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Note on coefficient fields of complete local rings*

By

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Let R and R' be complete local rings (which may not be Noetherian) such that R' is integral over R. Then it is known that if the residue class field of R' is separable over that of R then any coefficient ring of R is extendable to that of R'.

The purpose of the present paper is to prove the following

Theorem. *Assume that R contains a field of characteristic $p \neq 0$ and that $R'^p \subseteq R$. Then there is a coefficient field of R which is extendable to that of R'.*

We shall give later some remarks on the case where $R'^{p^n} \subseteq R$.

The following fact was proved by Cohen[1]:

Existence lemma. *Let R' be a complete local ring, with residue class field K', containing a field of characteristic $p \neq 0$. Let $\{\bar{c}_\sigma\}$ be a p-basis of K' and let $\{c_\sigma\}$ be a set of representatives of $\{\bar{c}_\sigma\}$ in R'. Then there exists a coefficient field of R which contains all the c_σ.*

Using this existence lemma, we shall prove the theorem. Let K and K' be the residue class fields of R and R' respectively. Since $R'^p \subseteq R$, we have $K'^p \subseteq K$. Let $\{\bar{c}_\sigma\}$ be a maximal subset of K among those which are p-independent over K'^p and let $\{\bar{c}'_{\sigma'}\}$ be such that $\{\bar{c}_\sigma, \bar{c}'_{\sigma'}\}$ forms a p-basis of K'. Then it is obvious that K is generated by the $\bar{c}_\sigma$ over K'^p. Let k' be a coefficient field of R' containing representatives c_σ of $\bar{c}_\sigma$ in R. Then $k'^p(\{c_\sigma\})$ is a coefficient field of R contained in $k,'$ which proves the theorem.

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1) I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc. vol. 59 (1946), pp. 54-106.

Assume now that R and R' are complete local rings which contain a field of characteristic $p\neq 0$ such that (i) $R\subseteq R'$ and (ii)$R'^{p^n}\subseteq R$ for some n. Then, denoting by K and K' the residue class fields of R and R' respectively:

(1) If there exists a coefficient field of R which is extendable to that of R', then the following must be true; if $\bar{a}^{p^m}\in K$ $(\bar{a}\in K')$, then there exists a representative a of $\bar{a}$ in R' such that $a^{p^m}\in R$.

(2) If there exists a p-basis $\{\bar{c}_\sigma\}$ of K' such that for each σ there is a power $n(\sigma)$ of p such that K is generated by all the $\bar{c}_\sigma^{n(\sigma)}$ over K'^{p^n}, then there exists a coefficient field of R which is extendable to that of R'.

Proof of (1) is nearly trivial, while (2) can be proved by the same idea as in the proof of the theorem.

We shall show by an example that the necessary condition stated in (1) is not sufficient.

Example. Let k be a perfect field of characteristic $p\neq 0$ and let t, u, v, w be algebraically independent elements over k. Let x be an indeterminates and set $R'=k(t, u, v, w)\{x\}$ (formal power series ring). Let R be the subring of R' generated by t, u and tv^p+uw^p+x over R'^{p^2}.

(い) Assume that for an element $\bar{a}$ of the residue class field K' of R' $\bar{a}^{p^m}$ is in the residue class field K of R and we shall show that there is a representative a of $\bar{a}$ such that $a^{p^m}\in R$. If $m=0$, then it is obvious. If $m\geq 2$, then it is also obvious. Hence we consider the case where $m=1$. Since $\bar{a}^p\in K$, we see that $\bar{a}\in K(K'^p)$ by our construction, hence $\bar{a}$ has a representative a in $R[R'^p]$, whence $a^p\in R$.

(ろ) We shall show now that there is no coefficient field of R which is extendable to that of R'. Assume the existence of a coefficient field K^* of R which is extendable to a coefficient field $K^{*\prime}$ of R'. Let t^*, u^*, v^*, w^*, a^* be the representatives of t, u, v, w, tv^p+uw^p+x in $K^{*\prime}$ respectively. Since t, u and tv^p+uw^p+x are in R, t^*-t, u^*-u and $a^*-tv^p-uw^p-x$ are in the maximal ideal of R, hence they are in x^pR'. a^* must be $t^*v^{*p}+u^*w^{*p}$. Thus we have $tv^p+uw^p+x\equiv a^*=t^*v^{*p}+u^*w^{*p}\equiv tv^p+uw^p$ (modulo x^pR'), hence $x\in x^pR'$, which is a contradiction. Thus the non-existence is proved.

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