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On the theory of Henselian rings, III.*

By

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In the paper [1], we defined the notion of Henselizations of normal quasi-local rings and proved generalized Hensel lemma in Henselian valuation rings, and in the paper [2] we proved the properties of Henselizations of normal quasi-local rings and of quasi-local integral domains.

In the present paper, we shall define the Henselization of an arbitrary quasi-local rings and we shall prove that if a Henselian ring $\mathfrak{h}$ dominates a quasi-local ring $\mathfrak{o}$ then there exists one and only one $\mathfrak{o}$-homomorphism from the Henselization of $\mathfrak{o}$ into $\mathfrak{h}$. Besides some other properties of Henselizations, we shall discuss unramifiedness. On the other hand, since the paper [2] contains some errors, corrections to the paper will be given in § 1.

§ 1. Corrections to the paper [2].

(1) In § 4 ([2, Chap. II]), we stated 4 lemmes (Lemmas 4-7). Among them, Lemma 6 is not correct (the others are correct).

What we should prove in § 4 are really as follows:

Let $\mathfrak{o}$ be a normal quasi-local ring with maximal ideal $\mathfrak{p}$ and let $\mathfrak{q}$ be a prime ideal of $\mathfrak{o}$. Let $\bar{\mathfrak{o}}$ be an almost finite separable normal extension of $\mathfrak{o}$ with Galois group G and let $\bar{\mathfrak{p}}$ be a maximal ideal of $\bar{\mathfrak{o}}$. Let $\tilde{\mathfrak{o}}$ be the decomposition ring of $\bar{\mathfrak{p}}$ and set $\tilde{\mathfrak{p}}=\bar{\mathfrak{p}}\cap\tilde{\mathfrak{o}}$, $\mathfrak{o}^*=\tilde{\mathfrak{o}}_{\tilde{\mathfrak{p}}}$. We denote by $\mathfrak{q}^*$ and S an arbitrary prime divisor of $\mathfrak{q}\mathfrak{o}^*$ and the complement of $\mathfrak{q}$ in $\mathfrak{o}$. Then, (i) $\mathfrak{q}^*\cap\mathfrak{o}=\mathfrak{q}$, (ii) $\mathfrak{q}\mathfrak{o}^*_{\mathfrak{q}^*}=\mathfrak{q}^*\mathfrak{o}^*_{\mathfrak{q}^*}$, (iii) $\mathfrak{o}^*_S/\mathfrak{q}\mathfrak{o}^*_S$ *is Noetherian and* (iv) $\mathfrak{q}\mathfrak{o}^*$ *is the intersection of all the* $\mathfrak{q}^*$.

(i) was proved in Lemma 4 (in a more general form) and the

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proof of Lemma 4 is good. If (ii) and (iii) are proved, then the proof of Lemma 7 becomes good and (iv) is proved. Thus we shall prove (ii) and (iii). Let a be an element of $\tilde{\mathfrak{p}}$ which is in none of the other maximal ideals of $\tilde{\mathfrak{o}}$ and let $f(x)$ be the monic polynomial over $\mathfrak{o}$ which has a as a root. Set $\mathfrak{J}=\mathfrak{o}[a]$, $\mathfrak{m}=\tilde{\mathfrak{p}}\cap\mathfrak{J}$. Then Corollary 1 to Theorem 1 in [2] shows that $\mathfrak{o}^*=\mathfrak{J}_{\mathfrak{m}}$. By our choice of a, $f(x)$ modulo $\mathfrak{q}^*$ has (a modulo $\mathfrak{q}^*$) as a simple root. Hence $\mathfrak{o}^*_S/\mathfrak{q}^*\mathfrak{o}^*_S$, which is a field, is a direct summand of $(\mathfrak{o}_{\mathfrak{q}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{q}})[x]/(f(x)$ modulo $\mathfrak{q})=\mathfrak{o}^*_S/\mathfrak{q}\mathfrak{o}^*_S$. Since this is true for any $\mathfrak{q}^*$, we see that $\mathfrak{q}^*_S/\mathfrak{q}\mathfrak{o}^*_S$ is the direct sum of a finite number of fields, which proves (ii) and (iii).

REMARK. We see that $\mathfrak{o}^*/\mathfrak{q}^*$ is separable over $\mathfrak{o}/\mathfrak{q}$ by the above proof.

(2) In §6, III), we stated two lemmas and one corollary to these lemmas. But these lemmas are to be stated under the additional condition that a or b in Lemma A or Lemma B respectively is in the decomposition ring of $\bar{\mathfrak{p}}$. Under this additional assumption, the proofs there work well. (The corollary should be omitted).

(3) In order to derive Corollary 1 to Theorem 1 (in [2]), we used an alternative form of Lemma 2 in [1] without explicit formulation of the lemma. Therefore the corrected Lemma B above (or the alternative form of Lemma 2 along the line of Lemma B) should be stated at the end of §1 or at the beginning of §2.

(4) Among the words added in proof (at the end of [2]), "Lemma 13" should be read as "lemmas stated in the introduction".

§2. Henselizations of arbitrary quasi-local rings.

Let $\mathfrak{o}$ be a quasi-local ring with maximal ideal $\mathfrak{m}$. We shall define the *Henselization* $\mathfrak{o}^*$ of $\mathfrak{o}$ as follows; the uniqueness will be proved later (Theorem 3):

Let R be a normal quasi-local ring with an ideal $\mathfrak{a}$ such that $R/\mathfrak{a}=\mathfrak{o}$ and let R^* be the Henselization of R. Then $\mathfrak{o}^*=R^*/\mathfrak{a}R^*$ is the Henselization of $\mathfrak{o}$.

Until the uniqueness of $\mathfrak{o}^*$ will be proved, we shall fix R so that $\mathfrak{o}^*$ is unique. We denote by $\mathfrak{M}$ the maximal ideal of R.

Theorem 1. *$\mathfrak{o}^*$ is a Henselian ring dominating $\mathfrak{o}$.*

Proof. Since R^* is Henselian, $\mathfrak{o}^*$ is Henselian, because any homomorphic image of a Henselian ring is Henselian. Since R^* dominates R, in order to prove that $\mathfrak{o}^*$ dominates $\mathfrak{o}$, it is sufficient to prove that $\mathfrak{a}R^* \cap R = \mathfrak{a}$. Let b be an arbitrary element of R which is in $\mathfrak{a}R^* \cap R$. Let R' be the integral closure of R in R^* and set $\mathfrak{m}' = \mathfrak{M}R^* \cap R'$. Then there are elements $c_0, \cdots, c_n$ of R' and elements $a_1, \cdots, a_n$ of $\mathfrak{a}$ such that (i) $c_0 \notin \mathfrak{m}'$ and (ii) $c_0 b = \sum c_i a_i$. Let S be a local ring dominated by R such that (i) S is of finitely generated type[1] over the prime integral domain of R and (ii) S contains b and all the a_i and (iii) all the c_i are integral over S. By the finiteness of derived normal rings of local integral domains of finitely generated type (see, for instance [5]) and by the fact each c_i is in a finite quasi-decompositional extension of R with characteristic prime contained in $\mathfrak{m}'$, we can extend S preserving the conditions stated above so that S is normal and that c_i/c_0 are in the Henselization S^* of S dominated by R^*. Then $b \in (\mathfrak{a} \cap S)S^* \cap S$. Since S is a normal Noetherian local ring, S is a dense subspace of S^* and $(\mathfrak{a} \cap S)S^* \cap S = \mathfrak{a} \cap S$, which shows that $b \in \mathfrak{a}$. Thus Theorem 1 is proved.

We have proved that if $\mathfrak{b}$ is an ideal of R, then $\mathfrak{b}R^* \cap R = \mathfrak{b}$. If we apply this fact to the case where $\mathfrak{b}$ contains $\mathfrak{a}$, we have

Corollary 1. *If $\mathfrak{b}$ is an ideal of $\mathfrak{o}$ and if $\mathfrak{o}^*$ is the Henselization of $\mathfrak{o}$, then $\mathfrak{b}\mathfrak{o}^* \cap \mathfrak{o} = \mathfrak{b}$.*

We apply Corollary 1 to the case where $\mathfrak{o}$ is an integral domain and $\mathfrak{b}$ is a principal ideal $b\mathfrak{o}$ ($b \in \mathfrak{o}$). Let K be the field of quotients of $\mathfrak{o}$. Then K can be imbedded in the total quotient ring of $\mathfrak{o}^*$ by Theorem 4 in [2]. $K \cap \mathfrak{o}^*$ contains $\mathfrak{o}$ obviously. If c/b ($b, c \in \mathfrak{o}$) is in $K \cap \mathfrak{o}^*$, then $c\mathfrak{o}^* \subseteq b\mathfrak{o}^*$ and $c\mathfrak{o} \subseteq b\mathfrak{o}$ by the above observation, hence $c/b \in \mathfrak{o}$.

The same observation can be applied even if $\mathfrak{o}$ is not an integral demain. Namely, we take K to be the total quotient ring of $\mathfrak{o}$, proving

Proposition 1. *If a is not a zero divisor in $\mathfrak{o}$, then a is not a zero divisor in $\mathfrak{o}^*$.*

1) We say that a ring R is of finitely generated type over another ring S if R is a ring of quotients of a finitely generated ring over S.

Proof. Assume that $ab=0$ $(b \in \mathfrak{o}^*)$. As in the proof of Theorem 1, we can reduce to the case where R is of finitely generated type over the prime integral domain, then R is Noetherian and therefore $b=0$.

Thus we have

Corollary 2. *If K is the total quotient ring of $\mathfrak{o}$, then $K \cap \mathfrak{o}^* = \mathfrak{o}$. In particular, if $\mathfrak{o}^*$ is a normal ring, then $\mathfrak{o}$ is normal.*

The technique we used for the proofs of Theorem 1 and Proposition 1 gives many results on correspondence between ideals of $\mathfrak{o}$ and of Henselization $\mathfrak{o}^*$ (under certain finiteness condition depending on the assertion), as in the case of Noetherian ring and its completion.

For example,

Proposition 2. *If $\mathfrak{b}$ is an ideal in $\mathfrak{o}$ and if $b \in \mathfrak{o}$, then $\mathfrak{b}\mathfrak{o}^* : b\mathfrak{o}^* = (\mathfrak{b} : b)\mathfrak{o}^*$.*

Proof. We may assume that $\mathfrak{b}=0$, because, by our definition, $\mathfrak{o}^*/\mathfrak{b}\mathfrak{o}^*$ is the Henselization of $\mathfrak{o}/\mathfrak{b}$. Then we can reduce to the Noetherian case and prove the assertion.

REMARK. Proposition 1 can be obtained as a corollary to Proposition 2.

The following can be obtained as a corollary to Proposition 1:

Proposition 3. *A maximal prime divisor of zero in $\mathfrak{o}^*$ lies over that in $\mathfrak{o}$.*

REMARK. Adaptation of the case of completions to the case of Henselizations of Noetherian local rings is rather trivial because of Theorem 5 which will be stated later.

Theorem 2. *If a Henselian ring $\mathfrak{h}$ dominates $\mathfrak{o}$, then there exists one and only one $\mathfrak{o}$-homomorphism ϕ from the Henselization $\mathfrak{o}^*$ of $\mathfrak{o}$ into $\mathfrak{h}$.*[2)]

Proof. Let F be the set of pairs (S, σ) of subrings S and homomorphisms σ such that (1) S is a quasi-local normal ring dominated by R^* and containing R and (2) σ is a homomorphism from S into $\mathfrak{h}$ whose restriction on R coincides with the natural homomorphism ϕ_0 from R onto $\mathfrak{o}$. Let F' be the subset of F

2) We shall understand here that an $\mathfrak{o}$-homomorphism from a ring containing $\mathfrak{o}$ into another ring containing $\mathfrak{o}$ is a ring homomorphism whose restriction on $\mathfrak{o}$ is the identity.

defined by $F'=\{(S, \sigma); (S, \sigma)\in F$ and if $(S, \sigma')\in F$ then $\sigma=\sigma'\}$. Introducing partial order in F as usual, we see that F' is an inductive set. Let (S^*, σ^*) be a maximal member of F'. Assume that $S^*\neq R^*$. Then S^* is not Henselian, therefore there exists a monic polynomial $f(x)=x^r+a_1x^{r-1}+\cdots+a_r$ which is irreducible over S^* such that $a_i\in S^*$, $a_r\in \mathfrak{M}R^*\wedge S^*$, $a_{r-1}\notin \mathfrak{M}R^*\wedge S^*$. Since $\mathfrak{h}$ is Henselian, $\mathfrak{h}$ has a root $\bar{a}$ of $\sigma^*(f(x))$ such that $\bar{a}$ is in the maximal ideal $\mathfrak{n}$ of $\mathfrak{h}$, hence $\sigma^*(f(x))=(x-\bar{a})g^*(x)$ such that $g^*(x)\in\mathfrak{h}[x]$ and $g^*(0)\notin\mathfrak{n}$. By the existense of $\bar{a}$, we can extend σ^* to the homomorphism $\sigma^{*\prime}$ from $S^{*\prime}=S^*[a]_{(\mathfrak{M}R^*\cap S^*[a])}$ (a being the root of $f(x)$ which is in $\mathfrak{M}R^*$) so that $\sigma^{*\prime}(a)=\bar{a}$. Thus $(S^{*\prime}, \sigma^{*\prime})\in F$. By the maximality of (S^*, σ^*) in F', there is a member $(S^{*\prime}, \sigma^{**})$ of F such that $\sigma^{*\prime}\neq\sigma^{**}$. Since $(S^*, \sigma^*)\in F'$, the restriction of σ^{**} on S^* is equal to σ^*. Therefore $\sigma^{**}(a)\neq\bar{a}$. Since $a\in\mathfrak{M}R^*\wedge S^{*\prime}$, $\sigma^{**}(a)$ must be in the maximal ideal of $\mathfrak{h}$, hence $g(\sigma^{**}(a))$ is a unit in $\mathfrak{h}$. Since $f(a)=0$, it follows that $\sigma^{**}(a)-\bar{a}=0$, which is contradiction. Thus $S^*=R^*$. Now the uniqueness of σ^* shows in particular the assertion.

Corollary. *If ϕ_0 is a homomorphism from a quasi-local ring $\mathfrak{o}$ into a Henselian ring $\mathfrak{h}$, then there exists one and only one homomorphism ϕ from the Henselization of $\mathfrak{o}$ into $\mathfrak{h}$, provided that the restriction of ϕ on $\mathfrak{o}$ coincides with ϕ_0.*

Now we prove the uniqueness of the Henselization. Let $\mathfrak{o}^{*\prime}$ be the Henselization of $\mathfrak{o}$ defined by another R. Applying Theorem 2, we see that there are $\mathfrak{o}$-homomorphisms ϕ, ϕ' from $\mathfrak{o}^*$ into $\mathfrak{o}^{*\prime}$ and from $\mathfrak{o}^{*\prime}$ into $\mathfrak{o}^*$ respectively. Consider the product $\phi'\cdot\phi$. This is an $\mathfrak{o}$-homomorphism from $\mathfrak{o}^*$ into $\mathfrak{o}^*$ itself, hence $\phi'\cdot\phi$ is identity by Theorem 2. Similarly, $\phi\cdot\phi'$ is identity. Therefore $\mathfrak{o}^*$ and $\mathfrak{o}^{*\prime}$ is isomorphic. Thus we have proved, by virtue of Theorem 2, the following

Theorem 3. *If $\mathfrak{o}^*$ and $\mathfrak{o}^{*\prime}$ are Henselizations of a given quasi-local ring $\mathfrak{o}$, then $\mathfrak{o}^*$ and $\mathfrak{o}^{*\prime}$ are canonically isomorphic. Furthermore, any $\mathfrak{o}$-homomorphism from $\mathfrak{o}^*$ into $\mathfrak{o}^{*\prime}$ is the canonical isomorphism.*

As a corollary to Theorem 2, we have

Theorem 4. *Let $\mathfrak{o}$ be a quasi-local integral domain such that the derived normal ring of $\mathfrak{o}$ is again quasi-local. If a Henselian ring $\mathfrak{h}$ dominates $\mathfrak{o}$, then $\mathfrak{h}$ contains the Henselization $\mathfrak{o}^*$ of $\mathfrak{o}$ (up to isomorphism).*

Proof. Let ϕ be the $\mathfrak{o}$-homomophism from $\mathfrak{o}^*$ into $\mathfrak{h}$ and let $\mathfrak{a}^*$ be the kernel of ϕ. By Theorem 6 in [2], $\mathfrak{o}^*$ is an integral domain. Therefore, if $\mathfrak{a}^* \neq 0$, then $\mathfrak{a}^* \cap \mathfrak{o} \neq 0$, which is not the case. Therefore $\mathfrak{a}^* = 0$ and ϕ is an isomorphism.

We shall remark the following

Theorem 5. *If $\mathfrak{o}$ is a local ring, then the Henselization $\mathfrak{o}^*$ of $\mathfrak{o}$ is a local ring and $\mathfrak{o}$ is a dense subspace of $\mathfrak{o}^*$. If $\mathfrak{o}$ is a Noetherian local ring, then $\mathfrak{o}^*$ is also Noetherian.*

Proof. Using R and R^* as before, let F be the set of pairs (S, σ) as in the proof of Theorem 2 in the case where $\mathfrak{h} = \mathfrak{o}^*$. F' be the subset of F consisting of all pairs (S, σ) such that $\sigma(S)$ is a local ring containing $\mathfrak{o}$ as a dense subspace. Then F' is an inductive set. Let (S^*, σ^*) be a maximal member in F'. If $S^* \neq R^*$, then by the proof of Theorem 1 in [2] (the first step), we have a contradiction, which proves the first half of the assertion. The last half of the assertion is proved by the same way as the proof of Theorem 3 in [2].

We say that a ring R is of *finite type* over another ring S if R is a ring of quotients of a ring which is a finite module over S. Then the following is easily seen:

Theorem 6. *If a quasi-local ring $\mathfrak{o}'$ is of finite type over another quasi-local ring $\mathfrak{o}$ dominated by $\mathfrak{o}'$, then the Henselization $\mathfrak{o}'^*$ is a finite module over the image of the Henselization of $\mathfrak{o}$ under the canonical homomorphism given by the corollary to Theorem 2.*

(The uniqueness of homomorphisms (Corollary to Theorem 2) is the key of the proof.)

§ 3. Unramifiedness.

There are many notions which are called unramifiedness. We shall consider two of them in the case of finite type extensions.

Let $\mathfrak{o}$ and $\mathfrak{o}'$ be quasi-local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ respectively. Assume that $\mathfrak{o}'$ dominates $\mathfrak{o}$, and is of finite type over $\mathfrak{o}$.

Though we shall restrict ourselves to the case where $\mathfrak{o}'$ is of finite type over $\mathfrak{o}$, the conditions we shall state below can be considered in a more general cases.

Each of the following (U1), (U2) gives *unramifiedness* and (U2) is obviously stronger than (U1):

$\mathfrak{o}'$ is unramified over $\mathfrak{o}$ if and only if:

(U1) $\mathfrak{m}' = \mathfrak{m}\mathfrak{o}'$ and $\mathfrak{o}'/\mathfrak{m}'$ is separable over $\mathfrak{o}/\mathfrak{m}$, or

(U2) The Henselization of $\mathfrak{o}'$ is a (separable) inertia extension of the Henselization of $\mathfrak{o}$.

One convenient property of the unramifiedness in the sense of (U1) lies in the validity of the following criterion, which is called Zariski's criterion of unramifiedness:

Theorem 7. (*Under the assumption that* $\mathfrak{o}'$ *is of finite type over* $\mathfrak{o}$),

(1) *If* $\mathfrak{o}'$ *is a ring of quotients of* $\mathfrak{o}[u]$, *with an element* u *of* $\mathfrak{o}'$ *which is a root of a polynomial* $f(x)$ *over* $\mathfrak{o}$ *such that denoting by* $f'(x)$ *the derivative of* $f(x)$, $f'(u) \notin \mathfrak{m}'$, *then* $\mathfrak{o}'$ *is unramified over* $\mathfrak{o}$.

(2) *Conversely, assume that* $\mathfrak{o}'$ *is unramified over* $\mathfrak{o}$ *and let* $\mathfrak{o}''$ *be a finitely generated subring of* $\mathfrak{o}'$ *over* $\mathfrak{o}$ *such that* (i) $\mathfrak{o}'$ *is a ring of quotients of* $\mathfrak{o}''$ *and* (ii) $\mathfrak{o}''$ *is integral over* $\mathfrak{o}$. *Let* $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ *be the maximal ideals of* $\mathfrak{o}''$, *where* $\mathfrak{o}''_{\mathfrak{m}_1} = \mathfrak{o}'$. *Let* u *be an element of* $\mathfrak{o}''$ *such that* (i) u *modulo* $\mathfrak{m}_1$ *generates the residue class field of* $\mathfrak{o}'$ *over that of* $\mathfrak{o}$ *and* (ii) *denoting by* $f_i(x)$ *a monic polynomial over* $\mathfrak{o}$ *such that* $f_i(x)$ *modulo* $\mathfrak{m}$ *is the irreducible monic polynomial for* u *modulo* $\mathfrak{m}_i$ *over* $\mathfrak{o}/\mathfrak{m}$, $f_1(x)$ *is not congruent to* $f_j(x)$ $(j \neq 1)$ *modulo* $\mathfrak{m}$ (*i.e.*, $f_1(u) \notin \mathfrak{m}_j$). *Then* $\mathfrak{o}'$ *is a ring of quotients of* $\mathfrak{o}[u]$ *and* (ii) u *is a root of a monic polynomial* $f(x)$ *over* $\mathfrak{o}$ *such that, for some natural numbers* $n_2, \cdots, n_r$, $f - f_1 f_2^{n_2} \cdots f_r^{n_r} \in \mathfrak{m}\mathfrak{o}[x]$, *hence in particular, if* $f'(x)$ *is the derivative of* $f(x)$, *then* $f'(u)$ *is not in* $\mathfrak{m}$.

For the proof and references, see [6].

Corollary. *If* $\mathfrak{o}'$ *is unramified over* $\mathfrak{o}$ *in the sense of* (U1) *and if* $\mathfrak{p}'$ *is a prime ideal of* $\mathfrak{o}'$, *then* $\mathfrak{o}'_{\mathfrak{p}'}$ *is unramified over* $\mathfrak{o}_{(\mathfrak{p}' \cap \mathfrak{o})}$ *in the sense of* (U1).

Now we shall consider some cases where these two notions coincide.

Theorem 8. *Assume that the derived normal ring of* $\mathfrak{o}$ *is quasi-local. Then* $\mathfrak{o}'$ *is unramified in the sense of* (U1) (*if and*) *only if it is unramified in the sense of* (U2).

In this case, if $\mathfrak{o}$ *is normal, then* $\mathfrak{o}'$ *is also normal and for any prime ideal* $\mathfrak{p}'$ *of* $\mathfrak{o}'$, $\mathfrak{o}'_{\mathfrak{p}'}$ *is unramified over* $\mathfrak{o}_{(\mathfrak{p}' \cap \mathfrak{o})}$.

Proof. By Theorem 4, the Henselization $\mathfrak{o}'^*$ of $\mathfrak{o}'$ contains the Henselization $\mathfrak{o}^*$ of $\mathfrak{o}$ and $\mathfrak{o}'^*$ is a finite $\mathfrak{o}^*$-module by Theorem 6. Since $\mathfrak{o}'^*$ is Henselian, there exists an inertia extension $\mathfrak{o}^{**}$ of $\mathfrak{o}^*$

whose residue class field coincides with that of $\mathfrak{o}'^{*}$. Therefore, denoting by $\mathfrak{m}^{**}$ the maximal ideal of $\mathfrak{o}^{**}$, we have $\mathfrak{o}'^{*}=\mathfrak{o}^{**}+\mathfrak{m}^{**}\mathfrak{o}'^{*}$, which implies $\mathfrak{o}'^{*}=\mathfrak{o}^{**}$ by Krull-Azumaya's lemma (Corollary to Lemma 1 in [2]). Thus unramifiedness of $\mathfrak{o}'$ in the sense of (U1) implies that of (U2). (The converse is trivial). Now we assume that $\mathfrak{o}$ is normal. The last assertion is a consequence of Corollary to Theorem 7 and what we proved above. The normality of $\mathfrak{o}'$ can be proved by Corollary 2 to Theorem 1.

Now we shall show how the finiteness assumption of $\mathfrak{o}'$ over $\mathfrak{o}$ is important in Theorem 8:

REMARK 1. Even if we assume that $\mathfrak{o}$ is normal, if we only assume that, $\mathfrak{o}'$ is a ring of quotients of an almost finite separable integral extension of $\mathfrak{o}$ instead of assuming to be of finite type, Theorem 8 becomes false. (Observe that in Theorem 8, we did not assume the separability, separability is a consequence of Theorem 7).

This can be seen easily considering suitable non-discrete valuation rings of rank 1.

REMARK 2. Even if we assume that $\mathfrak{o}$ is a discrete valuation ring of rank 1, if we assume only that $\mathfrak{o}'$ is a ring of quotients of an almost finite integral extension of $\mathfrak{o}$ instead of assuming to be of finite type, then Theorem 8 becomes false.

For, there exists a discrete valuation ring $\mathfrak{o}$ such that the completion $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ is an extension of degree p, p being the characteristic of $\mathfrak{o}$, as was given in [1, Appendix (II)].

By the way, we shall give a simple example, which shows that the condition on $\mathfrak{o}$ in Theorem 8 is important, even if we assume that $\mathfrak{o}$ is integral domain and $\mathfrak{o}'$ is separable over $\mathfrak{o}$:

Let P be an ordinary double point of an algebraic curve C and let P' be a point of the derived normal variety of C which corresponds to P. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be local rings of P and P' (over a field k over which P and P' are rational). Then $\mathfrak{o}'$ is unramified over $\mathfrak{o}$ in the sense of (U1). But, since the Henselization of $\mathfrak{o}$ is not an integral domain and since the Henselization of $\mathfrak{o}'$ is a valuation ring, $\mathfrak{o}'$ is not unramified in the sense of (U2).

Theorem 9. *Assume that $\mathfrak{o}$ is a local ring and that $\mathfrak{o}'$ is unramified over $\mathfrak{o}$ in the sense of* (U1). *Then $\mathfrak{o}'$ is unramified in the sense of* (U2) *if and only if $\mathfrak{o}$ is a subspace of $\mathfrak{o}'$.*

Proof. Assume that $\mathfrak{o}$ is a subspace of $\mathfrak{o}'$. Then the completion of $\mathfrak{o}'$ contains the completion of $\mathfrak{o}$. Therefore we see that the Henselization of $\mathfrak{o}'$ contains the Henselization of $\mathfrak{o}$. Therefore we prove the assertion by the same proof as in Theorem 8.

Theorem 10. *Assume that $\mathfrak{o}$ is a Noetherian local ring and that $\mathfrak{o}'$ is unramified over $\mathfrak{o}$ in the sense of* (U1). *Assume furthermore that, for any prime divisor $\mathfrak{p}$ of zero of $\mathfrak{o}$ and for any finite integral extension $\mathfrak{S}$ of $\mathfrak{o}/\mathfrak{p}$, every maximal ideal of $\mathfrak{S}$ has rank equal to rank $\mathfrak{o}$. Then $\mathfrak{o}'$ is unramified in the sense of* (U2) *if and only if for any, or equivalently for a suitable, primary ideal $\mathfrak{q}$ belonging to $\mathfrak{m}$, the multiplicity $e(\mathfrak{q})$ is equal to the multiplicity $e(\mathfrak{q}\mathfrak{o}')$.*

Proof. Let $\mathfrak{o}^*$ and $\mathfrak{o}'^*$ be the Henselizations of $\mathfrak{o}$ and $\mathfrak{o}'$ resepctively and let ϕ be the $\mathfrak{o}$-homomorphism from $\mathfrak{o}^*$ into $\mathfrak{o}'^*$. Then, by the proof of Theorem 8, $\mathfrak{o}'^*$ is an inertia extension of $\phi(\mathfrak{o}^*)$, hence $e(\phi(\mathfrak{q}\mathfrak{o}^*))=e(\mathfrak{q}\mathfrak{o}'^*)$ by the extension formula for multiplicities (see [3]). Since $\mathfrak{o}'$ is a dense subspace of $\mathfrak{o}'^*$, $e(\mathfrak{q}\mathfrak{o}')=e(\mathfrak{q}\mathfrak{o}'^*)$. By the assumption on $\mathfrak{o}$, zero ideal has no imbedded prime divisor in $\mathfrak{o}$, hence in $\mathfrak{o}^*$ as is easily seen by virtue of Proposition 3, and for any prime divisor $\mathfrak{p}^*$ of zero in $\mathfrak{o}^*$, rank $\mathfrak{o}^*/\mathfrak{p}^*=\text{rank}\,\mathfrak{o}$. Therefore, by the additivity of multiplicities (Corollary 1 to Theorem 9 in [3]), we see that $e(\mathfrak{q})=e(\phi(\mathfrak{q}\mathfrak{o}^*))$ if and only if the kernel of ϕ is zero, which proves the assertion.

REMARK. If we omit the assumption on prime divisors of zero of $\mathfrak{o}$, then even if we assume that $\mathfrak{o}$ is an integral domain, Theorem 10 becomes false.

We can get such an example by an example in [4].

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