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On the strong stability and boundedness of solutions of ordinary differential equations

By

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In recent years many authors[1] have studied the problem of determining gauge functions, that is, Lyapunov functions for various types of stability and boundedness of solutions of ordinary differential equations. In this paper we shall show that the function $D(P, Q)$, introduced by H. Okamura[2] in connection with the uniqueness problem in the theory of ordinary differential equations (cf. Definition 1), will work as the above mentioned gauge function.

In §1 we shall define the Okamura function $D(P, Q)$. In §2 we shall obtain a necessary and sufficient condition for the trivial solution $x=0$ of the differential equation (1) to be strongly stable[3] in terms of the Okamura function. It should be noted that the Okamura function can be determined concretely by the given differential equation itself, though it may not be easy. In §3 we shall prove a regularization theorem which will connect our condition in §2 with that of well-known form. In §4 we shall discuss the strong boundedness problem by the same idea as for the strong stability in §2.

1. Okamura function.

In §1, §2 and §3 we consider the differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

1) cf., for instant, Antosiewicz [1].
2) cf. Okamura [6], [7] and [8].
3) cf. Okamura [9] and cf. Yoshizawa [10].

where t is real and x a real n-dimensional vector and where $f(t, x)$ is a continuous function on $E\times F$ to R^n, where $E=\{t: 0\leqq t<+\infty\}$ and $F=\{x: |x|\leqq b\}$[4]. Suppose also $f(t, 0)=0$ for $t\in E$.

Let $P=(t_P, x_P)$ and $Q=(t_Q, x_Q)$ be two points in $E\times F$. When $t_P<t_Q$ we denote by $\boldsymbol{U}_{PQ}$ the set of all the absolutely continuous functions $u(t)$ on the interval $t_P\leqq t\leqq t_Q$ to F such that $u(t_P)=x_P$ and $u(t_Q)=x_Q$.

Definition 1[5]. *Put*

$$(2)\qquad D(P, Q)=\begin{cases}\inf\limits_{u\in \boldsymbol{U}_{PQ}}\displaystyle\int_{t_Q}^{t_P}|u'(t)-f(t, u(t))|\,dt & \text{if } t_P<t_Q,\\ D(Q, P) & \text{if } t_P>t_Q,\\ |x_Q-x_P| & \text{if } t_P=t_Q.\end{cases}$$

$D(P, Q)$ is called the Okamura function with respect to the differential equation (1).

It is easily seen that if P and Q are on a solution of (1) we have $D(P, Q)=0$ and if not $D(P, Q)>0$. We have also

$$|D(P, Q)-D(P, R)|\leqq\left|\int_{t_Q}^{t_R}M(\tau)\,d\tau\right|+|x_Q-x_R|$$

where $P=(t_P, x_P)$, $Q=(t_Q, x_Q)$, $R=(t_R, x_R)\in E\times F$ and where $M(t)=\max\limits_{x\in F}|f(t, x)|$. If $t_P\leqq t_Q\leqq t_R$ we have

$$(3)\qquad D(P, R)\leqq D(P, Q)+D(Q, R)$$

so that, if Q and R are on a solution of (1), (3) is reduced to

$$(4)\qquad D(P, R)\leqq D(P, Q).$$

Let O be the point $(0, 0)$ and P a variable point (t, x) in $E\times F$. Then we can define a function of (t, x) by

$$(5)\qquad U(t, x)=D(O, P).$$

Since $x=0$ is a solution of (1) we have for all $t\in E$

$$(6)\qquad U(t, 0)=0.$$

We have also for $(t, x), (t', x')\in E\times F$

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4) $x=(x_1, x_2, \cdots, x_n)$ and $|x|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$.

5) cf. Hayashi and Yoshizaw [2] and cf. Hayashi [3].

$$(7) \qquad |U(t', x')-U(t, x)| \leqq \left| \int_t^{t'} M(\tau)d\tau \right| + |x'-x|.$$

Since $0 \leqq U(t, x) \leqq U(t, 0) + |U(t, x)-U(t, 0)| \leqq |x| \leqq b$ $U(t, x)$ is a non-negative bounded continuous function on $E \times F$. Let $u(t)$ be a solution of (1) then by (4) $U(t, u(t))$ is a non-increasing function of t.

2. Strong stability.

Definition 2. *Given a positive $\varepsilon > 0$ and a point $P=(t_0, x_0) \in E \times F$, a function $u(t)$, defined on a subinterval $I(\ni t_0)$ of the interval $t_0 \leqq t < +\infty$ with values in F, which is absolutely continuous on each compact subinterval of I*[6] *is said to be an ε-solution of* (1) *starting at P if $u(t_0)=x_0$ and*

$$(8) \qquad \int_I |u'(t)-f(t, u(t))|dt < \varepsilon.$$

Clearly for any $\varepsilon > 0$ every solution of (1) is an ε-solution.

Definition 3. (Strong stability in the sense of Okamura.) *The solution $x=0$ of* (1) *is said to be strongly stable if for any $\eta > 0$ there exist an $\varepsilon > 0$ and a $\delta > 0$ such that for any ε-solution $u(t)$ starting at $(0, x_0)$ provided $|x_0| < \delta$ we have $|u(t)| < \eta$ in the whole of the interval on which $u(t)$ is defined.*

Theorem 1. *A necessary and sufficient condition for the solution $x=0$ of* (1) *to be strongly stable is that for any η such that $0 < \eta \leqq b$ we have*

$$(9) \qquad \inf_{\substack{\eta \leqq |x| \leqq b \\ t \in E}} U(t, x) > 0$$

where $U(t, x)$ is the function defined by the formula (5).

Proof. Put $W(\eta) = \inf_{\substack{\eta \leqq |x| \leqq b \\ t \in E}} U(t, x)$ and suppose that there is a positive η such that $0 < \eta \leqq b$ and $W(\eta)=0$. Then there exists a sequence $\{P_m\}$ such that $P_m=(t_m, x_m) \in E \times F$, $|x_m| \geqq \eta$ and $\lim_{m \to +\infty} U(t_m, x_m)=0$. If we put $U(t_m, x_m)=\frac{\varepsilon_m}{2}$ we have $\varepsilon_m \geqq 0$ and

6) Even though we read "which is continuous and has a continuous derivative" for "which is absolutely continuous on each compact subinterval of I" our discussion holds good throughout the present paper.

$\varepsilon_m \to 0$ $(m \to +\infty)$. Since $U(t_m, x_m) = \inf_{u \in U_{OP_m}} \int_0^{t_m} |u'(t) - f(t, u(t))| dt$, for every m there is an ε_m-solution $u_m(t)$ of (1) starting at (0, 0) such that $|u_m(t_m)| \geqq \eta$. Since $u_m(0) = 0$ and $\varepsilon_m \to 0$ $(m \to +\infty)$ the solution $x=0$ of (1) is not strongly stable. This completes the proof of the necessity.

Now suppose that we have $W(\eta) > 0$ in $0 < \eta \leqq b$. Let η be an arbitrary constant such that $0 < \eta < b$. Then there are an $\varepsilon > 0$ and a $\delta > 0$ such that $|x_0| < \delta$ implies $U(0, x_0) + \varepsilon < W(\eta)$. Let $u(t)$ be an ε-solution of (1) starting at $(0, x_0)$ defined on an I. Since $\int_I |u'(t) - f(t, u(t))| dt < \varepsilon$, if we put

$$u(t) = x_0 + \int_0^t f(t, u(t)) dt + \sigma(t),$$

$\sigma(t)$ is absolutely continuous on each compact subinterval of I and $\sigma(0) = 0$ and $\int_I |\sigma'(t)| dt < \varepsilon$. If $|x_0| < \delta$ and if there is a value of t such that $|u(t)| \geqq \eta$, then there is a subinterval $0 \leqq t \leqq t_1$ of I such that $|u(t)| < \eta$ in $0 \leqq t < t_1$ and $|u(t_1)| = \eta$. By (7) $U(t, u(t))$ is absolutely continuous in $0 \leqq t \leqq t_1$, so that $\frac{d}{dt} U(t, u(t))$ exists almost everywhere in $0 \leqq t \leqq t_1$. For any τ such that $0 \leqq \tau \leqq t_1$ there exists a solution $v(t)$ of (1) starting at $(\tau, u(\tau))$. Therefore for any small $h > 0$ we have

$$\begin{aligned} U(\tau+h, u(\tau+h)) - U(\tau, u(\tau)) &\leqq |U(\tau+h, u(\tau+h)) - U(\tau+h, v(\tau+h))| \\ &\quad + U(\tau+h, v(\tau+h)) - U(\tau, v(\tau)) \leqq |u(\tau+h) - v(\tau+h)| \\ &= |\{u(\tau+h) - u(\tau)\} - \{v(\tau+h) - v(\tau)\}|. \end{aligned}$$

Hence almost everywhere in $0 \leqq t \leqq t_1$ we have

$$\frac{d}{dt} U(t, u(t)) \leqq |u'(t) - v'(t)| = |u'(t) - f(t, u(t))| = |\sigma'(t)|,$$

and therefore we have in $0 \leqq t \leqq t_1$

$$U(t, u(t)) < U(0, x_0) + \varepsilon < W(\eta)$$

so that we have $U(t_1, u(t_1)) < W(\eta)$ that is $|u(t_1)| < \eta$. Thus there arises a contradiction. This completes the proof of the sufficiency.

q. e. d.

Now we obtain the following

Theorem 2. *If the solution $x=0$ of (1) is strongly stable, $x=0$ is uniformly stable, that is, for any $\eta>0$ there exists a $\delta>0$ such that $t_0\in E$ and $|x_0|<\delta$ imply that for any solution $u(t)$ starting at (t_0, x_0) we have $|u(t)|<\eta$ in the whole of the interval on which $u(t)$ is defined.*

Proof. Suppose $x=0$ to be strongly stable. Given any η such that $0<\eta\leqq b$, put $\delta=\inf_{\substack{\eta\leqq|x|\leqq b\\ t\in E}} U(t, x)$. Then we have $\delta>0$ by Theorem 1. Since $U(t, x)\leqq|x|$ for $(t, x)\in E\times F$, for any solution $u(t)$ starting at (t_0, x_0) provided that $t_0\in E$ and $|x_0|<\delta$ we have, on the whole of the interval on which $u(t)$ is defined, $U(t, u(t))<\delta$ so that $|u(t)|<\eta$.

q. e. d.

3. Regularization theorem.

Lemma 1. *Let $p(t, x)$ be a real continuous function on $D=E\times(F_1-\{0\})=\{(t, x): 0\leqq t<+\infty,\ 0<|x|<b\}$ and $q(t, x)$ a positive continuous function on D. Then there exists a positive continuous function $\delta(t, x)$ on D such that, for any (t, x), $(t', x')\in D$, $|t'-t|<\delta(t, x)$ and $|x'-x|<\delta(t, x)$ imply $|p(t', x')-p(t, x)|<q(t, x)$.*

Proof. Put $D_m=\left\{(t, x): 0\leqq t\leqq m,\ \frac{1}{m}\leqq|x|\leqq b-\frac{1}{m}\right\}$ $(m=1, 2, \cdots)$. Since every D_m is a compact subset of D there exists a positive ε_m such that $0<\varepsilon_m<\frac{1}{m}$ and that, for any $(t, x)\in D_m$ and any $(t', x')\in D$, $|t'-t|<\varepsilon_m$ and $|x'-x|<\delta_m$ imply $|p(t', x')-p(t, x)|<q(t,x)$. Let $\eta_m(t, x)$ be a non-negative continuous function on D such that $0<\eta_m(t, x)<\varepsilon_m$ in the interior of D_m and $\eta_m(t, x)=0$ in $D-D_m$. If we put $\delta_m(t, x)=\max_{1\leqq\nu\leqq m}\{\eta_\nu(t, x)\}$ $(m=1, 2, \cdots)$, we obtain a uniformly convergent sequence of continuous functions $\{\delta_m(t, x)\}$. Therefore $\delta(t, x)=\lim_{m\to+\infty}\delta_m(t, x)$ is a positive continuous function on D. It is easily verified that $|t'-t|<\delta(t, x)$ and $|x'-x|<\delta(t, x)$ imply $|p(t' x')-p(t, x)|<q(t, x)$.

q. e. d.

Lemma 2. *Let $p_1(t, x)$ and $q_1(t, x)$ be positive and continuous in D then there exists a positive continuous function $\gamma(t, x)$ on D such that $\frac{\partial\gamma}{\partial t}$, $\frac{\partial\gamma}{\partial x_i}$ $(i=1, 2, \cdots, n)$ are continuous in D and that for*

$(t, x)\in D$ we have $0<\gamma(t, x)<p_1(t, x)$, $\left|\frac{\partial\gamma}{\partial t}\right|<q_1(t, x)$ and $\left|\frac{\partial\gamma}{\partial x_i}\right|<q_1(t, x)$ $(i=1, 2, \cdots, n)$.

Proof. Let $h_m(t, x)$ be a non-negative continuous function on D such that $\frac{\partial h_m}{\partial t}$, $\frac{\partial h_m}{\partial x_i}$ $(i=1, 2, \cdots, n)$ are continuous in D and that $h_m(t, x)>0$ in the interior of D_m and $h_m(t, x)=0$ in $D-D_m$. Let c_m be a positive constant such that $c_m h_m(t, x)<2^{-m}p_1(t, x)$ and $c_m\left|\frac{\partial h_m}{\partial t}\right|$, $c_m\left|\frac{\partial h_m}{\partial x_1}\right|$, $\cdots$, $c_m\left|\frac{\partial h_m}{\partial x_n}\right|<2^{-m}q_1(t, x)$. Then all the series $\sum_{m=1}^{+\infty} c_m h_m(t, x)$, $\sum_{m=1}^{+\infty} c_m\frac{\partial h_m}{\partial t}$, $\sum_{m=1}^{+\infty} c_m\frac{\partial h_m}{\partial x_1}$, $\cdots$, $\sum_{m=1}^{+\infty} c_m\frac{\partial h_m}{\partial x_n}$ converge uniformly in any compact subset of D, so that if we put $\gamma(t, x)=\sum_{m=1}^{+\infty} c_m h_m(t, x)$, $\gamma(t, x)$ and its first partial derivatives are continuous in D and we have $0<\gamma(t, x)<p_1(t, x)$, $\left|\frac{\partial\gamma}{\partial t}\right|<q_1(t, x)$ and $\left|\frac{\partial\gamma}{\partial x_i}\right|<q_1(t, x)$ $(i=1, 2, \cdots, n)$ in D.

q. e. d.

Regularization Theorem. *Let $\varphi(t, x)$ be a non-negative continuous function on $E\times F_1=\{(t, x): 0\leqq t<+\infty, |x|<b\}$ such that*

(a) $\varphi(t, 0)=0$ for $t\in E$,

(b) $\varphi(t, x)\geqq\alpha(|x|)$ for $(t, x)\in D=E\times(F_1-\{0\})$ where $\alpha(r)$ is a positive non-decreasing function in $0<r<b$,

(c) $|\varphi(t', x')-\varphi(t, x)|\leqq\left|\int_t^{t'} N(\tau)d\tau\right|+L|x'-x|$ for (t, x), $(t', x')\in E\times F_1$ where $N(t)$ is a non-negative function of t continuous in E and L a positive constant,

(d) for any solution $u(t)$ of (1) $\varphi(t, u(t))$ is non-increasing function of t.

Then there exists a non-negative continuous function $\Phi(t, x)$ on $E\times F_1$ such that

(a) $\Phi(t, 0)=0$ for $t\in E$,

(b) $\Phi(t, x)\geqq\beta(|x|)$ for $(t, x)\in D$ where $\beta(r)$ is a positive non-decreasing function in $0<r<b$,

(c) $\frac{\partial\Phi}{\partial t}$ is continuous in $E\times F_1$,

(d) $\frac{\partial\Phi}{\partial x_i}$ $(i=1, 2, \cdots, n)$ are continuous and bounded in $E\times F_1$,

(e) $\dfrac{\partial\Phi}{\partial t}+\sum_{i=1}^{n}\dfrac{\partial\Phi}{\partial x_i}f_i(t, x)\leqq 0$ *in* $E\times F_1$.

Proof. Let $u(t)$ be a solution of (1) starting at $(t, x)\in E\times F_1$, then we have $\overline{\lim\limits_{h\to+0}}\dfrac{1}{h}\{\varphi(t+h, u(t+h))-\varphi(t, x)\}\leqq 0$. On the other hand we have for any small $h>0$

$$\begin{aligned}&\frac{1}{h}\{\varphi(t+h, x+hf(t, x))-\varphi(t, x)\}\\ &\quad=\frac{1}{h}\{\varphi(t+h, x+hf(t, x))-\varphi(t+h, u(t+h))\}\\ &\quad+\frac{1}{h}\varphi(t+h, u(t+h))-\varphi(t, x)\}\ .\end{aligned}$$

Here, since $\lim\limits_{h\to+0}\dfrac{1}{h}\{u(t+h)-x\}=u'(t)=f(t, x)$, we have

$$\begin{aligned}&\frac{1}{h}|\varphi(t+h, x+hf(t, x))-\varphi(t+h, u(t+h))|\leqq\frac{L}{h}|x+hf(t, x)-u(t+h)|\\ &\quad=L|\frac{1}{h}\{u(t+h)-x\}-f(t, x)|\to 0 \qquad (h\to+0)\ .\end{aligned}$$

Therefore we obtain for any $(t, x)\in E\times F_1$

$$\overline{\lim_{h\to+0}}\frac{1}{h}\{\varphi(t+h, x+hf(t, x))-\varphi(t, x)\}\leqq 0\ .$$

Now, if we put $\psi(t, x)=(1+e^{-t})\varphi(t, x)$, we have for any $(t, x)\in E\times F_1$

$$\begin{aligned}&\overline{\lim_{h\to+0}}\frac{1}{h}\{\psi(t+h, x+hf(t, x))-\psi(t, x)\}\\ &\quad=(1+e^{-t})\overline{\lim_{h\to+0}}\frac{1}{h}\{\varphi(t+h, x+hf(t, x)-\varphi(t, x)\}\\ &\quad-e^{-t}\varphi(t, x)\leqq -e^{-t}\varphi(t, x)\end{aligned}$$

and also we have

$$\begin{aligned}&|\psi(t', x')-\psi(t, x)|\leqq|e^{-t'}-e^{-t}|\varphi(t', x')+(1+e^{-t})|\varphi(t', x')-\varphi(t, x)|\\ &\quad\leqq Lb|\int_t^{t'}e^{-\tau}d\tau|+2|\int_t^{t'}N(\tau)d\tau|+2L|x'-x|\end{aligned}$$

since $\varphi(t', x')\leqq\varphi(t', 0)+|\varphi(t', x')-\varphi(t', 0|\leqq L|x'|\leqq Lb$. Let $N_1(t)=2N(t)+Lbe^{-t}$ and $2L=K$, then

$$|\psi(t', x')-\psi(t, x)|\leqq\left|\int_t^{t'} N_1(\tau)d\tau\right|+K|x'-x|.$$

Clearly $\psi(t, 0)=0$ for $t\in E$ and $\psi(t, x)\geqq\alpha(|x|)$ for $(t, x)\in D$.

Since $\varphi(t, x)>0$ in D, by Lemma 1 there exists a positive continuous function $\delta(t, x)$ on D such that, for any (t, x), $(t', x')\in D$, $|t'-t|<\delta(t, x)$ and $|x'-x|<\delta(t, x)$ imply $|e^{-t'}\varphi(t', x')-e^{-t}\varphi(t, x)|<\frac{1}{4}e^{-t}\varphi(t, x)$ and also $|f(t', x')-f(t, x)|<\frac{1}{4K}e^{-t}\varphi(t, x)$. Let $\delta_1(t, x)=\min\left\{\frac{1}{n}\delta(t, x), \frac{1}{2n}|x|, \frac{1}{n}(b-|x|), 1\right\}$ for $(t, x)\in D$, then $\delta_1(t, x)$ is continuous and positive in D and $\delta_1(t, x)\to 0$ as $x\to 0$ uniformly on $t\in E$. Since $N_2(t)=\max_{0\leqq s\leqq 1}N_1(t+s)$ is also continuous, by Lemma 2 there exists a positive continuous function $\rho(t, x)$ on D such that its first partial derivatives are continuous in D and that

$$0<\rho(t, x)<\delta_1(t, x),$$
$$\left|\frac{\partial\rho}{\partial t}\right|, \left|\frac{\partial\rho}{\partial x_1}\right|, \cdots, \left|\frac{\partial\rho}{\partial x_n}\right|<\frac{e^{-t}\varphi(t, x)}{4\{1+n|f(t, x)|\}\{N_2(t)+nK\}}.$$

Now if we put

$$\varphi_1(t, x)=\begin{cases}\dfrac{1}{\rho^{n+1}}\displaystyle\int_t^{t+\rho}\int_{x_1}^{x_1+\rho}\cdots\int_{x_n}^{x_n+\rho}\psi(\tau, \xi)d\tau d\xi_1\cdots d\xi_n & \text{for } (t, x)\in D,\\ 0 & \text{for } (t, x)\in E\times\{0\},\end{cases}$$

then $\varphi_1(t, x)$ is a non-negative function continuous in $E\times F_1$ such that

(a) $\varphi_1(t, 0)=0$ for $t\in E$,

(b) $\varphi_1(t, x)\geqq\alpha\left(\frac{1}{2}|x|\right)$ for $(t, x)\in D$,

(c) $\frac{\partial\varphi_1}{\partial t}, \frac{\partial\varphi_1}{\partial x_1}, \cdots, \frac{\partial\varphi_1}{\partial x_n}$ are continuous in D.

Since we may write in D

$$\varphi_1(t, x)=\int_0^1\int_0^1\cdots\int_0^1\psi(t+\rho\sigma, x+\rho\theta)d\sigma d\theta$$

where $\theta=(\theta_1, \theta_2, \cdots, \theta_n)$ and $d\theta=d\theta_1 d\theta_2\cdots d\theta_n$, we have in D for any small $h>0$

$$
\begin{aligned}
&\frac{1}{h}\{\varphi_1(t+h,\ x+hf(t,\ x))-\varphi_1(t,\ x)\}\\
&\quad=\frac{1}{h}\int_0^1\cdots\int_0^1\{\psi(t+h+\bar{\rho}\sigma,\ x+hf+\bar{\rho}\theta)-\psi(t+\rho\sigma,\ x+\rho\theta)\}\,d\sigma\,d\theta\\
&\quad\leqq\frac{1}{h}\int_0^1\cdots\int_0^1|\psi(t+h+\bar{\rho}\sigma,\ x+hf+\bar{\rho}\theta)-\psi(t+\rho\sigma+h,\ x+\rho\theta+h\bar{f})|\,d\sigma\,d\theta\\
&\quad+\frac{1}{h}\int_0^1\cdots\int_0^1\{\psi(t+\rho\sigma+h,\ x+\rho\theta+h\bar{f})-\psi(t+\rho\sigma,\ x+\rho\theta)\}\,d\sigma\,d\theta\\
&\quad\leqq\left|\frac{1}{h}\int_0^1\cdots\int_0^1 d\sigma\,d\theta\int_{t+h+\rho\sigma}^{t+h+\bar{\rho}\sigma}N_1(\tau)\,d\tau\right|+K|\bar{f}-f|+\frac{nK}{h}|\bar{\rho}-\rho|\\
&\quad+\frac{1}{h}\int_0^1\cdots\int_0^1\{\psi(t+\rho\sigma+h,\ x+\rho\theta+h\bar{f})-\psi(t+\rho\sigma,\ x+\rho\theta)\}\,d\sigma\,d\theta
\end{aligned}
$$

where $f=f(t,\ x)$, $\bar{f}=f(t+\rho\sigma,\ x+\rho\theta)$, $\rho=\rho(t,\ x)$ and $\bar{\rho}=\rho(t+h,\ x+hf(t,\ x))$. Then we have

$$
\begin{aligned}
\lim_{h\to+0}\frac{1}{h}\{\varphi_1(t+h,\ x+hf(t,\ x))-\varphi_1(t,\ x)\}&\leqq\left|\int_0^1 N_1(t+\rho\sigma)\,\sigma\,d\sigma\right|\left|\frac{d\rho}{dt}\right|\\
&+K|f(t+\rho\sigma,\ x+\rho\theta)-f(t,\ x)|+nK\left|\frac{d\rho}{dt}\right|\\
&-\int_0^1\cdots\int_0^1 e^{-(t+\rho\sigma)}\varphi(t+\rho\sigma,\ x+\rho\theta)\,d\sigma\,d\theta\,,
\end{aligned}
$$

where $\dfrac{d\rho}{dt}=\dfrac{\partial\rho}{\partial t}+\sum_{i=1}^{n}\dfrac{\partial\rho}{\partial x_i}f_i$ so that

$$
\left|\frac{d\rho}{dt}\right|\leqq\{1+n|f(t,\ x)|\}\frac{e^{-t}\varphi(t,\ x)}{4\{1+n|f(t,\ x)|\}\{N_2(t)+nK\}}=\frac{e^{-t}\varphi(t,\ x)}{4\{N_2(t)+nK\}}\,.
$$

Since $0\leqq\rho\sigma<\min\{\delta(t,\ x),\ 1\}$ and $|\rho\theta|\leqq n\rho<\delta(t,\ x)$, we have

$$
\left|\int_0^1 N_1(t+\rho\sigma)\,\sigma\,d\sigma\right|\left|\frac{d\rho}{dt}\right|+nK\left|\frac{d\rho}{dt}\right|\leqq\{N_2(t)+nK\}\left|\frac{d\rho}{dt}\right|<\frac{1}{4}e^{-t}\varphi(t,\ x)
$$

and

$$
K|f(t+\rho\sigma,\ x+\rho\theta)-f(t,\ x)|<\frac{1}{4}e^{-t}\varphi(t,\ x)\,.
$$

Hence we have

$$
\begin{aligned}
&\lim_{h\to+0}\frac{1}{h}\{\varphi_1(t+h,\ x+hf(t,\ x))-\varphi_1(t,\ x)\}\\
&\qquad<\frac{1}{2}e^{-t}\varphi(t,\ x)-\int_0^1\cdots\int_0^1 e^{-(t+\rho\sigma)}\varphi(t+\rho\sigma,\ x+\rho\theta)\,d\sigma\,d\theta
\end{aligned}
$$

$$\leqq \frac{1}{2}e^{-t}\varphi(t, x)-e^{-t}\varphi(t, x)+\int_0^1\cdots\int_0^1 |e^{-(t+\rho\sigma)}\varphi(t+\rho\sigma, x+\rho\theta) \\ -e^{-t}\varphi(t, x)|\, d\sigma d\theta$$

$$<\frac{1}{2}e^{-t}\varphi(t, x)-e^{-t}\varphi(t, x)+\frac{1}{4}e^{-t}\varphi(t, x) = -\frac{1}{4}e^{-t}\varphi(t, x)<0.$$

Moreover it may be proved easily that $\frac{\partial\varphi_1}{\partial t}$ is dominated in D by a continuous function of t alone and that $\frac{\partial\varphi_1}{\partial x_i}$ $(i=1, 2, \cdots, n)$ are bounded in D. Therefore if we set $\Phi(t, x)=\{\varphi_1(t, x)\}^2$ and $\beta(r)=\left\{\alpha\left(\frac{1}{2}r\right)\right\}^2$, $\Phi(t, x)$ is the function desired in the theorem.

q. e. d.

If $U(t, x)$ satisfies the inequality (9) in Theorem 1, it has the same properties as $\varphi(t, x)$ in the above theorem. Therefore we can state a stability theorem of the well-known form as follows:

Theorem 3. *A necessary and sufficient condition for the solution* $x=0$ *of* (1) *to be strongly stable is that there exists a non-negative continuous function* $V(t, x)$ *on* $E\times F_1$ *where* $F_1=\{x : |x|<b\}$, *such that*

(a) $V(t, 0)=0$ *for* $t\in E$,

(b) $V(t, x)>0$ *for* $(t, x)\in E\times(F_1-\{0\})$,

(c) $\frac{\partial V}{\partial t}$ *is continuous in* $E\times F_1$,

(d) $\frac{\partial V}{\partial x_i}$ $(i=1, 2, \cdots, n)$ *are continuous and bounded in* $E\times F_1$,

(e) $V(t, x)$ *is positive definite on* $E\times F_1$, *that is, there exists a positive function* $W(r)$ *of* r *defined in the interval* $0<r<b$ *such that* $V(t, x)\geqq W(|x|)$ *for* $(t, x)\in E\times(F_1-\{0\})$,

(f) $\frac{\partial V}{\partial t}+\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}f_i(t, x)\leqq 0$[7] *for* $(t, x)\in E\times F_1$.

Proof. The necessity follows at once from Theorem 1 and Regularization Theorem. The proof of the sufficiency is very similar to that of Theorem 1 and therefore is omitted.

q. e. d.

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7) $f=(f_1, f_2, \cdots f_n)$.

If we replace E by a compact interval, Theorem 3 coincides with the uniqueness theorem due to Okamura[8].

4. Strong boundedness.

Hereafter we consider the differential equation

$$\frac{dy}{dt} = f(t, y) \tag{10}$$

where t is real and y a real n-dimensional vector and where $f(t, y)$ is a continuous function on $E\times R^n$ to R^n, where $E=\{t: 0\leqq t<+\infty\}$.

In the foregoing paper [5] (Theorem 7, pp. 18-22), we have verified that (10) may be transformed into

$$\frac{dY}{dt} = h(t, Y) \tag{11}$$

where $\sum_{i=1}^{n+1} Y_i h_i(t, Y)=0$[9] for $(t, Y)\in E\times S^n$, where $h(t, Y)$ is a continuous function on $E\times S^n$ to R^{n+1}, S^n being the unit sphere in R^{n+1}, and where $h(t, N)=0$, N being $(0, 0, \cdots, 0, 1)\in R^{n+1}$, by the topological mapping $Y=\Phi(y)$ of $R^n \cup \{\infty\}$ onto S^n where

$$\Phi(y) = \begin{cases} \dfrac{2\lambda(|y|)}{\{\lambda(|y|)\}^2|y|^2+1}y+\dfrac{\{\lambda(|y|)\}^2|y|^2-1}{\{\lambda(|y|)\}^2|y|^2+1}N & \text{if } y\in R^n, \\ N & \text{if } y=\infty, \end{cases} \tag{12}$$

R^n being considered as the hyperplane $Y_{n+1}=0$ orthogonal to the vector N.

Now let $P=(t_P, Y_P)$ and $Q=(t_Q, Y_Q)$ be two points in $E\times S^n$. When $t_P<t_Q$ we denote by $\boldsymbol{V}_{PQ}$ the set of all the absolutely continuous functions $v(t)$ on the interval $t_P\leqq t\leqq t_Q$ to S^n such that $v(t_P)=Y_P$ and $v(t_Q)=Y_Q$.

In the present case we define the Okamura function $D(P, Q)$ by

$$D(P, Q) = \begin{cases} \inf\limits_{v\in \boldsymbol{V}_{PQ}} \displaystyle\int_{t_P}^{t_Q} |v'(t)-h(t, v(t))|\,dt & \text{if } t_P<t_Q, \\ D(Q, P) & \text{if } t_P>t_Q, \\ \operatorname{dis}(Y_P, Y_Q) & \text{if } t_P=t_Q \end{cases} \tag{13}$$

8) cf. Okamura [6], pp. 229-231.

9) $Y=(Y_1, Y_2, \cdots, Y_{n+1})$, $|Y|=\sqrt{Y_1^2+Y_2^2+\cdots+Y_{n+1}^2}$ and $h=(h_1, h_2, \cdots, h_{n+1})$.

where $\operatorname{dis}(Y_P, Y_Q)$ is the geodesic distance on S^n between Y_P and Y_Q.

If P and Q are on a solution of (11) we have $D(P, Q)=0$ and if not $D(P, Q)>0$. Let $P=(t_P, Y_P)$, $Q=(t_Q, Y_Q)$ and $R=(t_R, Y_R)$ be three points in $E\times S^n$ we obtain

$$|D(P, R)-D(Q, R)|\leqq\left|\int_{t_P}^{t_Q} N(\tau)\,d\tau\right| +\operatorname{dis}(Y_P, Y_Q)$$

where $N(t)=\max_{Y\in S^n}|h(t, Y)|$. If $t_P\leqq t_Q\leqq t_R$ we have

$$D(P, R)\leqq D(P,Q)+D(Q, R)\,, \tag{14}$$

so that, if Q and R are on a solution of (11), (14) is reduced to

$$D(P, Q)\geqq D(P, R) \tag{15}$$

and that, if P and Q are on a solution of (11), (14) is reduced to

$$D(P, R)\leqq D(Q, R) \tag{16}$$

Let $P=(t, Y)$ and $Q_1=(t_1, N)$. Since $Y=N$ is a solution of (11), $D(P, Q_1)$ is non-increasing with respect to t_1 whenever $t\leqq t_1$. Now we put

$$U(t, Y) = \lim_{t_1\to+\infty} D(P, Q_1)\,. \tag{17}$$

Then we have that $U(t, N)=0$ for all $t\in E$ and that for (t, Y), $(t', Y')\in E\times S^n$

$$|U(t', Y')-U(t, Y)|\leqq\left|\int_{t}^{t'} N(\tau)\,d\tau\right| +\operatorname{dis}(Y, Y')\,. \tag{18}$$

Since $0\leqq U(t, Y)\leqq U(t, N)+|U(t, Y)-U(t, N)|\leqq\operatorname{dis}(Y, N)\leqq\pi$, $U(t, Y)$ is a non-negative bounded continuous function on $E\times S^n$. If $v(t)$ is a solution of (11) then by (16) $U(t, v(t))$ is a non-decreasing function of t.

Now if we put

$$V(t, y) = U(t, \Phi(y))\,, \tag{19}$$

it is easily verified that

(a) $V(t, y)$ is non-negative, continuous and bounded in $E\times R^n$,
(b) $V(t, y)\to 0$ as $y\to\infty$ uniformly on $t\in E$,
(c) $|V(t', y')-V(t, y)|\leqq\left|\int_t^{t'} N(\tau)\,d\tau\right| +K|y'-y|$ for (t, y), $(t', y')\in E\times R^n$ where K is a positive constant,

(d) for any solution $u(t)$ of (10) $V(t, u(t))$ is a non-decreasing function of t.

Definition 4. (Strong boundedness.) *Every solution of* (10) *is said to be strongly bounded if for any* $(t_0, y_0) \in E \times R^n$ *there exist an* $\varepsilon > 0$ *and a* $B > 0$ *such that for any* ε*-solution* $u(t)$ *of* (10) *starting at* (t_0, y_0) *we have* $|u(t)| < B$ *in the whole of the interval on which* $u(t)$ *is defined.*

Theorem 4. *A necessary and sufficient condition for every solution of* (10) *to be strongly bounded is that for all* $(t, y) \in E \times R^n$ *we have*

$$V(t, y) > 0 \tag{20}$$

where $V(t, y)$ *is the function defined by the formula* (19).

Proof. Let (t_0, y_0) be a point in $E \times R^n$. Then by (12) $Y_0 = \Phi(y_0) \in S^n - \{N\}$. If every solution of (10) is strongly bounded there exist an $\varepsilon > 0$ and a $B > 0$ mentioned in Definition 4. On the other hand there exists an $\eta > 0$ such that the subset $|y| \geqq B$ of R^n is mapped by $Y = \Phi(y)$ onto the subset $Y_{n+1} \geqq 1 - \eta$ of $S^n - \{N\}$. Let $P = (t_0, Y_0)$ and $Q_1 = (t_1, N)$ where $t_0 < t_1$. For any $v(t) \in \boldsymbol{V}_{PQ_1}$ there exists a t_2 such that $t_0 < t_2 < t_1$ and that $v_{n+1}(t) < 1 - \eta$[10] in $t_0 \leqq t < t_2$ and $v_{n+1}(t_2) = 1 - \eta$. If we set $u(t) = \Phi^{-1}(v(t))$, $u(t)$ is also an absolutely continuous function on $t_0 \leqq t \leqq t_2$ and $u(t_0) = y_0$, $|u(t)| < B$ in $t_0 \leqq t < t_2$ and $|u(t_2)| = B$. Hence we have

$$\int_{t_0}^{t_2} |u'(t) - f(t, u(t))| \, dt \geqq \varepsilon .$$

On the other hand by (12) we obtain

$$|v'(t) - h(t, v(t))| \geqq \frac{2\lambda(|u|)}{\{\lambda(|u|)\}^2 |u|^2 + 1} |u'(t) - f(t, u(t))| .$$

If we put $C_B = \min\limits_{0 \leqq r \leqq B} \dfrac{2\lambda(r)}{\{\lambda(r)\}^2 r^2 + 1}$ we have

$$\begin{aligned} \int_{t_0}^{t_1} |v'(t) - h(t, v(t))| \, dt &\geqq \int_{t_0}^{t_2} |v'(t) - h(t, v(t))| \, dt \\ &\geqq C_B \int_{t_0}^{t_2} |u'(t) - f(t, u(t))| \, dt \geqq C_B \varepsilon , \end{aligned}$$

10) $v = (v_1, v_2, \cdots, v_{n+1})$.

so that we have $D(P, Q_1) \geqq C_B \varepsilon$. Since C_B is positive and independent of t_1 we have $U(t_0, Y_0) \geqq C_B \varepsilon > 0$, that is, $V(t_0, y_0) > 0$.

Next we will prove the sufficiency. Let (t_0, y_0) be a point in $E \times R^n$. Since $V(t_0, y_0) > 0$ there exist an $\varepsilon > 0$ and a $\mu > 0$ such that $V(t_0, y_0) - K\varepsilon > \mu$. Then for any ε-solution $u(t)$ of (10) starting at (t_0, y_0) defined on any I we have $V(t, u(t)) > V(t_0, y_0) - K\varepsilon > \mu$ in I. Since $V(t, y) \to 0$ as $y \to \infty$ uniformly on $t \in E$, there exists a $B > 0$ such that $|u(t)| < B$ in I. This completes the proof of the sufficiency.

q. e. d.

In the present case we can also replace $V(t, y)$ by a more regular function. Though the range of y is not bounded, yet $V(t, y)$ is bounded in the whole of $E \times R^n$ and tends to zero as $y \to \infty$ uniformly on $t \in E$ so that the proof is very similar to that of Regularization Theorem in §3 and therefore is omitted.

Now we obtain the following

Theorem 5. *A necessary and sufficient condition for every solution of* (10) *to be strongly bounded is that there exists a positive continuous function* $V(t, y)$ *on* $E \times R^n$ *such that*

(a) $\dfrac{\partial V}{\partial t}$ *is continuous on* $E \times R^n$,

(b) $\dfrac{\partial V}{\partial y_i}$ $(i = 1, 2, \cdots, n)$ *are continuous and bounded in* $E \times R^n$,

(c) $V(t, y) \to 0$ *as* $y \to \infty$ *uniformly on* $t \in E$,

(d) $\dfrac{\partial V}{\partial t} + \sum_{i=1}^{n} \dfrac{\partial V}{\partial y_i} f_i(t, y) \geqq 0$ *for* $(t, y) \in E \times R^n$.

If we replace E by a compact interval, Theorem 5 coincides with the boundedness theorem due to Okamura[11].

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11) cf. Hayashi [4], Theorem 2, p. 317.

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Errata

On quasi-equicontinuous sets—Sets of solutions of a differential equation—

By

Kyuzo HAYASHI

(these memoirs **31** (1958), pp. 9-23)

Page	Line	For	Read
13	28	on I	on each compact subinterval of I
15	19	$x \in I$	$(x, y) \in I \times D$
15	30	$\|f(x, y)\| \leqq M(x)$	$\|f(x, y)\| \leqq M(x)$ in $U \times F_1$
19	9	$m(E_m - e_m) < \delta_m$	$m(E_m - e_m) < \delta_m$ $(e_1 \subset e_2 \subset e_3 \subset \cdots)$
19	14	$\int_{E_m - e_m} M(x)\,dx < \frac{1}{2^m}$	$\int_{E_m - e_m} M_m(x)\,dx < \frac{1}{2^m}$
19	18	$\max_{x \in e_m} M_m(x)$	$\max_{\substack{x \in e_m \\ \|y\| \leqq m}} \|f(x, y)\|$
19	19	$k_1(\|y\|) = \max_{x \in e_m} M_m(x) \geqq \|f(x, y)\|$	$k_1(\|y\|) \geqq \|f(x, y)\|$
19	21	positive continuous	positive non-decreasing continuous