\tilde{k}(t, 4L)$ is summable and hence $G(t, \eta)$ is bounded by a periodic summable function. Similarly if $\eta \leqq L$, $G_{\eta}(t, \eta)$ is bounded by a periodic summable function. Here L is arbitrary.

Now we put

$$\varphi(t, x)=\int_t^\infty G(\tau, |x(\tau; x, t)|)d\tau. \tag{8}$$

Then $\varphi(t, x)$ is positive when $|x|>B$. It is seen easily that it has the property A, since if $|x|\leqq\alpha$, we have

$$\varphi(t, x)\leqq\int_t^{t+T} G(\tau, f(\alpha))d\tau.$$

Next we will show that $\varphi(t, x)$ satisfies the condition 2°. Since $\tilde{k}(t, \eta)\geqq\eta$, we can choose ρ such as

$$\int_t^{t+\rho}\tilde{k}(s, |x|)ds=\frac{\sqrt{|x|}}{2\sqrt{n}}.$$

From the equality

$$\int_t^{t+\rho}\tilde{k}(s, |x|)ds=\int_t^{t+\rho}[k(s, |x|)+|x|]\,ds$$

$$=\int_t^{t+\rho}k(s, |x|)ds+|x|\rho,$$

it follows that we have

$$\rho\leqq\frac{1}{2\sqrt{n}\sqrt{|x|}}.$$

Therefore if $|x|$ is sufficiently great, we have $\rho<1$. On the other hand, we have for $t\leqq\tau\leqq t+\rho$

$$|x_i(\tau; x, t)-x_i|\leqq\int_t^\tau g(s, |x|)ds\leqq\int_t^{t+\rho}\tilde{k}(s, |x|)ds=\frac{\sqrt{|x|}}{2\sqrt{n}}$$

and from this we have

$$|x(\tau; x, t)|\geqq\frac{|x|}{4}.$$

Hence for sufficiently great $|x|$ we have

$$\varphi(t, x)\geqq\int_t^{t+\rho}G\left(\tau, \frac{|x|}{4}\right)d\tau$$

$$\geqq\left(\frac{|x|}{4}-B\right)\left\{\int_t^{t+\rho}\tilde{k}(\tau, |x|)d\tau-\int_t^{t+\rho}\tilde{k}(\tau, 4B)d\tau\right\}$$

$$\geqq\left(\frac{|x|}{4}-B\right)\left\{\frac{\sqrt{|x|}}{2\sqrt{n}}-\int_t^{t+p}\tilde{k}(\tau,\ 4B)\,d\tau\right\},$$

while we have

$$\int_t^{t+p}\tilde{k}(\tau,\ 4B)\,d\tau\leqq\int_t^{t+1}\tilde{k}(\tau,\ 4B)\,d\tau=\int_0^1\tilde{k}(\tau,\ 4B)\,d\tau=\text{const.}\,.$$

Thus we can see that $\varphi(t, x)$ tends to infinity uniformly as $|x|\to\infty$.

To prove the condition 3°, we consider at first

$$|\varphi(t, x)-\varphi(t, x')|\leqq\int_t^{t+T}|G(\tau,\ |x(\tau;\ x,\ t)|)-G(\tau,\ |x(\tau;\ x',\ t)|)|\,d\tau\,.$$

Since $G(t, \eta)$ is differentiable with respect to η, we have

$$\begin{aligned}&|G(\tau,\ |x(\tau;\ x,\ t)|)-G(\tau,\ |x(\tau;\ x',\ t)|)|\\&=||x(\tau;\ x,\ t)|-|x(\tau;\ x',\ t)||\\&\times G_{\eta}(\tau,\ |x(\tau;\ x,\ t)|+\theta(|x(\tau;\ x,\ t)|-|x(\tau;\ x',\ t)|))\\&\leqq C\sqrt{|x-x'|}\cdot M(\tau),^{(1)}\end{aligned}$$

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(1) In general, we consider a system of Carathéodory's type, $x'=F(t, x)$, where $F(t, x)$ is defined for $\alpha\leqq t\leqq\beta$, $|x|\leqq\gamma$. Now to simplify the statement, let $|x|$ be $\sqrt{\sum_{i=1}^{n}x_i^2}$. We assume that $|F(t, x)-F(t, \bar{x})|\leqq K(t)|x-\bar{x}|$, where $K(t)$ is summable for $\alpha\leqq t\leqq\beta$. Let $x=x(t)$ and $x=\bar{x}(t)$ be two solutions, then if $x(\alpha)=\bar{x}(\alpha)$, we have $x(t)=\bar{x}(t)$ by the Lipschitz condition. Now we assume that $|x(\alpha)-\bar{x}(\alpha)|\neq 0$ and hence if we put $x_i(t)-\bar{x}_i(t)=\eta_i(t)$, we have $|\eta(t)|\neq 0$, where $|\eta(t)|=\sqrt{\sum_{i=1}^{n}\eta_i(t)^2}$. For almost all t

$$\begin{aligned}\sum_{i=1}^{n}\eta_i(t)\eta_i'(t)&=\sum_{i=1}^{n}\eta_i(t)\{F_i(t,\ x(t))-F_i(t,\ \bar{x}(t))\}\\&\leqq|\eta(t)||F(t,\ x(t))-F(t,\ \bar{x}(t))|\quad\text{(by Cauchy's inequality)}\\&\leqq|\eta(t)|^2K(t)\end{aligned}$$

and hence we have

$$\int_\alpha^t|\eta(t)|^{-2}\sum_{i=1}^{n}\eta_i(t)\eta_i'(t)\,dt\leqq\int_\alpha^t K(t)\,dt\,.$$

Since $|\eta(t)|^2$ is absolutely continuous, we have

$$\begin{aligned}\log|\eta(t)|^2-\log|\eta(\alpha)|^2&=\int_{|\eta(\alpha)|^2}^{|\eta(t)|^2}\frac{1}{t}\,dt=\int_\alpha^t|\eta(t)|^{-2}\sum_{i=1}^{n}2\eta_i(t)\eta_i'(t)\,dt\\&\leqq 2\int_\alpha^t K(t)\,dt\,.\end{aligned}$$

Therefore we have

$$|\eta(t)|\leqq|\eta(\alpha)|\,e^{\int_\alpha^t K(s)ds},$$

where C is a suitable constant and $M(\tau)$ is a periodic summable function. Then we have

$$|\varphi(t, x) - \varphi(t, x')| \leqq C\sqrt{|x-x'|}\int_t^{t+T} M(\tau)d\tau ,$$

i.e., we have

$$|\varphi(t, x) - \varphi(t, x')| \leqq \bar{C}\sqrt{|x-x'|} .$$

Next if we assume $t<t'$, then from

$$\varphi(t, x) - \varphi(t', x) = \int_t^{t'} G(\tau, |x(\tau; x, t)|)d\tau$$
$$+\int_{t'}^{\infty}[G(\tau, |x(\tau; x, t)|) - G(\tau, |x(\tau; x, t')|)]d\tau ,$$

we have

$$|\varphi(t, x) - \varphi(t', x)| \leqq \int_t^{t'} G(\tau, |x(\tau; x, t)|)d\tau$$
$$+\int_{t'}^{t'+T}|G(\tau, |x(\tau; x, t)|) - G(\tau, |x(\tau; x, t')|)|d\tau$$

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that is, we obtain an inequality

$$|x(t) - \bar{x}(t)| \leqq |x(\alpha) - \bar{x}(\alpha)| e^{\int_\alpha^t K(s)ds}.$$

In our case, we have

$$||x(\tau; x, t)| - |x(\tau; x', t)|| \leqq \sum_{i=1}^{n} |x_i(\tau; x, t) + x_i(\tau; x', t)||x_i(\tau; x, t) - x_i(\tau; x', t)|$$
$$\leqq A\sqrt{|x(\tau; x, t) - x(\tau; x', t)|}$$

(A be a suitable constant).

From the above-mentioned, we have

$$\sqrt{|x(\tau; x, t) - x(\tau; x', t)|} \leqq \sqrt{|x-x'|}\, e^{\int_t^\tau K(s)ds},$$

while for $t \leqq \tau \leqq t+T$

$$e^{\int_t^\tau K(s)ds} \leqq e^{\int_t^{t+T} K(s)ds} \leqq \bar{A} .$$

Here $\bar{A}$ is a suitable constant and it depends on $|x|$, but it is independent of t, because we can assume that $K(s)$ is periodic. Consequently we have

$$\sqrt{|x(\tau; x, t) - x(\tau; x', t)|} \leqq \bar{A}\sqrt{|x-x'|}$$

and hence we obtain

$$||x(\tau; x, t)| - |x(\tau; x', t)|| \leqq C\sqrt{|x-x'|} .$$

$$\leqq\int_t^{t'}G(\tau,\beta)d\tau+\int_{t'}^{t'+T}||x(\tau;x,t)|-|x(\tau;x,t')||$$

$$\times G_\eta(\tau,|x(\tau;x,t)|+\theta(|x(\tau;x,t)|-|x(\tau;x,t')|))d\tau$$

$$\leqq\int_t^{t'}G(\tau,\beta)d\tau+\int_{t'}^{t'+T}||x(\tau;x,t)|-|x(\tau;x,t')||\overline{M}(\tau)d\tau$$

$$\leqq\int_t^{t'}G(\tau,\beta)d\tau+\int_{t'}^{t'+T}2\beta\overline{M}(\tau)\sum_{i=1}^n|x_i(\tau;x,t)-x_i(\tau;x,t')|d\tau,$$

where $\beta=f(|x|)$ and $\overline{M}(\tau)$ is a suitable periodic summable function. Now we consider the case $t'\leqq\tau\leqq t'+T$. Since the solution is unique by the Lipschitz condition, if we put

$$x_i(t';x,t)=X_i,$$

we have

$$x_i(\tau;X,t')=x_i(\tau;x,t).$$

On the other hand, from the equality

$$x_i(t';x,t)=x_i+\int_t^{t'}F_i(s,x(s;x,t))ds,$$

we obtain

$$X_i-x_i=\int_t^{t'}F_i(s,x(s;x,t))ds.$$

Hence we have

$$|X_i-x_i|\leqq\int_t^{t'}|F_i(s,x(s;x,t))|ds\leqq\int_t^{t'}\widetilde{M}(s)ds,$$

where $\widetilde{M}(s)$ is a summable function determined by $g(s,|x|)$. Since we have

$$\sum_{i=1}^n|x_i(\tau;x,t)-x_i(\tau;x,t')|=\sum_{i=1}^n|x_i(\tau;X,t')-x_i(\tau;x,t')|$$

$$\leqq\sqrt{n}\sqrt{|x(\tau;X,t')-x(\tau;x,t')|}$$

$$\leqq\sqrt{n}\,\bar{A}\sqrt{|X-x|}\ ^{(2)}$$

$$\leqq\sqrt{n}\,\bar{A}\sqrt{n}\int_t^{t'}\widetilde{M}(s)ds=n\bar{A}\int_t^{t'}\widetilde{M}(s)ds,$$

we obtain

(2) cf. (1).

$$|\varphi(t, x)-\varphi(t', x)| \leqq \int_t^{t'} G(\tau, \beta)d\tau + \int_{t'}^{t'+T}[2\beta\bar{M}(\tau)n\bar{A}\int_t^{t'}\widetilde{M}(s)ds]\,d\tau$$
$$\leqq \int_t^{t'}[G(\tau, \beta)+2\beta\gamma n\bar{A}\widetilde{M}(\tau)]\,d\tau\,,$$

where γ is a suitable constant such that $\int_{t'}^{t'+T}\bar{M}(\tau)d\tau<\gamma$. That is to say, we have

$$|\varphi(t, x)-\varphi(t', x)| \leqq \int_t^{t'} N(s)ds\,,$$

where $N(s)$ is summable and of course, it depends on the region of $|x|$. Therefore we obtain the inequality

$$|\varphi(t, x)-\varphi(t', x')| \leqq \int_t^{t'} N(s)ds + \bar{K}\sqrt{|x-x'|}\,.$$

From this it is seen that $\varphi(t, x)$ has the property a.c.u. (cf. 253-254 in [1]). At the same time the condition 4° is proved.

If the point (t, x) moves along a fixed solution, say the solution through (t_0, x_0), we have

$$\varphi(t, x(t)) = \int_t^{\infty} G(\tau, |x(\tau; x_0, t_0)|)\,d\tau$$

so that for almost all t

$$\frac{d}{dt}\varphi(t, x(t)) = -G(t, |x(t; x_0, t_0)|) \qquad a.e.$$
$$= -G(t, |x|) \qquad a.e.$$
$$\leqq -4(|x|-B)^2 \qquad a.e.$$

whence we can prove the condition 5° easily.

BIBLIOGRAPH

[1] Yoshizawa; "*Note on the solutions of a system of differential equations*", These Memoirs, Vol. 29 (1955), pp. 249-273.

[2] Yoshizawa; "*Note on the boundedness and the ultimate boundedness of solutions of $x'=F(t, x)$*", *ibid.*, pp. 275-291.