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Proof that any birational class of non-singular surfaces satisfies the descending chain condition.

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In our monograph "Introduction to the problem of minimal models in the theory of algebraic surfaces" (Publications of the Mathematical Society of Japan, no. 4; this monograph will be referred to as IMM) we have stated the proposition that *each birational class of non-singular varieties satisfies the descending chain condition* (see IMM, Proposition III. 1.3, p. 79), it being understood that the underlying partial ordering of the class is the one in which $V<V'$ if V' dominates V. In the quoted monograph we gave a proof based on the theorem of Neron-Severi. We have also mentioned the existence, in the case of surfaces, of a sheaf-theoretic proof due to Serre (a similar sheaf-theoretic proof has been given recently by Matsumura in an unpublished paper). Finally we have alluded in IMM to a forthcoming note in Mem. Col. Sci. of Kyoto University in which we proposed to prove the above descending chain condition for algebraic surfaces by elementary algebro-geometric considerations, using properties of *exceptional cycles* and the *anticanonical system* $|-K|$. This is the note in which we propose to give this proof.

§ 1. Exceptional cycles of the first kind.

Let F be a non-singular surface (over an algebraically closed

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ground field k) and let E be an exceptional curve of the 1st kind on F (E may be reducible). We shall associate with E a well-defined positive divisorial cycle $\mathcal{E}$ whose components are the irreducible components of E, counted to suitable (positive) multiplicities.

Let P be the (simple) contraction of E and let $\mathfrak{m}_P$ be the maximal ideal of the local ring $\mathfrak{o}_P$ of the point P. If v is any valuation of the function field $k(F)$ of F and if v is non-negative on $\mathfrak{o}_P$ (i.e., if $v(z) \geqq 0$ for all z in $\mathfrak{o}_P$) then from the fact that $\mathfrak{m}_P$ has a finite basis it follows that min $\{v(z), z \in \mathfrak{m}_P\}$ exists. We denote this minimum by $v(\mathfrak{m}_P)$. It is clear that $v(\mathfrak{m}_P) \geqq 0$ and that $v(\mathfrak{m}_P) > 0$ if and only if P is the center of v (on the surface which carries the point P).

Let now $\Gamma_1, \Gamma_2, \cdots, \Gamma_h$ be the irreducible components of E and let v_{Γ_i} be the divisorial (discrete) valuation of $k(F)$ defined by the irreducible curve Γ_i. Since P is the contraction of E, P is the center of v_{Γ_i}, and thus $v_{\Gamma_i}(\mathfrak{m}_P)$ is defined (and is a *positive* integer). We set

$$\mathcal{E} = \sum_{i=1}^{h} v_{\Gamma_i}(\mathfrak{m}_P)\Gamma_i \,, \tag{1}$$

and we refer to $\mathcal{E}$ as the *cycle associated with the exceptional curve* E. We say that a divisorial cycle on F is an *exceptional cycle* (of the first kind) if it is the cycle associated with an exceptional curve E of the first kind. If P is the (simple) contraction of E we shall also refer to P as the contraction of the exceptional cycle $\mathcal{E}$.

Proposition 1. *If E is an irreducible exceptional curve then the exceptional cycle $\mathcal{E}$ associated with E is E itself.*

Proof. If E is irreducible then v_E is the principal P-adic divisor (IMM, p. 55 and Corollary II. 3. 2, p. 56), and hence $v_E(\mathfrak{m}_P) = 1$.

If X is any divisorial cycle on F we denote by $\langle X \rangle$ the *support* of X, i.e., the curve whose irreducible components are the prime components of X.

Proposition 2. *Let $\mathcal{E}$ be an exceptional cycle, let P be the contraction of $\mathcal{E}$ and let Q be a point of the support $\langle \mathcal{E} \rangle$ of $\mathcal{E}$. Then the ideal $\mathfrak{o}_Q\mathfrak{m}_P$ is principal, and if g is a generator of this ideal then $g = 0$ is a local equation of $\mathcal{E}$ at Q.*

Proof. Let x and y be regular parameters of the local ring $\mathfrak{o}_P$. Since $Q > P$ and $Q \neq P$, it follows that either y/x or x/y belongs $\mathfrak{o}_Q$ (IMM, Theorem II. 1. 2, p. 46). If, say $y/x \in \mathfrak{o}_Q$ then $\mathfrak{o}_Q \cdot \mathfrak{m}_P = \mathfrak{o}_Q \cdot x$. An irreducible curve Γ on F is a component of E if and only if $v_\Gamma(\mathfrak{m}_P) > 0$, hence (and assuming furthermore that $Q \in \Gamma$) if and only if $v_\Gamma(x) > 0$; and for any such curve Γ we have $v_\Gamma(\mathfrak{m}_P) = v_\Gamma(x)$. Hence $x = 0$ is a local equation of $\mathfrak{E}$ at Q. QED.

Let $T : G \to F$ be an antiregular birational transformation of a non-singular surface G onto a non-singular surface F. Let P be a fundamental point of T (on G). If $E = T\{P\}$ is the total T-transform of P (whence E is an exceptional curve on F, with contraction P), we denote by $T(P)$ the exceptional cycle $\mathfrak{E}$ associated with E.

Proposition 3. *Let $T_1 : H \to G$ and $T_2 : G \to F$ be antiregular birational transformations, the surfaces H, G, F being non-singular, and let $T = T_1 T_2 :$ $H \to F$. If P is a fundamental point of T_1 then $T(P) = T_2(T_1(P))$ [here $T_2(T_1(P))$ denotes the T_2-transform of the divisorial cycle $T_1(P)$].*

Proof. Since $T_1(P)$ is a positive cycle, the support of the T_2-transform of $T_1(P)$ coincides with the total T_2-transform of the support of $T_1(P)$ (IMM, Proposition II. 5. 1, p. 69). Hence $T(P)$ and $T_2(T_1(P))$ have the same support. Let now R be any point of $\langle T(P) \rangle$, let Q be the point of $\langle T_1(P) \rangle$ which corresponds to R, let x, y be uniformizing parameters at P and let, say, $y/x \in \mathfrak{o}_Q$. By Proposition 2, $x = 0$ is a local equation of $T(P)$ at R. By the same proposition, $x = 0$ is also a local equation of $T_1(P)$ at Q, and hence, by the definition of the T_2-transform of a cycle on G (IMM, Definition II. 5. 2, p. 70), $x = 0$ is also the local equation of $T_2(T_1(P))$ at R. Thus $T(P)$ and $T_2(T_1(P))$ have the same local equation at each point R of their common support. QED.

If P and Q are points of birationally equivalent surfaces, we write $P < Q$ if $\mathfrak{o}_P$ is a *proper* subring of $\mathfrak{o}_Q$; and if X and Y are two divisorial cycles on a surface F we write $X < Y$ if Y-X is a *strictly positive* cycle. In the latter case we say that X is a *proper sub-cycle* of Y.

Proposition 4. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be exceptional cycles of the 1st kind on a non-singular surface F and let P_1, P_2 be their contractions. Then the following relations are equivalent.*

(a) $\mathcal{E}_1 < \mathcal{E}_2$;
(b) $\langle \mathcal{E}_1 \rangle < \langle \mathcal{E}_2 \rangle$;
(c) $P_1 > P_2$.

Proof. If (a) is satisfied, then clearly $\langle \mathcal{E}_1 \rangle \subset \langle \mathcal{E}_2 \rangle$, and we cannot have $\langle \mathcal{E}_1 \rangle = \langle \mathcal{E}_2 \rangle$ since any exceptional curve determines uniquely the exceptional cycle associated with it. Thus (a) implies (b).

That (b) and (c) are equivalent has been proved in IMM (Lemma II. 3. 7, p. 58).

Now assume (c). If Γ is any prime component of $\mathcal{E}_1$ then P_1 is a center of v_Γ, and hence also P_2 is a center of v_Γ. Furthermore, we have $v_\Gamma(\mathfrak{m}_{P_1}) \leq v_\Gamma(\mathfrak{m}_{P_2})$ since $\mathfrak{m}_{P_2} \subset \mathfrak{m}_{P_1}$. This shows that $\mathcal{E}_1 \leqq \mathcal{E}_2$, and since equality is clearly impossible, the proof is complete.

With the notations of Proposition 4 we say that $\mathcal{E}_1$ is a *maximal exceptional sub-cycle of* $\mathcal{E}_2$ if $\mathcal{E}_1 < \mathcal{E}_2$ and if there exist no exceptional cycles $\mathcal{E}$ such that $\mathcal{E}_1 < \mathcal{E} < \mathcal{E}_2$.

Corollary 4. 1. *$\mathcal{E}_1$ is a maximal exceptional sub-cycle of $\mathcal{E}_2$ if and only if P_1 is a quadratic transform of P_2.*

This follows from Proposition 4 and Theorem II. 1. 1 of IMM, p. 44.

Corollary 4. 2. *Let $T: H \rightarrow F$ be an antiregular birational transformation of an non-singular surface H onto a non-singular surface F, let P be a fundamental point of T and let $\mathcal{E} = T(P)$. Let $T_1: H \rightarrow G$ be a locally quadratic transformation of H, with center P, let E_0 be the (irreducible) curve $T_1\{P\}$ on G, and let $T_2: G \rightarrow F$ be the antiregular birational transformation of G such that $T_1T_2 = T$. If $\mathcal{E}$ is not a prime cycle (or equivalently: if T_2 has fundamental points on E_0) and if $P_1', P_2', \cdots, P_g'$ denote the fundamental points of T_2 on E_0, then the g exceptional cycles $T_2(P_i')$ are the only maximal exceptional sub-cycles of $\mathcal{E}$.*

Obvious.

Proposition 5. *If $\mathcal{E}$ is an exceptional cycle on a non-singular surface F then $p(\mathcal{E}) = 0$ and $(\mathcal{E}^2) = -1$. If Γ is any prime component of $\mathcal{E}$, different from the principal component of $\langle \mathcal{E} \rangle$, then $(\mathcal{E} \cdot \Gamma) = 0$.*

Proof. Let P be the contraction of $\mathcal{E}$. There exists a non-singular surface H which carries the point P and such that $\mathcal{E} = T(P)$, where $T: H \rightarrow F$ is an anti-regular birational transformation of H

onto F (for instance, take $H=F-\langle\mathcal{E}\rangle+P$). Let $T_1: H\rightarrow G$ and $T_2: G\rightarrow F$ have the same meaning as in Corollary 4.2. Using the notations of that corollary, we have, by Proposition 3: $\mathcal{E}=T_2(E_0)$. Since $p(E_0)=0$ and $(E_0^2)=-1$ (E_0 being an *irreducible* exceptional curve of the first kind) and since anti-regular transformations preserve the arithmetic genus and the self-intersection number of any divisorial cycle, it follows that also $p(\mathcal{E})=0$ and $(\mathcal{E}^2)=-1$.

To prove the second part of the proposition we fix some *proper* exceptional sub-cycle $\mathcal{E}_1$ of $\mathcal{E}$ such that Γ is a component of $\mathcal{E}_1$ (the existence of $\mathcal{E}_1$ follows from IMM, Proposition II.3.3, p. 57). We replace in the preceding part of the proof $\mathcal{E}$ by $\mathcal{E}_1$. Let P', H', T' have the same meaning in relation to $\mathcal{E}_1$ as P, H and T had in relation to $\mathcal{E}$. Since $\mathcal{E}_1$ is a proper exceptional sub-cycle of $\mathcal{E}$, we have $H<H'$ (assuming, as we may, that $H=F-\langle\mathcal{E}\rangle+P$, $H'=F-\langle\mathcal{E}_1\rangle+P'$). Let $\mathcal{E}'$ be the exceptional cycle on H' which is the transform of the point P. By Proposition 3 (as applied to the surfaces H, H', F) we have $\mathcal{E}=T'(\mathcal{E}')$. From Proposition II.5.4 of IMM, p. 71, it now follows directly that $(\mathcal{E}.\Gamma)=0$.

Corollary 5.1. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are distinct exceptional sub-cycles of $\mathcal{E}$ then $(\mathcal{E}_1\cdot\mathcal{E}_2)=0$.*

By Proposition 5 it is sufficient to consider the case in which both $\mathcal{E}_1$ and $\mathcal{E}_2$ are proper sub-cycles of $\mathcal{E}$, for if, say, $\mathcal{E}_1=\mathcal{E}$ then $\mathcal{E}_2$ is a proper exceptional sub-cycle of $\mathcal{E}$ and therefore no prime component of $\mathcal{E}_2$ is the principal component of $\mathcal{E}$ (IMM, Corollary II.3.8, p. 58). Since the corollary is vacuous if $\langle\mathcal{E}\rangle$ is irreducible, we use induction with respect to the number of prime components of $\mathcal{E}$. We use the notations of the proof of Proposition 5 and we denote by $P_1', P_2' \cdots, P_g'$ the fundamental points of T_2 on E_0. By Corollary 4.1, each of the exceptional cycles $\mathcal{E}_1, \mathcal{E}_2$ is a sub-cycle of one of the exceptional cycles $T_2(P_i')$. Let, say $\mathcal{E}_1$ be a sub-cycle of $T_2(P_\alpha')$ and $\mathcal{E}_2$ a sub-cycle of $T_2(P_\beta')$. If $\alpha\neq\beta$, then $\langle T_2(P_\alpha')\rangle$ and $\langle T_2(P_\beta')\rangle$ have no common points, and the relation $(\mathcal{E}_1\cdot\mathcal{E}_2)=0$ is proved. If $\alpha=\beta$, then we observe that the number of prime components of $T_2(P_\alpha')$ is less than that of $\mathcal{E}$, and hence $(\mathcal{E}_1\cdot\mathcal{E}_2)=0$, by our induction hypothesis.

If $T: H\rightarrow F$ is an antiregular birational transformation of a non-singular surface H onto a non-singular surface F and if $X=\sum_{i=1}^{q} m_i\Gamma_i$ is any divisorial cycle on H whose distinct prime components are $\Gamma_1, \Gamma_2, \cdots, \Gamma_q$, then we denote by $T[X]$ the divisorial cycle $\sum_{i=1}^{q} m_i T[\Gamma_i]$, where $T[\Gamma_i]$ denotes the proper T-transform of Γ_i. This cycle $T[X]$ does not, in general, coincide with the T-transform $T(X)$ of X as defined in IMM, p. 70.

Proposition 6. *Let $T: H \rightarrow F$ be an anti-regular birational transformation of a non-singular surface H onto a non-singular surface F, let $P_1, P_2, \cdots, P_h$ be the fundamental points of T (on H) and let $\mathcal{E}_i = T(P_i)$, $i=1, 2, \cdots, h$. If $\{\mathcal{E}_{i,1}, \mathcal{E}_{i,2}, \cdots, \mathcal{E}_{i,s_i}\}$ is the set of all proper exceptional sub-cycles of $\mathcal{E}_i$, then for any divisorial cycle X on H we have*

$$T(X) = T[X] + \sum_{i=1}^{h} \lambda_i \mathcal{E}_i + \sum_{i=1}^{h} \sum_{j_i=1}^{s_i} \lambda_{i,j_i} \mathcal{E}_{i,j_i},$$

where the coefficients λ_i, λ_{i,j_i} are integers and where λ_i is the multiplicity of X at P_i. If $X>0$ then the λ_i, λ_{i,j_i} are non-negative.

Proof. It is obviously sufficient to prove the proposition under the assumption that $h=1$. The transformation T has in that case only one fnndamental point P. We set $\mathcal{E}=T(P)$ and we let $\{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_3\}$ be the set of all proper exceptional sub-cycles of $\mathcal{E}$. By IMM, Proposition II. 5. 8 (p. 73) the proposition is true if T is a locally quadratic transformation. We shall therefore use induction with respect to the number of prime components of $\mathcal{E}$. We use the notations of Corollary 4. 2. We have, by IMM, Proposition II. 5. 8 (p. 73),

$$T_1(X) = T_1[X] + \lambda E_0,$$

where λ is the multiplicity of X at P, and hence, by Proposition 3,

$$T(X) = T_2(T_1(X)) = T_2(T_1[X]) + \lambda\mathcal{E}.$$

The fundamental points of T_2 are the points $P_1', P_2', \cdots, P_g'$, and their T_2-transforms represent all the maximal exceptional sub-cycles of $\mathcal{E}$ (Corollary 4. 2). Hence, by our induction hypothesis, the proposition is applicable to T_2 and to any divisorial cycle on G (and, in particular, to the cycle $T_1[X]$). Since $T_2[T_1[X]] = T[X]$, the proposition is proved.

Corollary 6. 1. *Let $\mathcal{E}$ be an exceptional cycle of the first kind on a non-singular surface and let E_1 be the principal component of $\mathcal{E}$. If $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_s$ are the proper exceptional sub-cycles of $\mathcal{E}$, then*

$$\mathcal{E} = E_1 + \sum_{i=1}^{s} \lambda_i \mathcal{E}_i,$$

where the λ_i are non-negative integers and λ_i is positive if $\mathcal{E}_i$ is a maximal exceptional sub-cycle of $\mathcal{E}$.

In the notation of the proof of Proposition 6 we have $\mathcal{E}=T(P)=T_2(E_0)$, and the corollary follows by applying Proposition 6 to the transformation T_2 and the cycle E_0.

§ 2. The anticanonical system and exceptional cycles

Proposition 1. *If K is a canonical divisor on a non-singular surface F and $\mathcal{E}$ is an exceptional cycle on F, of the first kind, then $(K\cdot\mathcal{E})=-1$.*

Proof. The proposition follows directly from Proposition 5, § 1, in view of the equality $(K\cdot X)=2p(X)-2-(X^2)$ which holds for any divisorial cycle X on F.

Proposition 2. *If an anti-regular birational transformation $T: F'\rightarrow F$ of a non-singular surface F' onto a non-singular surface F is a product of n quadratic transformations then the dimension of the anticanonical system $|-K'|$ on F' satisfies the inequality*

$$\dim|-K'|\geqq n+(K^2)+P_a\,;$$

where K is a canonical divisor on F and where P_a is the arithmetic genus of F' (and of F).

Proof. Assume first that $n=1$. Let P' be the center of the locally quadratic transformation T and let $E=T\{P'\}$. Then it is known (see IMM, Proposition II. 5. 6, p. 72) that $T(K')+E$ is a canonical divisor K on F. Hence $(K^2)=(K'^2)-1$, since $(T(X')\cdot E)=0$ for any divisorial cycle X' of F' (Proposition II. 5. 5, IMM, p. 71) and since $(E^2)=-1$. By induction with respect to n we find that if T is a product of n quadratic transformations then $(K'^2)=(K^2)+n$. Since $p(-K')=1$ our proposition follows from the Riemann-Roch theorem on F'.

Theorem 1. *On a non-singular surface F there cannot exist an infinite strictly ascending chain $\mathcal{E}_1<\mathcal{E}_2<\cdots<\mathcal{E}_n<\cdots$ of exceptional cycles of the first kind.*

Proof. We shall assume that such a chain exists and we shall show that this assumption leads to a contradiction. Let $F_i=(F-\mathcal{E}_i)+P_i$, where P_i is the contraction of $\mathcal{E}_i$. Then F_i is a non-singular surface and we have $F>F_1>F_2>\cdots>F_n>\cdots$. We also have $P_1 > P_2 > \cdots > P_n > \cdots$, and each F_n carries an infinite strictly ascending chain of exceptional cycles of the first kind: namely, the cycles on F_n which are the transforms of the points $P_{n+1}, P_{n+2}, \cdots$ form such a chain. We therefore may replace in our proof the surface F by any of the surfaces F_n. Since the anti-regular birational transformation of F_n onto F is the product of at least n locally quadratic transformations, the dimension of the anticanonical system $|-K_n|$ on F_n satisfies the inequality: $\dim|-K_n| \geqq n + (K^2) + P_a$, where K is a canonical divisor on F and P_a is the arithmetic genus of F. Thus $\dim|-K_n| \geqq 1$ if n is sufficiently large, and *we may therefore assume* that $\dim|-K| \geqq 1$.

Let E_i be the principal component of $\mathscr{E}_i$. Then E_i is not a component of $\mathscr{E}_j$, $j < i$ (IMM, Corollary II. 3. 8, p. 58), and hence the irreducible curves $E_1, E_2, \cdots, E_n, \cdots$ are distinct.

By Corollary 6. 1, § 1, we have that $\mathscr{E}_i$ is the sum of E_i and a certain number ν_i of exceptional cycles of the first kind. Here $\nu_i \geqq 1$ except if $\mathscr{E}_i$ is a prime cycle (which can happen only for $i = 1$). Hence we may assume that $\nu_i \geqq 1$ for all i. By Proposition 1 it follows that

$$(1) \qquad (-K \cdot E_i) = 1 - \nu_i \leqq 0 .$$

Let N be an integer such that *no* E_i, $i \geqq N$, *is a fixed component of the linear system* $|-K|$. Then $(-K \cdot E_i) \geqq 0$ if $i \geqq N$, and hence by (1) we conclude that

$$(2) \qquad (-K \cdot E_i) = 0 , \qquad i \geqq N .$$

This shows that each E_i, $i \geqq N$, is a fundamental curve of $|-K|$, i.e., that the cycles in $|-K|$ which have E_i as component form a (linear) subsystem L_i of $|-K|$ the dimension of which is one less than the dimension of $|-K|$. Thus $|-K|$ has infinitely many fundamental curves. This implies that the rational transformation of F which is defined by the linear system $|-K|$ (IMM, p. 10) is necessarily a curve. In other words, if we denote by B the fixed cycle of $|-K|$, then the linear system obtained by deleting B from the members of $|-K|$ is composite with some irreducible pencil H. Since H contains at most a finite number of cycles which are not prime and since each E_i, $i \geqq N$, is a component of some member of H, it follows that some E_i is a member of H (actually, all but a finite number of the E_i must be members of

H). However, *we now show that* (E_i^2) *is negative* and therefore no E_i can be a member of a pencil. This contradiction will complete the proof.

Let, then, quite generally, $\mathcal{E}$ be an exceptional cycle of the first kind and let E_1 be the principal component of $\mathcal{E}$. We have then, by Corollary 6.1, §1:

$$(3) \qquad \mathcal{E} = E_1 + \sum_{i=1}^{s} \lambda_i \mathcal{E}_i ,$$

where the λ_i are non-negative integers and the $\mathcal{E}_i$ are proper exceptional sub-cycles of $\mathcal{E}$. Since $(\mathcal{E}\cdot\mathcal{E}_i)=0$ (Corollary 5.1, §1) and $(\mathcal{E}^2)=-1$, it follows from (3) that

$$(4) \qquad (\mathcal{E}\cdot E_1) = -1 .$$

For a fixed j, we intersect both sides of (3) with $\mathcal{E}_j$, and we note that $(\mathcal{E}_i\cdot\mathcal{E}_j)=0$ if $i \neq j$ (Corollary 5.1, §1). We thus obtain

$$(5) \qquad (\mathcal{E}_j\cdot E_1) = \lambda_j .$$

Intersecting both sides of (3) with E_1 we find in view of (4) and (5): $(E_1^2)=-1-\sum_{i=1}^{s}\lambda_i<0$. This completes the proof of the theorem.

Remark. After the relation (2) has been obtained, the rest of the proof admits another variation. From (1) and (2) it follows that $\nu_i=1$ if $i\geqq N$. It follows therefore from Corollary 6.1, §1, that each $\mathcal{E}_i$, $i\geqq N$, has only one maximal exceptional sub-cycle, say $\mathcal{E}_i'$, and that $\mathcal{E}_i=E_i+\mathcal{E}_i'$. By a refinement of the original sequence $\mathcal{E}_1<\mathcal{E}_2<\cdots$ we may arrange matters so that each $\mathcal{E}_i$ is a maximal exceptional sub-cycle of its successor $\mathcal{E}_{i+1}$. Then $\mathcal{E}_{i+1}'=\mathcal{E}_i$. We have then $0=(\mathcal{E}_i\cdot\mathcal{E}_{i+1})=(E_i\cdot E_{i+1})+(E_i\cdot\mathcal{E}_i)+(\mathcal{E}_{i-1}\cdot\mathcal{E}_{i+1})=(E_i\cdot E_{i+1})+(E_i\cdot\mathcal{E}_i)$. By relation (4), applied to $\mathcal{E}=\mathcal{E}_i$, we have $(E_i\cdot\mathcal{E}_i)=-1$. Hence $(E_i\cdot E_{i+1})=1$. Now let $i\geqq N$ and let L_i be the above sub-system of $|-K|$ whose members contain E_i as component. If D is any member of L_i, $D=E_i+D_i$, then $0=(D\cdot E_{i+1})=1+(D_i\cdot E_{i+1})$, i.e., $(D_i\cdot E_{i+1})=-1$. Hence E_{i+1} is a component of D_i, and if we set $D_i=E_{i+1}+D_{i+1}$, then from $(E_i\cdot E_{i+2})\geqq 0$, $(E_{i+1}\cdot E_{i+2})=1$ and $(D\cdot E_{i+2})=0$ follows at once that $(D_{i+1}\cdot E_{i+2})<0$ and that consequently E_{i+2} is a component of D_{i+1}. Proceeding in this fashion we see that all the curves $E_i, E_{i+1}, \cdots$ are components of D, and this is absurd.

§ 3. The descending chain condition in the birational class of F

We now come to our main object, i.e. to the proof of the following theorem:

Every strictly descending chain $F > F_1 > F_2 > \cdots$ of birationally equivalent non-singular surfaces is necessarily finite.

Proof. We shall assume that there exists an infinite strictly descending chain

$$F > F_1 > F_2 > \cdots > F_n > \cdots$$

of non-singular surfaces (each F_i dominating its successor F_{i+1}) and we shall show that this assumption leads to a contradiction.

We fix on each F_n ($n \geqq 1$) a fundamental point P_n of the antiregular birational transformation of F_n onto F_{n-1} and we denote by $\mathscr{E}_n$ the exceptional cycle of the first kind on F which corresponds to the point P_n in the antiregular birational transformation of F_n onto F. By Theorem 1, § 2, the infinite set $\{\mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_n, \cdots\}$ contains an infinite subset $\{\mathscr{E}_{i_1}, \mathscr{E}_{i_2}, \cdots, \mathscr{E}_{i_n}, \cdots\}$ consisting of maximal elements of the set. I assert that $\langle\mathscr{E}_{i_\alpha}\rangle \cap \langle\mathscr{E}_{i_\beta}\rangle = \emptyset$ *if* $\alpha \neq \beta$. For assume the contrary and let Q be a common point of $\langle\mathscr{E}_{i_\alpha}\rangle$ and $\langle\mathscr{E}_{i_\beta}\rangle$. Then Q corresponds to both points $P_{i_\alpha}, P_{i_\beta}$, and since $Q > P_{i_\alpha}$ and $Q > P_{i_\beta}$ it follows that P_{i_α} and P_{i_β} are corresponding points in the birational transformation between F_{i_α} and F_{i_β}. If, say, $\alpha < \beta$, then it follows that $P_{i_\alpha} > P_{i_\beta}$, whence $\mathscr{E}_{i_\alpha} < \mathscr{E}_{i_\beta}$, which is impossible. This proves our above assertion.

Any *minimal* exceptional sub-cycle of an exceptional cycle of the first kind is a prime cycle. We fix a minimal exceptional sub-cycle E_α of $\mathscr{E}_{i_\alpha}$, for each α. Then the E_α are *irreducible* exceptional curves of the first kind, and

$$E_\alpha \cap E_\beta = \emptyset \,. \tag{1}$$

We may assume that the anticanonical system $|-K|$ on F has dimension $\geqq 2$ (see § 2). We fix a linear subsystem L of $|-K|$ which has dimension 2. If D is any member of L we have

$$(D \cdot E_\alpha) = 1 \,, \qquad \text{all } \alpha \,. \tag{2}$$

Let B be the fixed cycle of L (if such a cycle exists) and let $L_1 = L - B$. If B meets a given E_α then $(D_1 \cdot E_\alpha) = 0$ for any D_1 in L_1 in view of (2), and thus E_α is a fundamental curve of L_1.

The assumption that L has infinitely many fundamental curves E_α would lead to the same contradiction as was reached in the proof of Theorem 1 (in view of $(E_\alpha^2)=-1$). Hence B meets at most a finite number of E_α. Omitting if necessary a finite number of the E_α we may therefore assume that B meets no E_α and that consequently $(D_1 \cdot E_\alpha)=1$ for all α. We replace L by L_1, and we may therefore assume that L has no fixed components without violating (2), and also that no E_α is fundamental for L. Since $\dim L=2$, it follows from (2) that for each α there exists one and only one cycle D_α in L such that E_α is a component of D_α. This cycle D_α cannot be E_α itself since (E_α^2) is negative. Hence D_α is not a prime cycle. Let M/k be the smallest algebraic sub-system of L/k which contains all the cycles D_α and let N be an irreducible component of M/k such that N contains infinitely many of the cycles D_α. Then it is clear that if D^* is a general member of N/k (the coördinates of the Chow point of D^* belonging to a universal domain), infinitely many of the curves E_α (regarded as cycles) will be specializations, over k, of one and the same prime component of D^*. Since this is a contradiction with the fact that the E_α have a negative self-intersection number, the proof of the theorem is complete.

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