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Determination of the second fundamental form of a hypersurface by its mean curvature

By

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It is well known [1, p. 200], [4, p. 188] that, if a hypersurface of an euclidean space is of type more than two, then the second fundamental form (II) is uniquely determined by its first fundamental form (I). On the other hand, in 1945, T. Y. THOMAS [5] shaw that the form (II) of a surface is determined in general by its form (I) and the mean curvature M. Therefore the imbedding of a 2-dimensional Riemannian space, which is assumed to be of type two, is uniquely determined by giving the mean curvature M, within rigid motions.

These results lead us to consider the imbedding of an $n(>2)$-dimensional Riemannian space of type two by giving the mean curvature M. Thus our problem is to *find the expression of the form (II) in terms of the form (I), the scalar M, and their derivatives.* The methods, by means of which Thomas deduced the expression of the form (II) of a surface, are not applicable to a hypersurface of general dimensional number, because he did not use the process of tensor-calculus, and further the simple equations (1.1) giving the curvature tensor of a surface do not hold good for a hypersurface, except when the hypersurface is of constant curvature.

In the first part of this paper the problem of Thomas [5] is treated by the process of tensor-calculus. We shall show that the determination of the form (II) will be done by solving a system of *linear* equations (1.13).

The second part of this paper is devoted to generalize the problem to the case of dimensions $n>2$. The expressions of the covariant derivatives of the second fundamental tensor H_{ij} are also obtained, but, in this time, their symmetry leads us to some equations, which are linear with respect to H_{ij}. We shall show that, in virtue of a system of those equations, H_{ij} shall be already determined in general, and we need not the conditions of integrability. Consequently, we see that the problem of possibility of imbedding an n-dimensional Riemannian space in an euclidean $(n+1)$-space rests in general upon the existence of a single scalar (the mean curvature M), which must satisfy a certain system of differential equations.

I. On the case of surfaces in an ordinary space.

§ 1. The covariant derivatives of the second fundamental tensor.

We consider a non-developable surface S of an euclidean 3-space and denote by

$$\text{(I)}\quad g_{ij}(x)\,dx^i dx^j\,, \qquad \text{(II)}\quad H_{ij}(x)\,dx^i dx^j\,, \quad (i, j = 1, 2)\,,$$

the first and second fundamental forms of S respectively. The curvature tensor R_{hijk} of S is written in the form

$$R_{hijk} = K\,(g_{hj}g_{ik} - g_{hk}g_{ij})\,, \tag{1.1}$$

where the scalar $K(x)$ is called the Gaussian total curvature of S, which does not vanish on S identically by our hypothesis. The scalar $g^{ab}H_{ab}$ is called the *mean curvature* of S and we denote it by M. It is well known that the components H_{ij} satisfy the *equations of Gauss and Codazzi* as follows.

$$R_{hijk} = H_{h[j}H_{ik]}{}^{*}\,, \qquad H_{i[j,k]} = 0\,.$$

We shall deduce some equations, which will be used in the following discussions. It follows from (1. 1) and the Gauss equation that

$$K g_{h[j}g_{ik]} = H_{h[j}H_{ik]}\,. \tag{1.2}$$

On contracting (1. 2) by g^{hj}, we have

$$K g_{ik} = MH_{ik} - H^a_i H_{ak}\,. \tag{1.3}$$

where we put $H^a_i = g^{ab}H_{bi}$. Multiply (1. 3) by H_{jl}, interchange the indices k, l, and subtract; using (1. 2), we obtain

$$g_{i[k}H_{jl]} = -H_{i[k}g_{jl]} + Mg_{i[k}g_{jl]}\,. \tag{1.4}$$

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* The symbols $[ik]$ and (ik) are used to express, for instance,

$$H_{h[j}H_{ik]} = H_{hj}H_{ik} - H_{hk}H_{ij}\,,$$
$$H^a_{(i}H_{ak),l} = H^a_i H_{ak,l} + H^a_k H_{ai,l}\,.$$

Mext, we take a skew-symmetric tensor S_{ij} arbitrarily. Contraction of (1.4) by $g^{ia}g^{jb}S_{ab}$ gives us easily

$$S_{[ka}H^{a}_{l]} = MS_{kl}. \tag{1.5}$$

Now, it follows from the Codazzi equation that $g^{ab}H_{ab,i} = g^{ab}H_{ia,b} = M_i$, in which, for simplicity, we have omitted the "comma" denoting covariant differentiation of the mean curvature M. Differentiating (1.3) covariantly by x^l, we obtain

$$K_l g_{ik} = M_l H_{ik} + MH_{ik,l} - H^{a}_{(i} H_{ak),l}^{*},$$

where K_l is the covariant derivative of the total curvature K. It follows from the above equations that

$$K_{(l}g_{i)k} - K_k g_{li} = MH_{ik,l} - 2H^{a}_{k}H_{ai,l} + M_{(l}H_{i)k} - M_k H_{li}. \tag{1.6}$$

Multiply (1.6) by H_{jm} and subtract from it the equation obtained by interchange of the indices k, m. Then, in virtue of (1.2), we have

$$\begin{aligned}&(K_{(l}g_{i)[k} - K_{[k}g_{li})H_{jm]} - KM_{(l}g_{i)[k}g_{jm]}\\ &-(MH_{i[k,l} - H_{il}M_{[k})H_{jm]} - 2Kg_{j[k}H_{m]i,l} = 0.\end{aligned}$$

We contract the above equation by g^{jk} and change the indices. The resulting equation is written in the form

$$\begin{aligned}&MH^{a}_{k}H_{ai,l} + (2K - M^2)H_{kl,i} + (MM_k - M_a H^{a}_{k})H_{il}\\ &-K_{(l}H_{i)k} + (MK_{(i} - KM_{(i})g_{l)k} + (K_a H^{a}_{k} - MK_k)g_{il} = 0.\end{aligned}$$

On eliminating the term $H^{a}_{k}H_{ai,l}$ from the above equation and (1.6), we then have

$$\begin{aligned}\theta H_{ij,k} = {}&(2KM_{(i} - MK_{(i})g_{j)k} + (MK_k - 2K_a H^{a}_{k})g_{ij}\\ &+(2K_{(i} - MM_{(i})H_{j)k} + (2M_a H^{a}_{k} - MM_k)H_{ij},\end{aligned} \tag{1.7}$$

where the scalar $\theta = 4K - M^2$.

Consequently, if θ does not vanish, then the covariant derivatives $H_{ij,k}$ are expressed by (1.7) in terms of the metric tensor g_{ij}, the mean curvature M, their derivatives, and H_{ij}. We notice here that *the equations* (1.7) *are tensorial expressions of the equations* (2.2) *and* (2.3) *of Thomas's paper* [5].

We can see easily that the right-hand member of (1.7) is symmetric with respect to the three indices. In order to verify the symmetry with respect to the indices j, k, we obtain from (1.7)

$$\frac{\theta}{2}H_{i[j,k]} = (KM_{[j}-MK_{[j})g_{k]i}+K_{[j}H_{k]i} +K_aH^a_{[j}g_{k]i}-M_aH^a_{[j}H_{k]i}.$$

The last two terms are rewritten, in virtue of (1.2) and (1.4), in the forms

$$K_aH^a_{[j}g_{k]i} = -K_{[j}H_{k]i}+MK_{[j}g_{k]i},$$
$$-M_aH^a_{[j}H_{k]i} = -M_{[j}g_{k]i}.$$

Therefore we establish $H_{i[j,k]}=0$.

§ 2. The conditions of integrability of the equations (1.7).

We now deduce the conditions of integrability derivable from the system (1.7). Using the formula of Ricci and (1.1), we find

$$H_{ij,[k,l]} = -Kg_{(i[k}H_{j)l]}.$$

Let us differentiate (1.7) covariantly by x^l, subtract from it the equation obtained by interchange of the indices k, l, and then make use of the equation of Codazzi and the above. When the substitution is made from (1.7), after considerable reduction, the resulting equation becomes

$$\begin{aligned}(1.8)\qquad &(MQ_{(i[k}-2P_{(i[k})H_{j)l]}+2(g_{ij}P_{a[k}-H_{ij}Q_{a[k})H^a_{l]}\\ &-\left(2KQ_{(i[k}-MP_{(i[k}-\frac{KM\theta^2}{2}g_{(i[k}+\theta S_{(i[k}\right)g_{j)l]}\\ &-2\theta S_{kl}g_{ij} = 0,\end{aligned}$$

where $S_{ij}=M_{[i}K_{j]}$, and symmetric tensors P_{ij} and Q_{ij} are defined by

$$P_{ij} = \theta K_{ij}-4K_iK_j+MM_{(i}K_{j)}-\frac{K\theta^2}{2}g_{ij},$$
$$Q_{ij} = \theta M_{ij}+2MM_iM_j-2M_{(i}K_{j)},$$

these being the same essentially as the one used by Thomas [5, p. 394].

It is desirable for the requirements of the following calculations to put the equation (1.8) into a more contracted form. To do this, we contract (1.8) by g^{jl} and then obtain

$$MQ_{a(k}H^a_{i)}-\theta Q_{ik}+(2P-MQ)H_{ik}$$
$$+(2KQ-MP)g_{ik}-2Q_{ak}H^a_bH^b_i+2Q_{ab}H^a_iH^b_k=0\,,$$

where putting $P=g^{ab}P_{ab}$ and $Q=g^{ab}Q_{ab}$. In virtue of (1.2) and (1.3), the last two terms are rewritten in the forms

$$Q_{ak}H^a_bH^b_i=MQ_{ak}H^a_i-Q_{ik}\,,$$
$$Q_{ab}H^a_iH^b_k=Q_{ab}H^{ab}H_{ik}+KQ_{ik}-KQg_{ik}\,.$$

Thus the above contracted equation becomes

$$(1.9)\qquad MQ_{a(i}H^a_{k)}=M^2Q_{ik}-MPg_{ik}+(2Q_{ab}H^{ab}+2P-MQ)H_{ik}\,.$$

Now, we introduce the quantities h^{ij}, which are the cofactors of the elements H_{ij} in the determinant $|H_{ab}|$ divided by $g=|g_{ab}|$. It follows from (1.2) that $H_{ik}h^{ij}=K\delta^j_k$. Contract (1.2) by h^{hj}: we then have

$$g_{ab}h^{ab}g_{ik}-g_{ai}g_{bk}h^{ab}=H_{ik}\,.$$

Furthermore, on contracting by g^{ik}, we find $g_{ab}h^{ab}=M$, and so the above equations give immediately

$$(1.10)\qquad H^{ik}=Mg^{ik}-h^{ik}\,.$$

We contract (1.9) by h^{ik}, and make use of those equations as above obtained. It follows that $Q_{ab}H^{ab}=MQ-P$. Hence the equation (1.9) becomes

$$(1.11)\qquad Q_{a(i}H^a_{k)}-QH_{ik}=MQ_{ik}-Pg_{ik}\,,$$

provided that *the mean curvature M does not vanish.*

We remark lastly that the equation $Q_{ab}H^{ab}=MQ-P$ becomes $Q_{ab}h^{ab}=P$, by means of (1.10). The latter equation was obtained by Thomas [5, (3.4) or (5.1)] as the integrability conditions of (1.7), and plaied a röle in order to determine H_{ij}. But we do not use this equation in the following.

§ 3. Explicit determination of the second fundamental form.

In this section we shall restrict our discussions to a region of the surface, where the mean curvature M does not vanish. Let us deduce the equations from the Gauss equation and (1.11), which will be used to determine H_{ij}. Multiply (1.11) by H_{jl}, interchange the indices k, l, and subtract; when we make use of of (1.2), then the equation

$$KQ_{i[k}g_{jl]}+Q_{a[k}H^{a}_{i}H_{jl]} = QKg_{i[k}g_{jl]}+A_{i[k}H_{jl]}$$

is obtained, where by definition A_{ik} is the right-hand member of (1.11). By making use of (1.2), the second term of the left-hand side of the above equation becomes

$$Q_{a[k}H^{a}_{i}H_{jl]} = Q_{a[k}H^{a}_{l]}H_{ij}+KQ_{i[k}g_{jl]}$$

Hence the equation as above shown is written as

$$Q_{a[k}H^{a}_{l]}H_{ij}-A_{i[k}H_{jl]} = K(Qg_{i[k}-2Q_{i[k})g_{jl]}\,. \tag{1.12}$$

We introduce so-called contravariant e-tensor e^{ij} [2, p. 77], components of which are $e^{11}=e^{22}=0$, $e^{12}=-e^{21}=\frac{1}{\sqrt{g}}$ $(g=|g_{ab}|)$. Then, on contracting (1.12) by e^{kl} we find

$$\lambda H_{ij}-\frac{2}{\sqrt{g}}(A_{i1}H_{j2}-A_{i2}H_{j1}) = 2KV_{ij}\,, \tag{1.13}$$

in which we put

$$\lambda = Q_{a[k}H^{a}_{l]}e^{kl}\,,$$

$$V_{ij} = \frac{1}{2}(Qg_{i[k}-2Q_{i[k})g_{jl]}e^{kl}\,.$$

If we choose a system of coördinates $(x)_p$, such that, at a point P, we have $g_{ij}=\delta_{ij}$, then the components V_{ij} are given in particular by

$$V_{11} = 2Q_{12}\,,\quad V_{12} = V_{21} = Q_{22}-Q_{11}\,,\quad V_{22} = -2Q_{12}$$

at the point P. Therefore V_{ij} can be also defined by

$$V_{ij} = -(Q_{ia}g_{jb}+Q_{ja}g_{ib})e^{ab}\,.$$

Hence this tensor V_{ij} is essencially the same as the one used by Thomas [5, p. 397].

Now, we solve the equations (1.13) with respect to H_{ij}, by Cramer's rule. First, when we take $i=j=1$ and $i=2$, $j=1$ in (1.13), we then obtain

$$H_{11} = -\frac{\lambda}{2W}V_{11}+\frac{1}{W\sqrt{g}}(V_{11}A_{12}-V_{12}A_{11})\,,$$

$$H_{12} = -\frac{\lambda}{2W}V_{12}+\frac{1}{W\sqrt{g}}(V_{11}A_{22}-V_{12}A_{12})\,,$$

where we put $W=-\frac{1}{2K}\left(\lambda^2+\frac{4|A_{ab}|}{g}\right)$. Next, taking $i=1$, $j=2$ and $i=j=2$, we then have

$$H_{12} = -\frac{\lambda}{2W}V_{12}+\frac{1}{W\sqrt{g}}(V_{12}A_{12}-V_{22}A_{11}),$$
$$H_{22} = -\frac{\lambda}{2W}V_{22}+\frac{1}{W\sqrt{g}}(V_{12}A_{22}-V_{22}A_{12}).$$

If we refer to the coördinates $(x)_p$ as above mentioned, then we see that, at the point P, we have

$$V_{11}A_{22}-2V_{12}A_{12}+V_{22}A_{11} = 0.$$

Hence the two equations giving H_{12} as thus found are written in the single form

$$H_{12} = -\frac{\lambda}{2W}V_{12}+\frac{1}{2W\sqrt{g}}(V_{11}A_{22}-V_{22}A_{11}).$$

As a consequence of these equations expressing H_{ij}, we have the invariantive form of H_{ij} as follows.

$$H_{ij} = -\frac{\lambda}{2W}V_{ij}+\frac{1}{W}V^*_{ij}, \tag{1.14}$$

where the last term is defined by

$$V^*_{ij} = \frac{1}{2}(V_{ia}A_{jb}+V_{ja}A_{ib})e^{ab}.$$

We show that V^*_{ij} are linear combinations of g_{ij} and Q_{ij}. In fact, if we refer again to the coördinates $(x)_p$, then the components V^*_{ij} are given, at the point P, by

$$V^*_{11} = PQ-2M|Q_{ab}|+(MQ-2P)Q_{11},$$
$$V^*_{12} = \qquad\qquad +(MQ-2P)Q_{12},$$
$$V^*_{22} = PQ-2M|Q_{ab}|+(MQ-2P)Q_{22}.$$

Thus the tensor is also expressible in the form

$$V^*_{ij} = Rg_{ij}+UQ_{ij}, \tag{1.15}$$

in which the coefficients R and U are defined by

$$R = PQ-2M\frac{|Q_{ab}|}{g}, \qquad U = MQ-2P.$$

The scalars λ and W in (1.14) are unknown yet. To determine these, we recall the definition of the mean curvature M. Then we contract (1.14) by g^{ij} and make use of $g^{ab}V_{ab}=0$ and (1.15). In virtue of the definitions of R and U, we obtain

$$W = Q^2 - 4\frac{|Q_{ab}|}{g}, \qquad \lambda = \pm\sqrt{-\theta W - U^2},$$

the latter being obtained from the defining equation of W.

Consequently, on substituting for λ and W in (1.14) from the expressions as above found, *we establish the explicit determination of the second fundamental tensor H_{ij} by the first fundamental tensor g_{ij}, the mean curvature M, and their derivatives.* We see from (1.15) that our expressions (1.14) are the same as the one obtained by Thomas [5, (5.10)].

§4. On the case of the mean curvature being constant.

The equations (1.11) were obtained under the restriction, that the mean curvature M does not vanish. And, if M is constant, we then see that Q_{ij} vanish, and hence $W=0$, so that the expressions (1.14) are of no avail.

Now we return to the equations (1.8) and suppose that the mean curvature M is constant. Then (1.8) becomes

$$-P_{(i[k}H_{j)l]}+g_{ij}P_{a[k}H^a_{l]} = 0.$$

On referring to the coördinates $(x)_p$, the above equations are written in the simple form

$$MP_{12}-PH_{12} = 0, \qquad P_{11}H_{22}-P_{22}H_{11} = 0$$

at the point P. From the second equation it follows that

$$MP_{11}-PH_{11} = 0, \qquad MP_{22}-PH_{22} = 0.$$

Then these equations can be combined into

$$MP_{ij} = PH_{ij}. \tag{1.16}$$

First, we consider a minimal surface S, which is characterized by $M=0$ identically. It follows from (1.16) that $PH_{ij}=0$, and hence we have $P=0$, because S is assumed to be non-developable. Thus we prove

Theorem 1. *The Gaussian total curvature K of a non-develop-*

able, minimal surface satisfies the differential equation

$$g^{ab}(KK_{ab}-K_aK_b) = 4K^3.$$

Next, we treate a surface, such that the mean curvature M is constant $\neq 0$. If the scalar P does not vanish, we then have $H_{ij}=\frac{M}{P}P_{ij}$. Besides, if $P=0$, then we see $P_{ij}=0$. Consequently we have

Theorem 2. *Let S be a non-developable, -minimal surface, such that the mean curvature M is constant. If*

$$P = g^{ab}(\theta K_{ab}-4K_aK_b)-K\theta^2 \neq 0,$$

then the second fundamental tensor H_{ij} is proportional to

$$P_{ij} = \theta K_{ij}-4K_iK_j-\frac{K\theta^2}{2}g_{ij}.$$

If $P=0$, then the tensor P_{ij} vanishs.

II. On the case of hypersurfaces of general dimensions.

§ 5. The covariant derivatives of the second fundamental tensor.

In the first place, we shall generalize the equations (1. 7) to the case of a hypersurface S^n $(n\geqq 3)$ of an euclidean $(n+1)$-space. We denote also by $g_{ij}(x)$ and $H_{ij}(x)$ the first and second fundamental tensors of S respectively. The components H_{ij} satisfy the equations of *Gauss* and *Codazzi* as follows.

$$R_{hijk} = H_{h[j}H_{ik]}, \qquad H_{i[j,k]} = 0.$$

The scalar $g^{ab}H_{ab}$ is called the *mean curvature* of S and is denoted by M.

Contraction of the Gauss equation by g^{hj} gives

$$R_{ik} = MH_{ik}-H_i^aH_{ak}, \tag{2.1}$$

where $R_{ik}=g^{ab}R_{aibk}$ is called the Ricci tensor of S. Differentiate (2. 1) covariantly by x^l; we then have

$$R_{ik,l} = M_lH_{ik}+MH_{ik,l}-H_{ai,l}H_k^a-H_{ak,l}H_i^a. \tag{2.2}$$

We introduce the tensor S_{ijk}, components of which are defined by

$$S_{ijk} = R_{ij,k}+R_{ik,j}-R_{jk,i}.$$

We remark that the tensor S_{ijk} is symmetric with respect to the last two indices. When the substitution is made from (2. 2) in the right-hand members of the above, the equation

$$(2.3)\qquad 2H^a_i H_{aj,k} = MH_{ij,k} + H_{i(j} M_{k)} - H_{jk} M_i - S_{ijk}$$

is obtained. Multiply (2. 3) by H_{hl}, interchange the indices i, l, and subtract; when use is made of the Gauss equation, we then obtain

$$(2.4)\qquad 2R_{h\cdot li}^{\ a} H_{aj,k} = MH_{h[l} H_{i]j,k} + M_{(j} R_{k)hil} - H_{jk} M_{[i} H_{l]h} - H_{h[l} S_{i]jk}\,.$$

First, we contract (2. 4) by g^{hl} and substitute from (2. 3). Then the resulting equation is written in the form

$$(2.5)\qquad 2R^a_i H_{aj,k} = \frac{M^2}{2} H_{ij,k} + M_a H^a_i H_{jk} - \frac{M}{2} M_{(i} H_{jk)}{}^{*} + M_{(j} R_{k)i} + H^a_i S_{ajk} - \frac{M}{2} S_{ijk}\,.$$

Next, we contract (2. 4) by g^{hl}, and substitute from (2. 1) and (2. 3). Then, after change of the indices, we have

$$(2.6)\qquad 2R_{i\cdot bj}^{\ a} H^b_{a,k} = \frac{M^2}{2} H_{ij,k} + M_a H^a_k H_{ij} - \frac{M}{2} M_{(i} H_{jk)} + H_{ij} R_{,k} + M_{(j} R_{k)i} + M_a R_{j\cdot ki}^{\ a} - H^a_i S_{jka} - \frac{M}{2} S_{ijk}\,,$$

where $R_{,k}$ is the covariant derivative of the scalar curvature $R = g^{ab} R_{ab}$ of S.

We shall use some equations, which will be deduced from the equation of Gauss directly. Those equations have been used by Thomas [4, (8. 2)] in the theory of Riemannian spaces of class one. That is, multiply the Gauss equation by H_{lm}, interchange the indices m, k, and subtract; when use is made of the Gauss equation, we then obtain

$$H_{l[m} R_{k]jih} - H_{j[h} R_{i]lmk} = 0\,.$$

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* We use the symbol (ijk), which means, for example,
$M_{(i} H_{jk)} = M_i H_{jk} + M_j H_{ki} + M_k H_{ij}$.

On contracting by $g^{im}g^{jh}$, we have

$$R_{ka}H^a_l - RH_{kl} + MR_{kl} + R_{k\cdot bl}^{\,a}H^b_a = 0\,.$$

Differentiate this equation covariantly by x^j and change the indices ; the resulting equation is

$$\begin{aligned} &R^a_i H_{aj,k} + R_{i\cdot bj}^{\,a} H^b_{a,k} + M_k R_{ij} - R,_k H_{ij} \\ &+ R_{ai,k} H^a_j + R_{i\cdot bj,k}^{\,a} H^b_a + MR_{ij,k} - RH_{ij,k} = 0\,. \end{aligned}$$

We substitute from (2.5) and (2.6) for the first two terms of the above equation, and then establish the final equation

$$\begin{aligned} (2.7)\qquad \theta H_{ij,k} = {} & 2M_a H^a_i H_{jk} - 2R_{kja} H^a_i + 2R_{ai,k} H^a_j \\ & + 2R_{i\cdot bj,k}^{\,a} H^b_a - MM_{(i} H_{jk)} - R,_k\ H_{ij} \\ & + MS_{jki} + 2M_{(j} R_{k)i}\,, \end{aligned}$$

where we put $R_{kja} = R_{k[j,a]}$ and $\theta = 2R - M^2$. Consequently, if the scalar θ does not vanish, then the covariant derivatives of the second fundamental tensor H_{ij} are expressed by (2.7) in terms of the first fundamental tensor g_{ij}, the mean curvature M, their derivatives, and H_{ij}.

The covariant derivatives $H_{ij,k}$ are *symmetric* with respect to the three indices, by means of the Codazzi equation. In this case of general dimensions, contrary to the remark at the end of the first section, the right-hand member is not symmetric automatically. First, the symmetry with respect to the indices i, j is given by the equation

$$(2.8)\qquad H^a_{[i} S_{j]ka} = -M_a R_{k\cdot ij}^{\,a} - M_{[i} R_{j]k} - MR_{kij}\,,$$

where we made use of the Gauss equation. Next, the symmetry with respect to the indices j, k is given by the equation

$$R_{ajk} H^a_i + R_{ai,[j} H^a_{k]} + R_{i\cdot jk,b}^{\,a} H^b_a - \frac{1}{2} R,_{[j} H_{k]i} = MR_{ijk}\,.$$

The latter equation can be written in the simpler form, making use of the equation (2.8). That is, adding to (2.8) similar equations obtained from (2.8) by cyclic permutation on the indices i, j, k, we obtain $R_{a(ij}H^a_{k)} = 0$. Hence the above equation is written in the form

$$R_{a[i,k}H^a_{j]}+R_{k\cdot ij,b}^{\cdot a}H^b_a-\frac{1}{2}R,_{[i}H_{j]k}=MR_{kij}\,. \tag{2.9}$$

We shall see that the former equations (2. 8) will play a röle in the following, to determine H_{ij}.

§ 6. The conditions of integrability of the equations (2. 7).

Let us rewrite (2. 7) in the form, which is symmetric with respect to the indices i, j. To do this, we sum (2. 7) and the equation obtained from it by interchange of the indices i, j. Making use of the Gauss equation, the resulting equation is

$$\begin{aligned}\theta H_{ij,k} = {} & 2M_aH^a_kH_{ij}+S_{ak(i}H^a_{j)}-R,_kH_{ij} \\ & +2R_{i\cdot bj,k}^{\cdot a}H^b_a-MM_{(i}H_{jk)}+T_{ijk}\,,\end{aligned} \tag{2.10}$$

where T_{ijk} do not involve explicitly the components H_{ij} of the second fundamental tensor, such that

$$T_{ijk}=-M_aR_{(i\cdot j)k}^{\cdot a}+MR_{i,jk}+2M_{(i}R_{jk)}\,.$$

The process by means of which from (1. 7) we obtained the conditions of integrability (1. 8) is also applied to (2. 10), but the resulting equations are rather complicated. In order to write these equations, we now introduce the following quantities.

$$\begin{aligned}B_{hijk} &= \theta S_{hi[j,k]}-S_{hi[j}\theta_{k]}+g^{ab}S_{ai[j}S_{hbk]}-\theta^2R_{hijk}\,.\\ F_{hijklm} &= \theta R_{hjki,[l,m]}-R_{hjki,[l}\theta_{m]}+2R_{k\cdot jb,[l}^{\cdot a}R_{h\cdot ia,m]}^{\cdot b}\,,\\ &\quad +S_{ha[l}R_{(j\cdot k)i,m]}^{\cdot a}+S_{a(j[l}R_{|h|\cdot k)i,m]}^{\cdot a}\,,\\ G_{hijk} &= \theta T_{hi[j,k]}+T_{hi[j}(3MM_{k]}-R,_{k]})\\ &\quad -2g^{bc}R_{h\cdot ib,[j}^{\cdot a}T_{cak]}+g^{ab}S_{a(h[j}T_{|b|i)k]}\\ &\quad +2MM_aM_{(h}R_{i)\cdot jk}^{\cdot a}\,,\end{aligned}$$

these being skew-symmetric with respect to the last two indices, and furthermore

$$\begin{aligned}A_{ij} &= \theta M_{ij}-M_i\theta_j\,, \qquad D_{ij}=MM_{[i}R,_{j]}\,,\\ C_{ij} &= -\theta MM_{ij}-2(R+M^2)M_iM_j+MM_iR,_j\\ &\quad +Mg^{ab}M_aS_{bij}\,,\\ E_{hijk} &= 2M_aR_{(i\cdot j)h,k}^{\cdot a}-2M_hT_{ijk}-MS_{hk(i}M_{j)}\,.\end{aligned}$$

Then the conditions of integrability are written in the forms

$$\begin{aligned}& 2A_{a[k}H^a_{l]}H_{ij}-B_{a(i|kl|}H^a_{j)}+C_{(i[k}H_{j)l]}\\ & -4D_{kl}H_{ij}-E_{aij[k}H^a_{l]}-2F_{abijkl}H^{ab}-G_{ijkl}=0\,.\end{aligned} \tag{2.11}$$

§7. The equations determining the second fundamental tensor.

We have the three systems of equations (2. 8), (2. 9) and (2. 11), which must be satisfied by the components H_{ij} of the second fundamental tensor and will be useful to determine H_{ij}. In this paper, we shall treate the first system (2. 8) and denote by U_{kij} the right-hand side of (2. 8), namely

$$U_{kij} = -MR_{k(i,j)} - M_a(R_{k\cdot ij}^{\;a} - R_{ki}\delta_j^a + R_{kj}\delta_i^a)\,.$$

First we shall deduce from (2. 8) the system, which is more useful to determine H_{ij} explicitly. Multiply (2. 8) by H_{hl} and take the sum of this and the two similar equations obtained from it by cyclic permutation of the indices i, j, l; when we make use of the Gauss equation, the equation

$$U_{k(ij}H_{l)h} = T_{hklij} \tag{2.12}$$

is obtained, where T_{hklij} are the components of the tensor, which is determined by the first fundamental tensor g_{ij}, such that

$$T_{hklij} = R_{h\cdot(ij}^{\;a}S_{l)ka}\,.$$

Next, multiply (2. 12) by H_{pm}, subtract from it the equation obtained by interchange of the indices h, m, and substitute from the Gauss equation. We then have

$$H_{i(k}T_{l)jabc} = Q_{lkijabc}\,, \tag{2.13}$$

where by definition

$$\begin{aligned} Q_{lkijabc} = -U_{j(ab}R_{c)ikl} &= MR_{j(ab}R_{c)ikl} \\ &+ M_d(R_{j\cdot(ab}^{\;d} - R_{j(a}\delta_b^d + R_{j(b}\delta_a^d)R_{c)ikl}\,. \end{aligned}$$

Throughout the remainder af this paper, *capital indices* $A, B, C, \cdots$, are used, for brevity, to show the permutations of four Latin indices, such that $T_{hklij} = T_{hA}$, $Q_{lkijabc} = Q_{lkiB}, \cdots$. By using these capital indices, the equation (2. 13) can be written in the form

$$H_{ik}T_{lA} - H_{il}T_{kA} = Q_{lkiA}\,. \tag{2.13'}$$

On multiplying (2.13′) by T_{jB}, the left hand member of the resulting equation becomes, in virtue of (2. 13′),

$$\begin{aligned} &H_{ik}T_{jB}T_{lA} - H_{il}T_{jB}T_{kA} \\ &= (H_{ij}T_{kB} + Q_{jkiB})T_{lA} - (H_{ij}T_{lB} + Q_{jliB})T_{kA}\,. \end{aligned}$$

Hence we obtain

$$(2.14)\qquad (T_{lA}T_{kB}-T_{lB}T_{kA})H_{ij} = T_{jB}Q_{lkiA}-T_{[kA}Q_{l]jiB}\,.$$

Furthermore, we shall deduce another equations from (2. 13′), which will be used to determine H_{ij}. Multiply (2. 13′) by H_{hm} and take the sum of this and two equations obtained from it by cyclic permutation of the indices l, k, m. When the substitution is made from the Gauss equation, we then obtain

$$H_{h(m}Q_{lk)iA} = R_{ih(ml}T_{k)A}\,.$$

Multiply (2. 13′) again by H_{hm}, interchange the indices k, m, and subtract. By means of the equation of Gauss, (2. 13′) and the equation as above found, we then have

$$(2.15)\qquad H_{li}Q_{kmhA}-H_{lh}Q_{kmiA} = T_{lA}R_{ihkm}\,.$$

The process by means of which from (2. 13′) we obtained (2. 14) is applied to (2. 15), and then the following equation is easily established.

$$(2.16)\qquad (T_{lB}Q_{kmhA}-T_{hB}Q_{kmlA})H_{ij} = T_{iA}T_{jB}R_{lhkm}+Q_{km[hA}Q_{l]ijB}\,.$$

Now, we obtained the systems (2. 14) and (2. 16) as the equations determining the second fundamental tensor H_{ij} explicitly. It is natural that, according as the coefficients of H_{ij} in these equations are to be equal to zero or not, the hypersurfaces under consideration are divided into the following four cases.

The case A: The tensor $T_{ijAB}=T_{iA}T_{jB}-T_{iB}T_{jA}$ does not vanish.

The case B: The tensor T_{iA} does not vanish, but the above tensor $T_{ijAB}=0$. And the tensor $T_{iA}Q_{jklB}-T_{lA}Q_{jkiB}$ does not vanish.

The case C: The tensor T_{iA} does not vanish also, but T_{ijAB} and $T_{iA}Q_{jklB}-T_{lA}Q_{jkiB}$ are equal to zero.

The case D: The tensor T_{iA} is equal to zero.

In the remainder of this paper, we shall treate these four cases separately. We shall see that hypersurfaces of type two shall play a röle in the following discussions.

§ 8. On the case A.

In this general case, there exists at least one component $T_{lkAB} \neq 0$, and hence all of the components H_{ij} of the second fundamental tensor are completely determined by the equations (2.14) in terms of the first fundamental tensor g_{ij}, the mean curvature M and their derivatives. The mean curvature M has to satisfy the differential equations

$$(2.17) \qquad \begin{aligned} &T_{lkAB}(T_{jD}Q_{qpiC} - T_{[pC}Q_{q]jiD}) \\ &- T_{qpCD}(T_{jB}Q_{lkiA} - T_{[kA}Q_{l]jiB}) = 0, \end{aligned}$$

because these equations mean the fact that (2.14) can admit the solutions H_{ij}. Then, by means of the definitions of T_{iA} and Q_{ijkA}, the components H_{ij} are expressible in the form

$$(2.18) \qquad H_{ij} = ML_{ij} + M_a L^a_{ij} ,$$

where L_{ij} and L^a_{ij} are components of the intrinsic tensors of S, namely, these are defined by the first fundamental tensor of S and their derivatives. The condition that H_{ij} as thus determined are symmetric is given by

$$(2.19) \qquad T_{[iA}Q_{klj]B} - T_{[kB}Q_{ijl]A} = 0 ,$$

in which we have used the properties $Q_{(ij)kA} = 0$ and $Q_{(ijk)A} = 0$. Consequently we obtain

Theorem 3. *In the case A, the second fundamental tensor H_{ij} of the hypersurface S is determined by the equations* (2.14). *The mean curvature M of S satisfies the differential equations* (2.17) *and* (2.19). *Then H_{ij} are expressible in the form* (2.18), *where L_{ij} and L^a_{ij} are components of the intrinsic tensors of S.*

We are solely interested here in determining the expressions of H_{ij}. It should be remarked moreover that we have the equations of Gauss and Codazzi, and the equation $M = g^{ab}H_{ab}$. These are now looked upon as the differential equations, which have to be satisfied by the mean curvature M, after substitution in these equations from (2.18). It is easy to write these equations explicitly, using (2.14). In fact, if we denote by Q^*_{ijlkAB} the right-hand member of (2.14), then the equations of Gauss are written in the form

$$R_{hijk}T_{pqAB}T_{rsCD} = Q^{*}_{[h|jpqAB}Q^{*}_{i|k]rsCD}\,.$$

The equations of Codazzi become

$$T_{pqAB,[k}Q^{*}_{ij]rsCD} = T_{rsCD}Q^{*}_{i[jpqAB,k]}\,,$$

Finally the equation defining the mean curvature M is

$$MT_{ijAB} = g^{ab}Q^{*}_{abijAB}\,.$$

§ 9. On the case B.

Since $T_{ijAB}=T_{iA}T_{jB}-T_{iB}T_{jA}=0$ in this case, we see clearly that the components of the tensor T_{iA} are decomposable to the form

(2. 20) $$T_{iA} = \lambda_i \mu_A\,.$$

In the following discussion, it is convenient to take an unit vector λ_i satisfying (2. 20), and we then show easily that such a vector λ_i is uniquely determined to within algebraic sign.

It follows from (2. 14)

(2. 21) $$T_{jB}Q_{lkiA}-T_{[kA}Q_{l]jiB} = 0\,.$$

Multiply (2. 21) by T_{hC}, interchange the indices A, C, and subtract; when use is made of (2. 20), we then have $\mu_{[C}Q_{lkiA]}=0$. Hence, Q_{ijkA} are also decomposed to the form

(2. 22) $$Q_{ijkA} = Q_{ijk}\mu_A\,.$$

from which it follows that (2. 21) becomes

(2. 23′) $$\lambda_{[i}Q_{jk]l} = 0\,.$$

Then we see that (2. 16) is written in the form

(2. 24) $$Q_{km[h}\lambda_{l]}H_{ij} = \lambda_i\lambda_j R_{lhkm}+Q_{km[h}Q_{l]ji}\,.$$

Because of the hypothesis that $Q_{km[h}\lambda_{l]}$ does not vanish, then H_{ij} are uniquely determined by the above equations, but it seems to be impossible that H_{ij} are expressed in the simple form similar to (2. 18). The conditions that (2. 24) admits the solutions H_{ij} are that

(2. 25′) $$Q_{km[h}\lambda_{l]}(\lambda_i\lambda_j R_{abcd}+Q_{cd[b}Q_{a]ji}) - Q_{cd[b}\lambda_{a]}(\lambda_i\lambda_j R_{lhkm}+Q_{km[h}Q_{l]ji}) = 0\,,$$

and further the symmetry of H_{ij} is given by

(2. 26) $$Q_{km[h}Q_{jil]} = 0\,.$$

Now, we show that the equations (2.23′) are written in the form

$$(2.23)\qquad Q_{jkl} = \lambda_j \tilde{Q}_{kl} - \lambda_k \tilde{Q}_{jl}\,.$$

In fact, contracting (2.23′) with λ^i, we have the above equations, where we putted $\tilde{Q}_{kl} = \lambda^a Q_{akl}$. Conversely we can easily see that (2.23′) is a consequence of (2.23). Next, contracting (2.26) with $\lambda^j \lambda^k$, we have $\tilde{Q}_{m[h} \tilde{Q}_{il]} = 0$, so that the matrix $(\tilde{Q}_{ij})$ is of rank one. We see immediately that the condition of the matrix $(\tilde{Q}_{ij})$ is equivalent to (2.26). Make use of (2.23); then the equations (2.25′) are written in the form

$$(2.25)\qquad \lambda_i (Q_{km[h} \lambda_{l]} R_{abcd} - Q_{cd[b} \lambda_{a]} R_{lhkm}) = Q_{km[h} Q_{l][ai} Q_{cdb]}\,.$$

Thus we conclude that

Theorem 4. *For the hypersurface S of the case B, the tensors* T_{iA} *and* Q_{ijkA} *are decomposable to the forms* (2.20) *and* (2.22) *respectively, and* Q_{ijk} *is written in the form* (2.23), *where the rank of the matrix* $(\tilde{Q}_{ij})$ *is one. The second fundamental tensor* H_{ij} *is uniquely determined by the equations* (2.24), *and the mean curvature* M *satisfies the differential equations* (2.25).

It should be remarked lastly that we have further differential equations satisfied by the mean curvature M, as mentioned at the end of the last section. But we are not interested now in writting these equations.

§ 10. On the case C.

We have, in this case also, the equations (2.20), (2.22) and (2.23′). Since the assumption $T_{iA} Q_{jklB} - T_{lA} Q_{jkiB} = 0$ is then written in the form $\lambda_{[i} Q_{jkl]} = 0$, the tensor Q_{jkl} is moreover decomposed to the form $Q_{jkl} = Q_{jk} \lambda_l$. Thus we have

$$(2.27)\qquad Q_{ijkA} = Q_{ijk} \mu_A\,, \qquad Q_{ijk} = Q_{ij} \lambda_k\,,$$

and the equation (2.23′) becomes

$$(2.28)\qquad \lambda_{[i} Q_{jk]} = 0\,.$$

The coefficients of H_{ij} in (2.24) vanish as a consequence of the hypothesis of the case C, from which it follows that the right-hand side of (2. 24) is equal to zero. Therefore, with the aid of (2. 27) and (2. 28), we obtain

$$R_{lhkm} = -Q_{lh}Q_{km}\,. \tag{2.29}$$

Consequently we see that the hypersurface under consideration is *of separated curvature* [3]. It is known [3] that *the skew-symmetric tensor* Q_{ij} *is uniquely determined by the intrinsic quantities of* S, *and the matrix* (Q_{ij}) *is of rank two.*

Now, the equations (2. 13′) and (2. 15) are written as

$$H_{ik}\lambda_l - H_{il}\lambda_k = -\lambda_i Q_{kl}\,. \tag{2.30}$$

Contract (2. 30) by $\lambda^l = g^{la}\lambda_a$; if we put $H_i = H_{ia}\lambda^a$ and $Q_i = Q_{ia}\lambda^a$, we then have

$$H_{ik} = H_i\lambda_k - \lambda_i Q_k\,,$$

because we choosed $\lambda_a\lambda^a = 1$. In virtue of the symmetry of H_{ik}, we obtain $H_{[i}\lambda_{k]} = \lambda_{[i}Q_{k]}$ from the above. Next, we contract (2. 30) by λ^i and we have $H_{[k}\lambda_{l]} = -Q_{kl}$. Therefore we obtain

$$\lambda_i Q_k - \lambda_k Q_i = -Q_{ik}\,. \tag{2.31}$$

By applying the equation (2. 30) we show that

$$\begin{aligned} H_{ij}\lambda_k\lambda_l &= (H_{ik}\lambda_j - \lambda_i Q_{jk})\lambda_l \\ &= (H_{kl}\lambda_i - \lambda_k Q_{il})\lambda_j - \lambda_i Q_{jk}\lambda_l\,. \end{aligned}$$

If we contract this by g^{kl} and recall the definition of the mean curvature, we then have

$$H_{ij} = M\lambda_i\lambda_j - Q_i\lambda_j - Q_j\lambda_i\,. \tag{2.32}$$

Gathering the foregoing results together, we have

Theorem 5. *The hypersurface* S *of the case* C *is necessarily of separated curvature. The tensors* T_{iA} *and* Q_{ijkA} *of* S *are decomposable to the form* (2. 20) *and* (2. 27) *respectively, where* Q_{ij} *is the intrinsic tensor of* S, *which is defined by* (2. 29). *Then the second fundamental tensor* H_{ij} *is completely determined by the equation* (2. 32), *where* $Q_i = Q_{ia}\lambda^a$.

It should be remarked that the components H_{ij} are given by the linear equations (2. 32), which do not involve the derivatives of the mean curvature M.

It is easy to verify that H_{ij} as thus determined by (2.32) satisfy the Gauss equation automatically, with the aid of (2.31), and furthermore the equation $M=g^{ab}H_{ab}$ by means of $Q_a\lambda^a =Q_{ab}\lambda^a\lambda^b=0$. On the other hand, the Codazzi equation will be looked upon as the differential equations, which have to be satisfied by M. These can be written explicitly by using the equations (2.32).

§ 11. On the case D.

We consider the exceptional case D. By means of the assumption $T_{iA}=0$, we have $Q_{ijkA}=0$ from (2.13′), and

$$U_{k(ij}H_{l)h} = 0, \tag{2.33}$$

in virtue of (2.12). Multiply (2.33) by H_{pm}, interchange the indices l, m, and subtract; when use is made of the Gauss equation and (2.33), we then obtain $U_{kij}R_{lmhp}=U_{klm}R_{ijhp}$. If the tensor U_{ijk} does not vanish, we then can choose a vector λ^i, such that the tensor $\lambda^a U_{aij}=U_{ij}$ does not vanish. Then we have $U_{ij}R_{lmhp} =U_{lm}R_{ijhp}$, from which it follows that $R_{hijk}=\rho U_{hi}U_{jk}$, where ρ is a scalar. Therefore we see that the hypersurface under consideration is *of separated curvature*, and so the curvature tensor is written in the form

$$R_{hijk} = eQ_{hi}Q_{jk}, \qquad (e = \pm 1). \tag{2.34}$$

The tensor U_{ij} is proportional to Q_{ij}, and there exists a vector μ_k such that

$$U_{kij} = \mu_k Q_{ij}. \tag{2.35}$$

Consequently we have

Theorem 6. *For the hypersurface S of the case D, the mean curvature M satisfies the differential equations* $Q_{ijkA}=0$. *If M does not satisfy the differential equations* $U_{ijk}=0$, *S is of separated curvature, and* U_{ijk} *is decomposable to the form* (2.35), *where* Q_{ij} *is defined by* (2.34).

It is to be noted that a hypersurface of the case D is not always to be of separated curvature. To show this fact, we consider a hypersurface S of *constant curvature K*. Then we have

$$R_{hijk} = Kg_{h[j}g_{ik]}, \qquad R_{ik} = (n-1)Kg_{ik}.$$

It follows that the Ricci tensor R_{ik} is covariant constant, and hence we obtain $S_{ijk}=0$. Thus we see that $T_{iA}=0$. It is well known that, for the hypersurface S of constant curvature, the second fundamental tensor H_{ij} is proportional to the metric tensor g_{ij}, such that $H_{ij}=\sqrt{K}\, g_{ij}$, and hence the mean curvature M is constant. Therefore we obtain $U_{ijk}=0$. Thus the hypersurface S belongs to the case D. Besides, S is of type $n\geqq 3$, and so is *not* of separated curvature.

In this case D, we have further equations (2. 9), (2. 11) and (2. 8). By means of these three systems of equations, H_{ij} may be determined. But such discussion will be rather complicated.

§ 12. On symmetric hypersurfaces.

In the final section we consider a hypersurface, which is symmetric in the sense of *E. Cartan*. That is characterized by the property, that the curvature tensor R_{hijk} is covariant constant. For the symmetric hypersurface S, the equation (2. 9) is satisfied identically, and we see that (2. 8) becomes

$$M_a R_{k\cdot ij}^{\;a} = R_{ki}M_j - R_{kj}M_i \,. \tag{2. 36}$$

Now, we have only one system (2. 11), which involves H_{ij}, and we shall study how to determine H_{ij} by means of this system.

It is clear that the tensors D_{ij} and F_{hijklm}, which appear in the equations (2. 11), are equal to zero, and further A_{ij}, B_{hijk} and E_{hijk} become respectively

$$A_{ij} = \theta M_{ij} + 2MM_iM_j \,,$$
$$B_{hijk} = -\theta^2 R_{hijk} \,,$$
$$E_{hijk} = -2M_h T_{ijk} \,.$$

The equations (2. 11) are now written in the form

$$2A_{a[k}H^a_{l]}H_{ij} + \theta^2 R_{kla(i}H^a_{j)} + C_{(i[k}H_{j)l]}$$
$$+2M_a T_{ij[k}H^a_{l]} - G_{ijkl} = 0 \,.$$

Making use of the Gauss equation and (2. 1), the second term is rewritten in the form

$$R_{kla(i}H^a_{j)} = H_{a[k}H^a_{(j}H_{i)l]}$$
$$= (MH_{[k(j} - R_{[k(j})H_{i)l]} = -R_{(i[k}H_{j)l]} \,.$$

Therefore we obtain

$$2A_{a[k}H^{a}_{l]}H_{ij}+C^{*}_{(i[k}H_{j)l]} + 2M_{a}T_{ij[k}H^{a}_{l]}-G_{ijkl} = 0\,, \tag{2.37}$$

where we put $C^{*}_{ik}=C_{ik}-\theta^{2}R_{ik}$. Multiply (2.37) by H_{hm} and take the sum of it and the similar two equations obtained from it by cyclic interchange of the indices k, l, m; where use is made of the Gauss equation, we then obtain

$$2A_{a(l}R^{a}_{\cdot|h|mk)}H_{ij}+C^{*}_{i(l}R_{mk)jh}+C^{*}_{j(l}R_{mk)ih} + 2T_{ij(l}R^{a}_{\cdot|h|mk)}M_{a}-G_{ij(kl}H_{m)h} = 0\,.$$

It follows from (2.36) that $M_{a}R^{a}_{k\cdot(ij}M_{h)}=0$, and further $M_{a(l}R_{|k|}{}^{a}_{\cdot ij)}=0$, in virtue of $R_{h\cdot jk,l}^{\;i}=0$. Hence the coefficients of H_{ij} in the above equations vanish, and we have

$$G_{ij(kl}H_{m)h} = C^{*}_{i(l}R_{mk)jh}+C^{*}_{j(l}R_{mk)ih} + 2T_{ij(l}R^{a}_{\cdot|h|mk)}M_{a}\,. \tag{2.38}$$

We observe that the equation (2.38) is clearly to be the similar form with the equation (2.12), and hence, the process by means of which from (2.12) we deduced (2.13), (2.14), (2.15) and (2.16) successively, is applied equally well to the equation (2.38). Thus it is not difficult to obtain the similar equations, which are useful in general to determine the second fundamental tensor of S explicitly. We will have no occasion to study the detail of those theories.

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[4] T. Y. Thomas: *Riemann spaces of class one and their characterization.* Acta Math., 67 (1936) 169-211.
[5] T. Y. Thomas: *Algebraic determination of the second fundamental form of a surface by its mean curvature.* Bull. Amer. Math. Soc., 51 (1945), 390-399.

REMARK. After writting the present paper, I was informed of the following paper in Mathematical Review.

[6] Hu, Hou-Sung: *On the deformation of a Riemannian metric V_m of class I which preserves the mean curvature.* Acta Math. Sinica, 6 (1956), 127-137. (Chinese, English summary)

According to the author's summary, I think that the purpose and method are rather different from the one of the present paper, though Thomas's paper [5] is the common starting point.