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On quasi-equicontinuous sets – Sets of solutions of a differential equation –

By

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In the previous papers [5], [6], we have studied some kinds of transformations of differential equations. In the present paper the same subject will be studied more systematically.

In §1 we introduce the new concept of "quasi-equicontinuity." In §2 we study the correspondence between "quasi-equicontinuous sets" and "equicontinuous sets". In §3 and §4 we shall find it convenient to introduce the new concept into the theory of differential equations. Theorem 7 in §4 is an extension of theorems discussed in the previous papers.

1. Notations and definitions.

Notations. 1) *Given two sets* E, F, $\boldsymbol{F}(E, F)$ *denotes the set of all functions defined on* E *with values in* F. $\boldsymbol{F}_1(E, F)$ *denotes the set of all functions each of which is defined on a subset of* E *with values in* F. *Then clearly* $\boldsymbol{F}(E, F) \subset \boldsymbol{F}_1(E, F)$. *For each* $u \in \boldsymbol{F}_1(E, F)$ A_u *denotes the subset of* E *on which* u *is defined. We denotes by* τ *such an element* u *of* $\boldsymbol{F}_1(E, F)$ *as* $A_u = \phi$[1].

2) *Given two topological spaces* E, F, $\boldsymbol{C}(E, F)$ *denotes the set of all continuous functions on* E *to* F. *Clearly* $\boldsymbol{C}(E, F) \subset \boldsymbol{F}(E, F)$. $\boldsymbol{C}_1(E, F)$ *denotes the subset of* $\boldsymbol{F}_1(E, F)$ *such that for each* $u \in \boldsymbol{C}_1(E, F)$.

a) A_u *is open,*

b) u *is continuous on* A_u,

c) *if* x_0 *belongs to* $\bar{A}_u$[2] *but not to* A_u, *there is no point*

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1) ϕ means the empty set.

2) $\bar{A}$ means the closure of A.

adherent[3] *to the filter-base* $u(\boldsymbol{F})$ *where* $\boldsymbol{F}$ *is the trace on* A_u *of the filter of neighborhoods of* x_0.

Clearly $\boldsymbol{C}(E, F) \subset \boldsymbol{C}_1(E, F) \subset \boldsymbol{F}_1(E, F)$.

If F is compact the adherence of any filter-base in F is not empty. Then for any $u \in \boldsymbol{C}_1(E, F)$ we have $\bar{A}_u = A_u$. Therefore, each element of $\boldsymbol{C}_1(E, F)$ is defined on an open and closed subset of E. Let α be any fixed point of F we can define the function φ on $\boldsymbol{C}_1(E, F)$ onto $\boldsymbol{C}(E, F)$ by letting

$$\varphi \circ u(x) = \begin{cases} u(x) & \text{if} \quad x \in A_u, \\ \alpha & \text{if} \quad x \in E - A_u \text{[4]}. \end{cases}$$

Moreover, if E is connected $\boldsymbol{C}_1(E, F)$ coincides with $\boldsymbol{C}(E, F)$ and φ is the identity.

Definition 1.[5] *Given a topological space* E *and a uniform space* F, *a subset* $\boldsymbol{H}$ *of* $\boldsymbol{F}(F, F)$ *is said to be equicontinuous at a point* $x_0 \in E$ *if for every entourage* V *of* F *there is a neighborhood* U *of* x_0 *such that* $(u(x_0), u(x)) \in V$ *if* $u \in \boldsymbol{H}$ *and if* $x \in U$.

$\boldsymbol{H}$ *is said to be equicontinuous if it is equicontinuous at every point of* E.

Definition 2. *Given two topological spaces* E, F, *a subset* $\boldsymbol{H}_1$ *of* $\boldsymbol{F}_1(E, F)$ *is said to be quasi-equicontinuous at a point* $x_0 \in E$ *if*

a) *for any compact*[6] *subset* F_1 *of* F *there are a compact subset* F_2 *of* F *and a neighborhood* U *of* x_0 *such that for any* $u \in \boldsymbol{H}_1$, $u(U \cap A_u) \cap F_1 \neq \phi$ *implies* $U \subset A_u$ *and* $u(U) \subset F_2$,

b) *for each compact subset* F_1 *of* F *there is a neighborhood* U *of* x_0 *provided that the set of restrictions to* U *of all* u, *such that* $u \in \boldsymbol{H}_1$, $U \subset A_u$ *and* $u(U) \subset F_1$, *is an equicontinuous subset of* $\boldsymbol{F}(U, F_1)$ *at* x_0.

$\boldsymbol{H}_1$ *is said to be quasi-equicontinuous if it is quasi-equicontinuous at every point of* E.

If F is compact it may be supposed as F_1 in the condition *a*). Let $\boldsymbol{H}_1$ be quasi-equicontinuous and suppose that $u \in \boldsymbol{H}_1$ and that $x_0 \in \bar{A}_u$. Since for any neighborhood U of x_0 we have $U \cap A_u \neq \phi$,

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3) cf. Bourbaki [1], Chap. I.

4) $A-B$ means the set of all the elements of A which are not contained in B.

5) cf. Bourbaki [1], Chap. X.

6) In this paper compact spaces are always supposed to be Hausdorff spaces. We also consider every compact space as a uniform space with the uniform structure of finite open coverings which is the unique structure compatible with its topology.

there is a neighborhood U_1 of x_0 such that $U_1 \subset A_u$. Hence A_u is open and $\bar{A}_u = A_u$. By the condition *b*) it is clear that $\boldsymbol{H}_1 \subset \boldsymbol{C}_1(E, F)$ ann $\varphi(\boldsymbol{H}_1)$ is an equicontinuous subset of $\boldsymbol{C}(E, F)$. Furthermore, if E is connected A_u is identical with E for each $u \in \boldsymbol{H}_1$. That is, the quasi-equicontinuity coincides with the equicontinuity.

Remark. If $\boldsymbol{H}_1$ satisfies the condition *a*) in Definition 2 for each $u \in \boldsymbol{H}_1$, A_u is open. Because for any $x_0 \in A_u$ if we suppose that $F_1 = \{u(x_0)\}$ there are a neighborhood U of x_0 and a compact subset F_2 of F such that $u(U) \subset F_2 \subset F$, *i.e.* $U \subset A_u$.

2. Correspondence between $\boldsymbol{F}_1(E, F)$ and $\boldsymbol{F}(E, F')$.

Let E be a topological space and let F be a locally compact but not compact space. Then we can construct the Alexandroff compactification F' of F by adding the point at infinity ω.

We can define the function φ on $\boldsymbol{F}_1(E, F)$ onto $\boldsymbol{F}(E, F')$ by letting

$$\varphi \circ u(x) = \begin{cases} u(x) & \text{if } x \in A_u, \\ \omega & \text{if } x \in E - A_u. \end{cases}$$

(Clearly $\varphi \;\; \tau(x) = \omega$ for all $x \in E$.) If $v \in \boldsymbol{F}(E, F')$ let A be the set of all x such that $v(x) \neq \omega$. Since A is a subset of E we get an element u of $\boldsymbol{F}_1(E, F)$ by setting $u(x) = v(x)$ for all $x \in A$. It is clear that u is the unique element of $\boldsymbol{F}_1(E, F)$ such that $\varphi(u) = v$. Therefore φ defines a one-to-one correspondence between $\boldsymbol{F}_1(E, F)$ and $\boldsymbol{F}(E, F')$.

Now suppose that $u \in \boldsymbol{C}_1(E, F)$ and $v = \varphi(u)$. Since A_u is open and $v(x) = u(x)$ for $x \in A_u$, $v(x)$ is continuous on A_u. And since $v(x) = \omega$ for $x \in E - \bar{A}_u$ and $E - \bar{A}_u$ is open, $v(x)$ is continuous on $E - \bar{A}_u$. Finally suppose $x_0 \in \bar{A}_u - A_u$ and let $\boldsymbol{u}(x_0)$ be the filter of neighborhoods of x_0. Since F' is compact the adherence of the filter-base $v(\boldsymbol{u}(x_0))$ in F' is not empty. On the other hand, since $u \in \boldsymbol{C}_1(E, F)$ any point of F cannot be adherent to $v(\boldsymbol{u}(x_0))$. Therefore $v(\boldsymbol{u}(x_0))$ converges to ω so that $v \in \boldsymbol{C}(E, F')$. Inversely if $v \in \boldsymbol{C}(E, F')$ it is easy to see that $\varphi^{-1}(v)$[7] $\in \boldsymbol{C}_1(E, F)$. Consequently φ defines a one-to-one correspondence between $\boldsymbol{C}_1(E, F)$ and $\boldsymbol{C}(E, F')$.

Theorem 1. *Let E be a topological space, let F be a locally*

7) φ^{-1} means the inverse of φ.

compact but not compact space and let F' be the compact space added the point at infinity ω to F. Then a necessary and sufficient condition for a subset $\boldsymbol{H}_1$ of $\boldsymbol{F}_1(E, F)$ to be quasi-equicontinuous at $x_0 \in E$ (or quasi-equicontinuous) is that the subset $\varphi(\boldsymbol{H}_1)$ of $\boldsymbol{F}(E, F')$ be equicontinuous at x_0 (or equicontinuous).

Proof. Let $\boldsymbol{H}_1$ be quasi-equicontinuous at x_0. If W is an entourage of F' there is a finite open covering $\boldsymbol{R}=(V_i)_{1\leqq i\leqq n}$ such that $\bigcup_{1\leqq i\leqq n}(V_i\times V_i)\subset W$[8]. Let V_ω denote the intersection of all V_i such that $\omega\in V_i$. Then V_ω is also an open neighborhood of ω and $F'-V_\omega$ is a compact subset of F. Let $F_1=F'-V_\omega$. Hence there are a compact subset F_2 of F and a neighborhood U_1 of x_0 such that for any $u\in\boldsymbol{H}_1$, $u(U_1\cap A_u)\cap F_1\neq\phi$ implies $U_1\subset A_u$ and $u(U_1)\subset F_2$. Now let $\boldsymbol{H}_2$ be the set of all the elements of $\boldsymbol{H}_1$ such that $U_1\subset A_u$ and $u(U_1)\subset F_2$. Then if $u\in\boldsymbol{H}_1-\boldsymbol{H}_2$ $u(U_1\cap A_u)\cap F_1=\phi$. If we write $\varphi(u)=v$ we have $v(x)=\omega$ for $x\in E-A_u$. Hence for any $x\in U_1$ we obtain $(v(x_0), v(x))\in V_\omega\times V_\omega\subset W$. Since the restriction of $\boldsymbol{H}_2$ to U_1 is an equicontinuous subset of $\boldsymbol{F}(U_1, F_2)$ at x_0 it is not difficult to assert that $\varphi(\boldsymbol{H}_2)$ is an equicontinuous subset of $\boldsymbol{F}(U_1, F')$. Then there is a neighborhood $U_2\subset U_1$ of x_0 such that for any $u\in\boldsymbol{H}_2$ and $x\in U_2$ we have $(v(x_0), v(x))\in W$ where $v=\varphi(u)$. Thus for any $u\in\boldsymbol{H}_1$ and $x\in U_2$ we have $(v(x_0), v(x))\in W$. That is, $\varphi(\boldsymbol{H}_1)$ is an equicontinuous subset of $\boldsymbol{F}(E, F')$ at x_0.

Conversely suppose that $\varphi(\boldsymbol{H}_1)$ is equicontinuous at x_0. Let F_1 be a compact subset of F. Since $F_1\subset F'$, $F_1\cap\{\omega\}=\phi$ and F' is compact, there is an open subset Q of F' such that $F_1\subset Q$ and $\overline{Q}\cap\{\omega\}=\phi$ where $\overline{Q}$ is the closure of Q in F'. If we put $\overline{Q}=F_2$ then F_2 is a compact subset of F[9]. Let $V_1=Q-F_1$ so that V_1 is an open subset of F'. Since F_1 and $F'-Q$ are compact without any common point there are two open subsets V_2, V_3, of F' such that $V_2\supset F_1$, $V_3\supset F'-Q$ and $V_2\cap V_3=\phi$. Let $\boldsymbol{R}=(V_i)_{1\leqq i\leqq 3}$ $\boldsymbol{R}$ is a finite open covering of F' so that $W=\bigcup_{1\leqq i\leqq 3}(V_i\times V_i)$ is an entourage of F'. Hence there is a neighborhood U of x_0 such that $\{v(x_0), v(x)\}\in W$ for any $v\in\varphi(\boldsymbol{H}_1)$ and $x\in U$. If there are an element $u\in\boldsymbol{H}_1$ and a point $x_1\in U$ such that $u(x_1)\in F_1$ we have $v(x_1)\in F_1$ where $v=\varphi(u)$. Therefore $v(x_0)\in V_2$ so that for any $x\in U$ we have

8) cf. Bourbaki [1], Chap. II.

9) In the present case F_2 in the condition *a*) in Definition 2 can be supposed as an arbitrary compact neighborhood of F_1.

$v(x) \in V_1 \cup V_2 = Q \subset F_2$ and then we have $U \subset A_u$ and $u(U) \subset F_2$. Thus it is proved that the condition *a*) is fulfilled. It is easily proved that the condition *b*) is fulfilled. q.e.d.

Corollary. *If $\boldsymbol{H}_1$ is a quasi-equicontinuous subset of $\boldsymbol{F}_1(E, F)$ we have $\boldsymbol{H}_1 \subset \boldsymbol{C}_1(E, F)$.*

Proof. By Theorem 1 $\varphi(\boldsymbol{H}_1)$ is an equicontinuous subset of $\boldsymbol{F}(E, F')$. Therefore $\varphi(\boldsymbol{H}_1) \subset \boldsymbol{C}(E, F')$ so that $\boldsymbol{H}_1 \subset \boldsymbol{C}_1(E, F)$. q.e.d.

Theorem 2. *Let E be a locally compact space, let F be a locally compact but not compact space and let F' be the compact space made by adding the point at infinity ω to F. Then a necessary and sufficient condition for a subset $\boldsymbol{H}_1$ of $\boldsymbol{C}_1(E, F)$ to be quasi-equicontinuous is that the closure of $\varphi(\boldsymbol{H}_1)$ in $\boldsymbol{C}(E, F')$, equipped with the topology of compact convergence*[10]*, be compact.*

Proof. Since F' is compact the theorem follows from Theorem 1 and the Ascoli theorem. q.e.d.

3. Sets of solutions of a differential equation.

Let E be a real interval[11] and R^n the real n-dimensional vector space, then both of them are locally compact relative to their usual topologies. Let F be an open subset of R^n then F is a locally compact but not compact subspace of R^n. And let F' be the Alexandroff compactification of F made by adding the point at infinity ω.

Definition 3. *Given a function $f(x, y)$ on $E \times F$ to R^n. For any $(x_0, y_0) \in E \times F$ a function $u(x)$ defined on a subinterval $I(\ni x_0)$ of E with values in F is said to be a solution on I of the differential equation*

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

equal to y_0 at the point x_0 if $f(x, u(x))$ is Lebesgue integrable on I and if for any $x \in I$ holds the following relation

$$u(x) = y_0 + \int_{x_0}^{x} f(t, u(t))dt\,.$$

10) cf. Bourbaki [1], Chap. X.

11) A real interval means a non-empty connected subset of the real line equipped with the usual topology.

From Definition 3 follows

Lemma 1. *If u is a solution of* (1) *on I equal to y_0 at x_0, for each $x_1 \in I$ u is a solution of* (1) *on I equal to $u(x_1)$ at x_1.*

Definition 3′. *For brevity, we call u in Definition* 3 *a solution of* (1) *on I or more simply a solution of* (1).

Notation. *We denote by $\boldsymbol{S}$ the set of all solutions of* (1).

For any $(x_0, y_0) \in E \times F$ there is such an element u of $\boldsymbol{S}$ that $A_u = \{x_0\}$, $u(x_0) = y_0$. Then u is called the trivial solution of (1) equal to y_0 at x_0. Thus $\boldsymbol{S}$ is a non-empty subset of $\boldsymbol{F}_1(E, F)$. Since any interval is supposed not to be empty it is clear that τ does not belong to $\boldsymbol{S}$.

Notation. *For each pair of elements u, v, of $\boldsymbol{S}$ we write $u \leqq v$ (or $v \geqq u$) if $A_u \subset A_v$ and if $v(x) = u(x)$ for all $x \in A_u$. $u < v$ (or $v > u$) means that $u \leqq v$ but that $u \neq v$.*

Lemma 2. *$\boldsymbol{S}$ is an ordered set*[12] *relative to the relation "$\leqq$".*

Proof. If $u \leqq v$ and $v \leqq w$ we have $A_u \subset A_v \subset A_w$. Then $w(x) = v(x) = u(x)$ if $x \in A_u$ so that $u \leqq w$. If $u \leqq v$ and $v \leqq u$ we have $A_u \subset A_v$ and $A_v \subset A_u$ so that $A_u = A_v$. Therefore we have $v(x) = u(x)$ for any $x \in A_u = A_v$, *i.e.* $u = v$. q.e.d.

Lemma 3. *Given $u \in \boldsymbol{S}$. Any totally ordered subset $\boldsymbol{S}_u$ of $\boldsymbol{S}$, consisting of elements v such that $u \leqq v$, has its supremum.*

Proof. Since for any $v \in \boldsymbol{S}_u$ we have $A_u \subset A_v$ $A = \bigcup_{v \in S_u} A_v$ is a subinterval of E. Let v, w, be a pair of elements of $\boldsymbol{S}_u$, we have $v \leqq w$ or $w \leqq v$. Then for any $x \in A_v \cap A_w$ we have $w(x) = v(x)$ so that we can define a function z on A to F such that for any $v \in \boldsymbol{S}_u$ we have $z(x) = v(x)$ for $x \in A_v$. If $x \in A$ there is an element v of $\boldsymbol{S}_u$ such that $x \in A_v$. On the other hand if $x_0 \in A_u$, x_0 belongs to A_v. Then we have $v(x) = u(x_0) + \int_{x_0}^{x} f(t, v(t))dt$. Since $z(t) = v(t)$ for $t \in A_v$ we get $z(x) = u(x_0) + \int_{x_0}^{x} f(t, z(t))dt$. Thus, it is clear that $z \in \boldsymbol{S}$ and that z is the supremum of $\boldsymbol{S}_u$. q.e.d.

Theorem 3. *For each $u \in \boldsymbol{S}$ there is a maximal element v of $\boldsymbol{S}$ such that $u \leqq v$.*

Proof. The subset of $\boldsymbol{S}$ consisting of all $w \in \boldsymbol{S}$ such that $u \leqq w$ has its least element u. Then, by Lemma 3 it is inductive.

12) An ordered set is often called a partially ordered set.

Consequently by the Zorn lemma it has a maximal element v. And clearly $u \leqq v$. q.e.d.

Each trivial solution of (1) is a minimal element of $\boldsymbol{S}$.

Definition 4. *A maximal (or minimal) element of $\boldsymbol{S}$ is called a maximal (or minimal) solution of* (1).

Notation. *We denote by $\boldsymbol{S}_M$ the set of all maximal solutions of* (1).

Corollary. *For any $(x_0, y_0) \in E \times F$ there is a maximal solution of* (1) *equal to y_0 at x_0.*

Proof. For any $(x_0, y_0) \in E \times F$ there exists the trivial solution of (1) equal to y_0 at x_0. Hence by Theorem 3 there is a maximal solution of (1) equal to y_0 at x_0. q.e.d.

Definition 5. *Let $f(x, y)$ be a function on $E \times F$ to R^n, Lebesgue measurable with respect to $x \in E$ for any fixed $y \in F$ and continuous with respect to $y \in F$ for any fixed $x \in E$. f is said to fulfill Carathéodory's condition locally in $E \times F$ if, given any compact subinterval I of E and any compact subset D of F, there is a non-negative Lebesgue integrable function $M(x)$ defined on I such that $|f(x, y)| \leqq M(x)$*[13] *for $x \in I$.*

Obviously any continuous function on $E \times F$ to R^n satisfies Carathéodory's condition locally in $E \times F$.

Remark. If f in (1) satisfies Carathéodory's condition locally in $E \times F$, by Scorza Dragoni's theorem[14] it is readily seen that there is such a subset e_0 of E of Lebesgue measure zero that if $u \in \boldsymbol{S}$ the relation $\frac{du(x)}{dx} = f(x, u(x))$ holds strictly whenever $x \in A_u - e_0$.

Theorem 4. *If f in* (1) *satisfies Carathéodory's condition locally in $E \times F$ $\boldsymbol{S}_M$ is a quasi-equicontinuous subset of $\boldsymbol{F}_1(E, F)$.*

Proof. Let x_0 be a point of E, F_1 a compact subset of F and U a compact interval neighborhood of x_0[15]. Then there is a non-negative integrable function $M(x)$ on U such that $|f(x, y)| \leqq M(x)$. Let $\boldsymbol{S}_1$ be the set of all the solutions u such that $u \in \boldsymbol{S}_M$, $U \subset A_u$, $u(U) \subset F_1$. Then if $u \in \boldsymbol{S}_1$, $x_1 \in U$, $x_2 \in U$, $x_1 \leqq x_2$, we have $|u(x_2) - u(x_1)| = |\int_{x_1}^{x_2} f(t, u(t))dt| \leqq \int_{x_1}^{x_2} M(t)dt$. Therefore the restriction of $\boldsymbol{S}_1$

13) $|f|$ means the Euclidean norm of f.

14) cf. Scorza Dragoni [8]; Hayashi [4].

15) It means a compact subinterval of E which includes x_0 in its interior.

to U is an equicontinuous subset of $\boldsymbol{F}(U, F_1)$. Thus $\boldsymbol{S}_M$ satisfies the condition *b*) in Definition 2.

Next let x_0 be a point of E and let F_1 be a compact subset of F. For some positive ρ let F_2 be the set of all $y \in F$ such that $d(y, F_1) \leqq \rho$[16], then F_2 is a compact subset of F. Let U_1 be a compact interval neighborhood of x_0. There exists a non-negative integrable function $M(x)$ on U_1 such that $|f(x, y)| \leqq M(x)$ in $U_1 \times F_2$. If we select some compact interval neighborhood U of x_0 such that $U \subset U_1$ we obtain $\int_U M(x)dx < \rho$. Hence if $u \in \boldsymbol{S}_M$ $u(U \cap A_u) \cap F_1 \neq \phi$ implies $u(U \cap A_u) \subset F_2$. Since $U \cap A_u$ is also an interval there is a monotone increasing sequence of compact subintervals of U, written $\{I_m\}_{m \in N}$[17], such that $\bigcup_{m \in N} I_m = U \cap A_u$. By Carathéodory's existence theorem[18] for each $m \in N$ there is a solution v_m of (1) on U such that $v_m(x) = u(x)$ for $x \in I_m$. Let x_0 be a point of $\bigcap_{m \in N} I_m$ we have, for any m, $v_m(x) = u(x_0) + \int_{x_0}^{x} f(t, v_m(t))dt$ for $x \in U$. Since $v_m(U) \subset F_2$ $\{v_m\}_{m \in N}$ is an equicontinuous subset of $\boldsymbol{F}(U, F_2)$. By the Ascoli theorem we can select a uniformly convergent subsequence of $\{v_m\}_{m \in N}$. Let v be its limit, by the Lebesgue theorem we have $v(x) = u(x_0) + \int_{x_0}^{x} f(t, v(t))dt$. Since $v(x) = u(x)$ in $U \cap A_u$, we can define a solution w of (1) on $U \cup A_u$ such that $u \leqq w$ by setting $w(x) = \begin{cases} u(x) & \text{if } x \in A_u, \\ v(x) & \text{if } x \in U - A_u. \end{cases}$ Since u is a maximal solution of (1) w must coincide with u so that $A_u = U \cup A_u$ and then $U \subset A_u$. That is, $\boldsymbol{S}_M$ satisfies the condition *a*) in Definition 2. q.e.d.

Corollary 1. *The closure of $\varphi(\boldsymbol{S}_M)$ in $\mathbf{C}$ (E, F'), equipped with the topology of compact convergence, is compact.*

Proof. This follows directly from Theorem 2 and Theorem 4. q.e.d.

Corollary 2. *If $u \in \boldsymbol{S}_M$, A_u is open. And if $x_0 \in \bar{A}_u - A_u$ we have* $\lim_{\substack{x \to x_0 \\ x \in A_u}} u(x) = \omega$.

16) $d(y, F_1)$ means the Euclidean distance between y and F_1.

17) N means the set of all positive integers.

18) cf. Carathéodory [2], pp. 665–672.

Proof. This follows from Theorem 4 and the corollary of Theorem 1. q.e.d.

4. Sets of generalized solutions.

Suppose f in (1) satisfies Carathéodory's condition locally in $E\times F$. Let $\{u_m\}_{m\in N'}$, where N' is some subset of N, be a sequence of elements of $\boldsymbol{S}_M$ such that any pair of two elements of the sequence $\{A_{u_m}\}_{m\in N'}$ has no common point. Then $\bigcup_{m\in N'} A_{u_m}$ is an open subset of E. Now we can define a function v on $\bigcup_{m\in N'} A_{u_m}$ to F by letting for each m $v(x)=u_m(x)$ if $x\in A_{u_m}$. If $N'=\phi$, v must coincide with τ.

Definition 6. *If f in* (1) *satisfies Carathéodory's condition locally in $E\times F$, the above-mentioned v is called a generalized solution of* (1).

Notation. $\boldsymbol{S}_G$ *means the set of all the generalized solutions of* (1).

It is clear that $\boldsymbol{S}_M\subset\boldsymbol{S}_G\subset\boldsymbol{C}_1(E, F)$. Since $\tau\in\boldsymbol{S}_G$, τ is called the trivial generalized solution of (1).

Theorem 5. *If f in* (1) *satisfies Carathéodory's condition locally in $E\times F$, $\varphi(\boldsymbol{S}_G)$ is compact relative to the topology of compact convergence.*

Proof. The quasi-equicontinuity of $\boldsymbol{S}_G$ is verified in the same way with that of $\boldsymbol{S}_M$. Then $\varphi(\boldsymbol{S}_G)$ is equicontinuous and the closure of $\varphi(\boldsymbol{S}_G)$ in $\boldsymbol{C}(E, F')$ equipped with the topology of compact convergence, written $\overline{\varphi(\boldsymbol{S}_G)}$, is compact. If $v_0\in\overline{\varphi(\boldsymbol{S}_G)}$, $v_0\in\boldsymbol{C}(E, F')$. Let $u_0=\varphi^{-1}(v_0)$, A_{u_0} is an open subset of E so that there is such a sequence of intervals $\{I_m\}_{m\in N'}$, where N' is some subset of N, that each I_m is open relative to E, $I_{m_1}\cap I_{m_2}=\phi$ if $m_1\neq m_2$, and $\bigcup_{m\in N'} I_m=A_{u_0}$. For any I_m there is a monotone increasing sequence of compact subintervals of I_m, written $\{E_i\}_{i\in N}$, such that $I_m=\bigcup_{i\in N} E_i$. Since E_i is compact $u_0(E_i)$ is a compact subset of F. Let $K_i=u_0(E_i)$ and let K_i' be a compact neighborhood of K_i in F. We denote by V_i the set of all (y, z) such that $y\in K_i'$, $z\in K_i'$ and $d(y, z)<\frac{1}{i}$ [19], and we write $K_i''=F'-K_i$. Then $W_i=V_i\cup(K_i''\times K_i'')$ is an entourage of F'. Since $v_0\in\overline{\varphi(\boldsymbol{S}_G)}$ there is an element v_i of $\varphi(\boldsymbol{S}_G)$ such

19) $d(y, z)$ means the Euclidean distance in F between y and z.

that $(v_0(x), v_i(x)) \in W_i$ for all $x \in E_i$. Consequently for any $x \in E_i$, we have $v_i(x) \in W_i(v_0(x)) \subset W_i(K_i) \subset K'_i \subset F$. Then we have for any $x \in E_i$, $u_i(x) \in K'_i$ and $|u_0(x) - u_i(x)| < \frac{1}{i}$. Hence the sequence $\{u_i\}_{i \in N}$ converges to u_0 uniformly in each E_i, so that the restriction of u_0 to E_i is a solution of (1). Since $I_m = \bigcup_{i \in N} E_i$ and $u_0 \in C_1(E, F)$, then the restriction of u_0 to each I_m is a maximal solution of (1). Now it is clear that $u_0 \in S_G$ so that $\overline{\varphi(S_G)} = \varphi(S_G)$. Therefore $\varphi(S_G)$ is compact. q.e.d.

Theorem 6. *Suppose f in* (1) *satisfies Carathéodory's condition locally in $E \times F$. Let x_0 be a point of E, D a closed subset of F, $\boldsymbol{H}_1$ the set of all the maximal solutions u of* (1) *such that $u(x_0) \in D$ and $\boldsymbol{H}$ the closure of $\varphi(\boldsymbol{H}_1)$ in $\boldsymbol{C}(E, F')$ equipped with the topology of compact conveagence. Then $\boldsymbol{H}$ is compact. And if u belongs to $\varphi^{-1}(\boldsymbol{H})$ but not to $\boldsymbol{H}_1$ then u is a generalized solution of* (1) *such that $u(x_0) \in D$ or $\varphi \circ u(x_0) = \omega$.*

Proof. This theorem follows directly from Theorem 5. q.e.d.

Corollary. *In particular, suppose D compact. Then if u belongs to $\varphi^{-1}(\boldsymbol{H})$ but not to $\boldsymbol{H}_1$ u is a generalized solution of* (1) *such that $u(x_0) \in D$.*

Theorem 7. *Suppose F coincides with R^n and f in* (1) *satisfies Carathéodory's condition locally in $E \times R^n$. Let R^{n+1} be the real $n+1$-dimensional vector space and S^n the unit sphere in R^{n+1}. There is such a homeomorphism of F' onto S^n as is a homeomorphism of $\varphi(S_G)$ equipped with the topology of compact convergence onto the set of all maximal solutions with values in S^n of the differential equation* (2) *equipped with the topology of compact convergence;*

$$\frac{dY}{dx} = h(x, Y) \tag{2}$$

where $h(x, Y)$ is a function on $E \times R^{n+1}$ to R^{n+1} and the conditions 1), 2), 3), 4), 5) *are fulfilled;*

1) *for any fixed $Y \in R^{n+1}$ h is a Lebesgue measurable function of x on E,*

2) *for any fixed $x \in E$ h is a continuous function of Y on R^{n+1},*

3) *let $P = (0, 0, \cdots, 0, 1)$[20] $\in S^n$ then $h(x, P) = 0$ for any $x \in E$,*

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20) $(y_1, y_2, \cdots, y_n)$ and $(Y_1, Y_2, \cdots, Y_{n+1})$ denote the Euclidean coordinates of $y \in R^n$ and $Y \in R^{n+1}$ respectively.

4) $(Yh)=0$[21] *for any* $(x, Y)\in E\times R^{n+1}$,

5) (Carathéodory's condition), *there exists a non-negative function* $L(x)$ *on* E *such that* $L(x)$ *is Lebesgue integrable on each compact subinterval of* E *and that* $|h(x, Y)|\leqq L(x)$ *for* $(x, Y)\in E\times R^{n+1}$.

Proof. Since E is an interval there is a monotone increasing sequence of compact intervals $\{E_m\}_{m\in N}$ such that $E_m\subset E$ and $\bigcup_{m\in N} E_m=E$. (If E is compact we may take E for every E_m.) By Scorza Dragoni's theorem[22] for any $m\in N$ and any positive δ_m there is a compact subset e_m of E_m such that $m(E_m-e_m)<\delta_m$[23] and $f(x, y)$ is continuous on $e_m\times R^n$. (If f is continuous we may take E_m for e_m.) On the other hand such a non-negative integrable function $M_m(x)$ may be defined on E_m that $|f(x, y)|\leqq M_m(x)$ whenever $|y|\leqq m$ and $x\in E_m$. If δ_m is sufficiently small we have

$$\int_{E_m-e_m} M(x)dx<\frac{1}{2^m}\,.$$

Moreover suppose $\delta_m\to 0$ as $m\to+\infty$ and write $\bigcup_{m\in N} e_m=e$. Then $m(E-e)=0$, for $E-e=\bigcup_{m\in N}(E_m-e)$ where $m(E_m-e)=0$ for all $m\in N$. Now we define a non-negative function $k_1(t)$ on the interval $0\leqq t<+\infty$ by setting $k_1(t)=\max_{x\in e_m} M_m(x)$ for $m-1\leqq t<m (m\in N)$. Then we have $k_1(|y|)=\max_{x\in e_m} M_m(x)\geqq|f(x,y)|$ if $x\in e_m$ and $m-1\leqq |y|<m$. Since $e_1\subset e_2\subset e_3\subset\cdots$, whenever $x\in e_m$ and $m-1\leqq|y|$ we have $|f(x, y)|\leqq k_1(|y|)$. Let $k(t)$ be a positive continuous function of t defined on the interval $0\leqq t<+\infty$ such that $k(t)\geqq\max[1, t, k_1(t)]$. Since $k_1(t)\leqq k(t)$, whenever $x\in e_m$ and $m-1\leqq|y|$ we have $|f(x, y)|\leqq k(|y|)$. If we put $\dfrac{1}{\lambda(r)}=\displaystyle\int_r^{r+1}\frac{dt}{[k(t)]^2}$, $\lambda(r)$ is a positive continuous function of r defined on the interval $0\leqq r<+\infty$. If we set

$$\eta=\lambda(|y|)\cdot y \tag{3}$$

we get a regular topological mapping from R^n onto itself.[24] By (3), x being unchanged, (1) is reduced to $\dfrac{d\eta}{dx}=\lambda(|y|)f(x, y)+$

21) $(Yh)=Y_1h_1+Y_2h_2+\cdots+Y_{n+1}h_{n+1}$.

22) cf. Scorza Dragoni [7], [8].

23) $m(E_m-e_m)$ means the Lebesgue measure of E_m-e_m.

24) cf. Hayashi [5], pp. 314-316.

$\frac{(yf)\lambda'(|y|)}{|y|}y$. If we consider its second member as a function of $(x, \eta) \in E \times R^n$, written $g(x, \eta)$, we obtain the equation

$$(4) \qquad \frac{d\eta}{dx} = g(x, \eta)$$

where the unknown is η. It is clear that $g(x, \eta)$ satisfies Carathéodory's condition locally in $E \times R^n$ and that (3) defines a one-to-one correspondence between the set of all generalized solutions of (1) and that of (4). Since $r \leqq k(r) \leqq \lambda(r)$ and $\lambda'(r) < \frac{[\lambda(r)]^2}{[k(r)]^2}$, whenever $x \in e_m$ and $m-1 \leqq |y|$ we have $|g(x, \eta)| \leqq |f(x, y)| [\lambda(|y|) + |y| \lambda'(|y|)] < 2[\lambda(|y|)]^2$ and $\frac{|g(x, \eta)|}{|\eta|^2} < \frac{2}{|y|^2}$. Hence if $x \in e$ we have

$$(5) \qquad \lim_{\substack{\eta \to \omega \\ \eta \in R^n}} \frac{|g(x, \eta)|}{|\eta|^2} = 0 .$$

Next if $x \in e_m$ and $m-1 \leqq |y|$ we have $\frac{|g(x, \eta)|}{1+|\eta|^2} < 2[\lambda(1)]^2 = \text{const.}$ $(=K)$. And for any $(x, \eta) \in E \times R^n$ we have $\frac{|g(x, \eta)|}{1+|\eta|^2} \leqq K|f(x, y)|$. Now if $m_1 > m$, $m \in N$, $m_1 \in N$, we have

$$\int_{E_m} \max_{|y| \leqq m_1} \frac{|g(x, \eta)|}{1+|\eta|^2} dx^{25)} \leqq \int_{E_m} K\, dx + \int_{E_m} K M_{m-1}(x) dx + \int_{E_m - e_m} K M_m(x) dx$$
$$+ \int_{E_m - e_{m+1}} K M_{m+1}(x) dx + \cdots + \int_{E_m - e_{m_1}} K M_{m_1}(x) dx$$
$$\leqq K[m(E_m) + \int_{E_m} M_{m-1}(x)\, dx + \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{m_1}}] \leqq K\, [m\, (E_m)$$
$$+ \int_{E_m} M_{m-1}(x) dx + 1].$$

If we put $M(x) = \sup_{\eta \in R^n} \frac{|g(x, \eta)|}{1+|\eta|^2} = \lim_{m \to +\infty} \max_{|y| \leqq m} \frac{|g(x, \eta)|}{1+|\eta|^2}$ then $M(x)$ is a non-negative integrable function of x in every E_m such that $\int_{E_m} M(x) dx \leqq K[m(E_m) + \int_{E_m} M_{m-1}(x) dx + 1]$ and that for any $(x, \eta) \in E \times R^n$ we have $\frac{|g(x, \eta)|}{1+|\eta|^2} \leqq M(x)$.

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25) It is readily seen that $\max_{|y| \leqq m_1} \frac{|g(x, \eta)|}{1+|\eta|^2}$ is measurable with respect to x. cf. Cesari [3], footnote 13).

If we consider $R^n(=F)$ as the hyperplane $Y_{n+1}=0$, orthogonal to the vector P, we obtain a homeomorphism of $S^n-\{P\}$ onto R^n by the stereographic projection $\eta=\frac{1}{1-Y_{n+1}}(Y-Y_{n+1}P)$, whose inverse is defined by

$$(6)\qquad Y=\frac{2}{|\eta|^2+1}\eta+\frac{|\eta|^2-1}{|\eta|^2+1}P.$$

Then (4) is transformed to

$$(7)\qquad \frac{dY}{dx}=h(x, Y)$$

where $h_i(x, Y)=\sum_{j=1}^{n}\frac{\partial Y_i}{\partial \eta_j}g_j(x, \eta)$ $(i=1, 2, \cdots, n+1)$ and

$$(8)\qquad (Yh)=0.$$

Since it is clear that for any $x\in e$ we have $\lim_{\substack{Y\to P\\ Y\in S^n-\{P\}}} h(x, Y)=0$, if we set $h(x, P)=0$ for all $x\in E$ then $h(x, Y)$ is a function defined on $E\times S^n$, measurable with respect to x in E for any fixed $Y\in S^n$ and continuous with respect to Y in S^n for any fixed $x\in e$. Since $m(E-e)=0$ we can make $h(x, Y)$ change to a continuous function of Y on S^n for any fixed x in $E-e$ without any ambiguity. We have also $|h(x, Y)|=2\frac{|g(x, \eta)|}{1+|\eta|^2}\leqq 2M(x)$ where $M(x)$ is a non-negative integrable function on any compact subinterval of E.

(**Remark.** If $f(x, y)$ is continuous on $E\times R^n$ $g(x, \eta)$ is also continuous on $E\times R^n$. Since (5) holds uniformly in every $E_m(=e_m)$ $h(x, Y)$ may be defined to be continuous on the whole of $E\times S^n$.)

The product of (3) and (6)

$$(9)\qquad Y=\begin{cases}\dfrac{2\lambda(|y|)}{[\lambda(|y|)]^2|y|^2+1}y+\dfrac{[\lambda(|y|)]^2|y|^2-1}{[\lambda(|y|)]^2|y|^2+1}P & \text{if } y\in R^n,\\ P & \text{if } y=\omega\end{cases}$$

maps topologically F' onto S^n. If we set

$$h(x, Y)=\begin{cases}h\left(x, \dfrac{Y}{|Y|}\right) & \text{if } 1\leqq |Y|,\\ |Y|h\left(x, \dfrac{Y}{|Y|}\right) & \text{if } 0<|Y|<1,\\ 0 & \text{if } Y=0\end{cases}$$

the range of definition of $h(x, Y)$ is extended to $E\times R^{n+1}$ and we have $|h(x, Y)|\leqq 2M(x)$ if $(x, Y)\in E\times R^{n+1}$. Then we obtain the differential equation

$$\frac{dY}{dx}=h(x, Y) \tag{10}$$

which satisfies all the conditions mentioned in the theorem.

From the condition 4) we see that every maximal solution of (10) is defined on the whole of E.

Since F' and S^n are compact the homeomorphism (9) is an isomorphism of the uniform structure of F' onto that of S^n. Therefore (9) is also regarded as a homeomorphism of $\varphi(\boldsymbol{S}_G)$ onto the set of all maximal solutions of (10). q.e.d.

Example. Let $n=1$, $F=R^1$ and $f(x, y)=y^2$ in (1), then we get

$$\frac{dy}{dx}=y^2. \tag{11}$$

By $Y_1=\frac{2y^3}{1+y^6}$, $Y_2=1-\frac{2}{1+y^6}$, (11) is reduced to

$$\begin{cases} \dfrac{dY_1}{dx}=-3Y_2\sqrt[3]{Y_1^2(1+Y_2)}, \\ \dfrac{dY_2}{dx}=3Y_1\sqrt[3]{Y_1^2(1+Y_2)}, \end{cases} \quad \text{where} \quad (x, Y)\in E\times R^2. \tag{12}$$

Then the set of all maximal solutions of (12) with values in the unit circle $Y_1^2+Y_2^2=1$ corresponds to $\varphi(\boldsymbol{S}_G)$ of (11). But by $Z_1=\frac{2y}{1+y^2}$, $Z_2=1-\frac{2}{1+y^2}$, (11) is reduced to

$$\begin{cases} \dfrac{dZ_1}{dx}=-Z_2(1+Z_2), \\ \dfrac{dZ_2}{dx}=Z_1(1+Z_2), \end{cases} \quad \text{where} \quad (x, Z)\in E\times R^2. \tag{13}$$

In this case the function defined by $(Z_1, Z_2)=(0, 1)$ for all $x\in E$, is not a solution. Let $\boldsymbol{S}_0$ be the subset of $\boldsymbol{S}_G$ of (11) whose each element is defined by just two maximal solutions u, v, such that $\overline{A_u\cup A_v}=E$ and $A_u\cap A_v=\phi$. Then the set of all maximal solutions of (13) with values in the unit circle $Z_1^2+Z_2^2=1$ corresponds to $\varphi(\boldsymbol{S}_0)$.

BIBLIOGRAPHY

[1] Bourbaki, N., *Topologie générale*, Paris (Hermann).

[2] Carathéodory, C., *Vorlesungen über reelle Funktionen*, Leipzig-Berlin (Teubner), 1918.

[3] Cesari, L., *Sul teorema di dentità in senso forte*, Ann. R. Sc. Norm. Sup. Pisa, (2) 8 (1939), pp. 301-307.

[4] Hayashi, K., *On the differential equation of Carathéodory's type*, these memoirs 28 (1954), pp. 129-132.

[5] Hayashi, K., *On transformations of differential equations*, these memoirs 28 (1954), pp. 313-325.

[6] Hayashi, K., *On transformations of differential equations whose second members are discontinuous*, these memoirs 29 (1955), pp. 131-137.

[7] Scorza Dragoni, G., *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Padova, 17 (1948), pp. 102-106.

[8] Scoaza Dragoni, G., *Una applicazione della quasicontinuità semiregolare della funzioni misurabili rispetto ad una e continue rispetto ad un'altra variabile*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat., (8) 12 (1952), pp. 55-61.