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Notes on meromorphic covariants

By

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The present paper is based essentially upon R. Nevanlinna's theory [1] on Abelian integrals. In § 1 we shall establish the first fundamental theorem of R. Nevanlinna (Theorem 2) for the meromorphic covariants of order $n (n=0, \pm 1, \pm 2, \cdots)$ (cf. sec. 4) on an arbitrary open Riemann surface and give some of its applications in § 2, where Theorem 3 corresponds to Shimizu-Ahlfors' theorem for covariants.

§ 1

1. Let R be an arbitrary open Riemann surface. To make our integration paths unchanged even when the exhausting domains of R vary, we shall introduce the following coordinates on R. Let $\{R_n\}$ $n=0, 1, \cdots$ be any exhaustion of R where R_n is the compact domain whose boundary Γ_n consists of a finite number of analytic Jordan closed curves. We fix a point $P_0 \in R_0$ and consider the Green function $g(P) = g(P, P_0)$ of R_0 with a pole P_0 and denote by $h(P) = h(P, P_0)$ the conjugate harmonic function of $-g(P)$. Next we construct the harmonic function $u_n(P)$ on $R_n - (R_{n-1} \cup \Gamma_{n-1})$ which vanishes on Γ_{n-1} and $=$const., say $\log \sigma_n (>0)$, on Γ_n where σ_n is chosen such that the period of the conjugate function $v_n(P)$ of u_n along Γ_{n-1} becomes 2π. Now we define the function $z(P) = x(P) + iy(P)$ as follows:

$$
(1) \qquad z(P) = \begin{cases} -g(P) + ih(P) & \text{for } P \in R_0, \\ u_n(P) + iv_n(P) + \sum_{i=1}^{n-1} \log \sigma_i, & \\ & \text{for } P \in R_n - R_{n-1} (n=1, 2, \cdots). \end{cases}
$$

If we put

$$S=\sum_{i=1}^{\infty}\log\sigma_i,$$

then $R\epsilon O_G$ (parabolic type) if and only if there exists an exhaustion such that $S=\infty$ (L. Sario [5], K. Noshiro [3]).

2. Let ψ be a non-negative covariant whose second derivatives are all continuous except the isolated zero points P_ν and singular points Q_μ where, for corresponding local parameters (in general $w=u+iv$ stands for the local parameter),

$$\psi=|w|^k\psi_1,\quad \psi=|w|^{-l}\psi_2\quad (k,\ l>0)$$

and

$$\iint \Delta\log\psi_i\,dudv\quad (i=1,2)$$

remain finite in respective neighbourhood.

Let L_x denote the level curves $\Re z=x$. Suppose now that there exists neither zero nor pole of ψ on a curve L_x. Then we have by Gauss-Bonnet's theorem (c. f. R. Nevanlinna [1])

$$(2)\quad n_\psi(x,\ 0)-n_\psi(x,\ \infty)=\chi(x)+\frac{1}{2\pi}K_\psi(x)+\frac{1}{2\pi}\int_{L_x}\frac{\partial\log\psi(z)}{\partial x}dy$$

where $n_\psi(x,0)$ and $n_\psi(x,\infty)$ are respectively the number of zeros and poles of ψ (counted with multiplicities) on the domain G_x: $\{-\infty\leqq\Re z<x\}$, $\chi(x)$ is the Euler characteristic of G_x, and $K_\psi(x)=\iint_{G_x}k_\psi\cdot\psi^2dxdy$ where k_ψ denotes Gauss' total curvature under the metric $d\sigma=\psi|dw|$. Integrating (2) from $x=x_0$ to x, we have

$$(3)\quad \int_{x_0}^{x}[n_\psi(x,\ 0)-n_\psi(x,\ \infty)]dx=\int_{x_0}^{x}\chi(x)\,dx+\frac{1}{2\pi}\int_{x_0}^{x}K_\psi(x)\,dx$$
$$+\frac{1}{2\pi}\int_{L_x}\log\psi(z)\,dy-\frac{1}{2\pi}\int_{L_{x_0}}\log\psi(z)\,dy.$$

3. Suppose that for the fixed local parameter $\zeta=\xi+i\eta$ at P_0

$$g(\zeta)=\log\frac{1}{|\zeta|}+\gamma_0+U(\zeta),\quad \psi(\zeta)=|\zeta|^m\psi_1(\zeta)\quad (m\gtreqless 0)$$

where $\psi_1(0)\neq 0,\ \infty$ and U is single-valued, harmonic at $\zeta=0$ and $U(0)=0$. Taking x_0 so small that L_{x_0} is contained in this neighbourhood of P_0, then we have

$$(4)\quad \int_{L_{x_0}} \log \psi(z)\,dy = \int_{-g(\zeta)=x_0} \log\left(\psi(\zeta)\left|\frac{d\zeta}{dz}\right|\right) dh(\zeta) = 2\pi \log \psi_1(0) +$$

$$+ \int_{-g=x_0} \log\left(|\zeta|^m \left|\frac{d\zeta}{dz}\right|\right) dh(\zeta) + \delta_1,\ \delta_1 \to 0 \text{ for } x_0 \to -\infty$$

and $\left|\frac{d\zeta}{dz}\right| = |\zeta| \cdot |1 + \text{reg. function}|$ at $\zeta = 0$. Since $\log|\zeta| = x_0 + \gamma_0 + O(|\zeta|)$ on the level curve $-g(\zeta) = x_0$, the integral (4) is equal to

$$(5)\qquad 2\pi[\log \psi_1(0) + (m+1)(\gamma_0 + x_0)] + \delta_2,$$

$$\delta_2 \to 0 \quad \text{for} \quad x_0 \to -\infty.$$

Let $n_\psi(-\infty, a)$ be the number of a-points ($a=0, \infty$) of ψ at P_0 and

$$N_\psi(x, a) = \int_{-\infty}^{x} [n_\psi(x, a) - n_\psi(-\infty, a)]\,dx + n_\psi(-\infty, a)\,x \quad (a=0, \infty),$$

then from (3) and (5) we have easily for $x_0 \to -\infty$ the fundamental

THEOREM 1 *For non-negative covariant ψ stated in sec. 2*

$$(6)\quad N_\psi(x, 0) - N_\psi(x, \infty) = \int_{-\infty}^{x} [\chi(x) + 1]\,dx + \frac{1}{2\pi}\int_{-\infty}^{x} K_\psi(x)\,dx +$$

$$+ \frac{1}{2\pi}\int_{L_x} \log \psi(z)\,dy - \gamma_0(1 + m_\psi) - x - c_\psi,$$

where $m_\psi = n_\psi(-\infty, 0) - n_\psi(-\infty, \infty)$ *and* $c_\psi = \lim_{\zeta \to 0} \log(\psi(\zeta)|\zeta|^{-m_\psi})$.

N. B. Since $\chi(x) + 1 = 0$ for x which is sufficiently near $-\infty$, all the integrals in (6) are finite.

4. Now we consider a (meromorphic) differential of order n on R, by which we shall understand the invariant form

$$f(w)\,dw^n$$

where $f(w)$ is meromorphic with respect to a local parameter w and is transformed under the change of local parameters by the rule $f(\zeta) = f(w)\left(\frac{dw}{d\zeta}\right)^n$. We call f the (meromorphic) covariant of order n. The differentials of order n are usually called as functions, differentials, quadratic differentials, reciprocal differentials and so on, according as $n = 0, 1, 2, -1$ and so on. In the expansion of f with respect to a local parameter w

$$f(w) = c_0 w^k + c_1 w^{k+1} + \cdots \quad (c_0 \neq 0),$$

we call $|k|$ the order of zeros (if $k>0$) and poles (if $k<0$) of $f(w)dw^n$ (or $f(w)$) respectively. This order is obviously invariant under the change of local parameters.

5. If $\varphi_1, \varphi_2, \cdots$ are meromorphic covariants (of order 1) and $\psi=\sqrt{\sum_i|\varphi_i|^2}$, we have

$$(7) \qquad -K_\psi(x)=\iint_{G_x}\frac{\sum_{i,j}|\varphi_i\varphi_j'-\varphi_i'\varphi_j|^2}{(\sum|\varphi_i|^2)^2}dxdy\equiv 2A_\psi(x)\geqq 0,$$

$$T_\psi(x)=\frac{1}{\pi}\int_{-\infty}^{x}A_\psi(x)\,dx$$

which vanish if and only if the space $(\varphi_1, \varphi_2, \cdots)$ is one dimensional. (R. Nevanlinna [1], [2]). Hence if $A_\psi\neq 0$, $T_\psi(x)\geqq O(x)$. (cp. (12))

6. THEOREM 2. *If φ is a meromorphic covariant of order $n\,(n=0, \pm1, \pm2, \cdots)$, we have*

$$(8) \qquad \begin{aligned} N_\varphi(x, 0)-N_\varphi(x, \infty)&=\frac{1}{2\pi}\int_{L_x}\log|\varphi(z)|\,dy-c_{|\varphi|}-m_\varphi\cdot\gamma_0 \\ &+n\Big[\int_{-\infty}^{x}(\chi(x)+1)\,dx-x-\gamma_0\Big] \quad (-\infty\leqq x<S). \end{aligned}$$

Proof. First we note that for any two covariants φ_1, φ_2 we have always for any $x\,(-\infty\leqq x<S)$

$$(9) \qquad \begin{aligned} n_{\varphi_1/\varphi_2}(x, 0)-n_{\varphi_1/\varphi_2}(x, \infty)&=n_{\varphi_1}(x, 0)-n_{\varphi_1}(x, \infty) \\ &-[n_{\varphi_2}(x, 0)-n_{\varphi_2}(x, \infty)]. \end{aligned}$$

Now for any given meromorphic covariant φ_1 of order $n+1\,(n>1)$ we take a meromorphic covariant φ of order n (the existence is well known), then since $\psi=|\varphi_1/\varphi|$ becomes a covariant (of order 1), we have under the remark of sec. 5 the formula (6) without a term $\int_{-\infty}^{x}K_\psi(x)\,dx$. Therefore, if we assume that the formula (8) holds for n, we see that it is valid for $(n+1)$ under the considerations of (9) and the relations

$$m_{\varphi/\varphi_1}+m_{\varphi_1}=m_\varphi, \quad c_{|\varphi/\varphi_1|}+c_{|\varphi_1|}=c_{|\varphi|}\,.$$

Hence we can conclude, by induction that (8) holds for any positive integer n, since it is valid for $n=1$ (i.e. (6)). We can analogously obtain the formula (8) for $n=0, -1, -2, \cdots$, q.e.d.

Remark 1°. $c_{|\varphi|}$ can be also expressed as

$$c_{|\varphi|}+m_{\varphi}\gamma_0=\lim_{\zeta\to 0}\left[\log|\varphi(\zeta)|+m_{\varphi}g(\zeta)\right].$$

Hence (8) just accords with (1), (1′) (p. 184) of Parreau [4], when $n=0$ (i.e. φ=function) and $R_0=R$ $(-\infty\leqq x<0)$. If $R\epsilon O_G$ and $R_0\to R$ as in Parrau [4], then the Robin constant γ_0 in (8) would tend to infinity and if R_0 is fixed, then the term x in the remainder increases to $S(\leqq\infty)$. (8) for $n=1$ and (6) are nothing else but R. Nevanlinna's formulas (10‴) and (10) in [1] resp., which only differ from ours in the remainder and the choice of coordinates.
2°. Starting from (2) we can also prove the following: for a meromorphic covariant φ of order n we have

$$(10)\qquad n_{\varphi}(x,0)-n_{\varphi}(x,\infty)=n\chi(x)+\frac{1}{2\pi}\int_{L_x}\frac{\partial\log|\varphi(z)|}{\partial x}dy$$

where it is supposed that φ has no zero and pole on L_x.

§ 2.

As an application of above results, some relations on the meromorphic covariants $\varphi_1,\varphi_2,\cdots$ (of order 1) will be obtained. For simplicity we take P_0 in this paragraph as different from anyone of zeros and poles of $\varphi_1,\varphi_2,\cdots$.
7. For $\varphi_1=p'\,(p=-g+ih)$, $\varphi_2=\varphi$ and $\psi=\sqrt{|p'|^2+|\varphi|^2}$ on R_0 it follows that from (7)

$$A_{\psi}(x)=\iint_{G_x}\frac{|\varphi'|^2}{(1+|\varphi|^2)^2}dxdy,\qquad T(x,\varphi)\equiv\frac{1}{\pi}\int_{-\infty}^{x}A_{\psi}(x)\,dx$$
$$(-\infty\leqq x<0)$$

and $c_{\psi}=0$, $m_{\psi}=-1$, $N_{\psi}(x,\infty)=N_{\varphi}(x,\infty)+x$, $0\leqq N_{\psi}(x,0)\leqq N_{p'}(x,0)$ $=\int_{-\infty}^{x}(\chi(x)+1)\,dx$, because $p'dz$ is the Schottky differential on every G_x, therefore (16) holds. Hence Theorem 1 reduces to

THEOREM 3. *For meromorphic covariant φ of order 1*

$$(11)\qquad \frac{1}{2\pi}\int_{L_x}\log\sqrt{1+|\varphi(z)|^2}\,dy+N_{\varphi}(x,\infty)$$
$$=T(x,\varphi)+s(x)\int_{-\infty}^{x}(\chi(x)+1)\,dx$$

where $-1\leqq s(x)\leqq 0$, $-\infty\leqq x<0$.

$s=-1$ if $\varphi\neq 0$ at zero points of p', and $s=0$ if the function φ/p' is regular at the same points. (11) corresponds to Shimizu-Ahlfors' theorem in the theory of meromorphic functions in the circle $(\chi(x)+1=0)$. To obtain the corresponding theorem for a meromorphic function f, it suffices to take the covariant $\psi=\sqrt{|p'|^2+|\varphi|^2}$ where $\varphi=fp'$. Then we see that $N_\psi(x, 0)-N_\psi(x, \infty)=N_{p'}(x, 0)-N_f(x, \infty)-x$, $m_\psi=-1$ and $c_\psi=\log\sqrt{1+|f(P_0)|^2}$. Hence we can analogously obtain the

THEOREM 3′. *For the meromorphic function f on R*

$$\frac{1}{2\pi}\int_{L_x}\log\sqrt{1+|f(z)|^2}\,dy+N_f(x, \infty)$$
$$=T(x,f)+\log\sqrt{1+|f(P_0)|^2}\quad(-\infty\leqq x<0). \text{ (cf. [4] p. 185 (3)).}$$

8. By (2), (6), (8) and (10) we have *for quadratic covariant* $\varphi=\sum_i(c_i\varphi_i)^2$ (c_i : *const.*) *and the covariant* $\psi=\sqrt{\sum_i|c_i\varphi_i|^2}$

$$(12)\quad N_1(x, \{c_i\varphi_i\})\leqq N_1(x, \{c_i\varphi_i\})+\frac{1}{2\pi}\int_{L_x}\log\frac{\psi^2(z)}{|\varphi(z)|}dy=2T_\psi(x)+C,$$

$$(13)\quad n_1(x, \{c_i\varphi_i\})=\frac{2}{\pi}A_\psi(x)+\frac{1}{2\pi}\int_{L_x}\frac{\partial}{\partial x}\log\frac{|\varphi(z)|}{\psi^2(z)}dy,$$

where $C=\log\frac{\psi^2(p_0)}{|\varphi(p_0)|}\geqq 0$ *and* $n_1(x, \{c_i\varphi_i\})=[n_\varphi(x, 0)-2n_\psi(x, 0)]+[2n_\psi(x, \infty)-n_\varphi(x, \infty)]\geqq 0$ in which the first term gives the number of zeros of φ which exceed those of ψ^2 and the second one the number of poles of ψ^2 which exceed those of φ, and $N_1(x, \{c_i\varphi_i\})=\int_{-\infty}^{x}n_1(x, \{c_i\varphi_i\})\,dx\geqq 0$.

9. Now we take a compact subregion W, e.g. $W=R_0$ and suppose that for $\varphi_1, \varphi_2, \cdots$ either the condition

$$(14)\quad \int_{L_0}\frac{\partial}{\partial x}\log\frac{|\varphi(z)|}{\psi^2(z)}dy\geqq 0\quad\text{or}\quad\int_{L_0}\log\frac{\psi^2(z)}{|\varphi(z)|}dy\leqq 2\pi C$$

is satisfied. From (12) or (13) it is seen that *under the condition* (14) $(\varphi_1, \varphi_2, \cdots)$ *is one dimensional* ($\varphi_i=$*const.* φ_1; $i=2, 3, \cdots$) *if and only if* $n_1(0, \{c_i\varphi_i\})=0$ *for* $c_1, c_2, \cdots$ *such that* $\varphi\not\equiv 0$. For instance, let $\mathfrak{S}$ be the space of differentials on W (real Schottky differentials on W) and $\varphi_1 dz, \varphi_2 dz, \cdots\in\mathfrak{S}$, $c_1, c_2, \cdots$ be real constants, then both conditions (14) are certainly fulfilled. Moreover by (10) (cf. (16)) $n_\varphi(0, 0)-n_\varphi(0, \infty)=2\chi_0$ (χ_0 ; Euler characteristic of W), hence *in the case of* $\varphi_i dz\in\mathfrak{S}$ $(i=1, 2, \cdots)$ $(\varphi_1, \varphi_2, \cdots)$ *is one dimensional if and only if for real constants* $c_i\neq 0$ *such that* $\varphi\not\equiv 0$,

(15) $$n_{\psi}(0, 0) - n_{\psi}(0, \infty) = \chi_0$$

where $n_{\psi}(0, 0)$ *is just the number of common zeros of* $\varphi_1, \varphi_2, \cdots$ *on* W *and* $n_{\psi}(0, \infty)$ *the number of all polcs of them.* We may here choose constants $c_1, c_2, \cdots$ such that φ has the same multiplicities at the zero points and poles as ψ^2 does. That is, let $P_1^{m_1}P_2^{m_2}\cdots P_l^{m_l}$ $(m_i \gtrless 0)$ be the divisor of ψ^2 and $\gamma_{ij} = \alpha_{ij} + \sqrt{-1}\,\beta_{ij}$ be the m_j-th coefficient in the expansion of $\varphi_i (i=1, \cdots, p)$ with respect to the fixed local parameter at each P_j. It is sufficient to take as $(c_1^2, c_2^2, \cdots)$ an arbitrary point of the first quadrant Q in the space $(x_1, \cdots, x_p)$ with the exception of at most l spaces of dimension $(\leqq p-1)$ each of which appears as the intersection of (hyper) plane $\sum_{i=1}^{p} x_i(\alpha_{ij}^2 - \beta_{ij}^2) = 0$ and $\sum_{i=1}^{p} x_i\alpha_{ij}\beta_{ij} = 0$. Hence the condition (15) can be stated also such that for any point $(c_1^2, c_2^2, \cdots)$ belonging to Q, φ has the same divisor as ψ^2.

10. Finally, we see by (10) immediately that *if* $\varphi\, dz$ *is a differential of order* n *which is real or imaginary on every contour of* L_0, *then*

(16) $$n_{\varphi}(0, 0) - n_{\varphi}(0, \infty) = n\chi_0$$

where the numbers in the left hand side should be counted with half its multiplicities if zeros or poles lie on L_0. Especially for $p = g - ih$, $ip'dz \in \mathfrak{S}$ and p' has a simple pole at P_0, hence *the number of multiple points of Green niveau curves on* W *is* $(2\rho + q - 1)$, *where* ρ *is the genus and* q *the number of contours of* W. This relation is reduced to a familiar one if W is a plane domain.

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