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The Bessel motion and a singular integral equation

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1. INTRODUCTION

Given a standard $n(\geq 2)$ dimensional Brownian motion with sample paths $\mathfrak{b}(t)$ $(t\geq 0)$ and generator $\mathfrak{G}=\frac{1}{2}\left(\frac{\partial^2}{\partial b_1^2}+\cdots+\frac{\partial^2}{\partial b_n^2}\right)$, its radial part $\mathfrak{r}(t)=|\mathfrak{b}(t)|\,(t\geq 0)$ is the Bessel motion with generator $\mathfrak{G}^+=\frac{1}{2}\left(\frac{d^2}{d\mathfrak{r}^2}+\frac{n-1}{\mathfrak{r}}\frac{d}{d\mathfrak{r}}\right)$; in fact, if $P_{\mathfrak{a}}(B)$ is the n-dimensional Wiener measure of the event B as a function of the starting point $\mathfrak{a}=\mathfrak{b}(0)$ of the Brownian path and if $t_1<t_2$, then

1.1
$$P.[\mathfrak{r}(t_2)\leq l\,|\,\mathfrak{r}(s):\ s\leq t_1] = \int_{|\mathfrak{b}-\mathfrak{a}|\leq l} (2\pi t)^{-n/2}e^{-|\mathfrak{b}-\mathfrak{a}|^2/2t}\,d\mathfrak{b}$$
$$t=t_2-t_1,\quad \mathfrak{a}=\mathfrak{b}(t_1)$$

depends upon $|\mathfrak{a}|=\mathfrak{r}(t_1)$ alone, *i.e.*, $[\mathfrak{r}, P.]$ is Markov, and the identification of its generator as $\mathfrak{G}^+$ (=the radial part of $\mathfrak{G}$) is immediate.

Because $\mathfrak{G}^+$ is the sum of the generator of the standard 1-dimensional Brownian motion and the generator of translation at speed $\dot{\mathfrak{r}}=\frac{n-1}{2}\mathfrak{r}^{-1}$, it is plausible that if $b(t)$ $(t\geq 0)$ is a 1-dimensional Brownian motion with $b(0)\geq 0$, then the solution of

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1) Fulbright grantee 1957-58, during which time section 2 of this paper was worked out.

1.2 $$\mathfrak{r}(t) = b(t)+\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds \qquad t\geq 0$$

should be a Bessel motion starting at $\mathfrak{r}(0)=b(0)$; this is not correct because, if $b(0)=0$, then 1.2 has both a non-positive and a non-negative solution, but it becomes correct, if, as will be understood below, *solution* means *non-negative solution*[2].

Given 2 non-negative solutions $\mathfrak{r}_1$ and $\mathfrak{r}_2$ of 1.2, their difference e satisfies $e=-\int_0^t e/\mathfrak{r}_1\mathfrak{r}_2\, ds$, and, using $\int_0^t |e/\mathfrak{r}_1\mathfrak{r}_2|ds\leq\int_0^t(\mathfrak{r}_1^{-1}+\mathfrak{r}_2^{-1})ds< +\infty$, it follows from $e\dot{e}=-e^2/\mathfrak{r}_1\mathfrak{r}_2\leq 0$, that $e^2\equiv 0$, *i.e.*, 2 has at most 1 non-negative solution.

Now the trick is to prove that if $\mathfrak{r}$ is a Bessel motion, then $b\equiv\mathfrak{r}-\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds$ is a Brownian motion and to use the 1:1 nature of the map $b\to\mathfrak{r}$ to conclude that, neglecting a class of Brownian paths b of Wiener measure 0, 1.2 has a non-negative Bessel distributed solution $\mathfrak{r}$.

2. PROVING THAT $b\equiv\mathfrak{r}-\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds$ IS BROWNIAN

Given a Bessel motion $[\mathfrak{r}, P.]$ as described above, an application of

2.1 $$E.\left[\int_0^t \mathfrak{r}^{-1}ds\right] = \int_0^t ds\int(2\pi s)^{-n/2}e^{-|\mathfrak{b}|^2/2s}|\mathfrak{b}|^{-1}d\mathfrak{b}$$
$$= \int_0^t \frac{ds}{\sqrt{s}}(2\pi)^{-n/2}e^{-|\mathfrak{b}|^2/2}|\mathfrak{b}|^{-1}d\mathfrak{b}$$
$$< +\infty$$

shows that $b\equiv\mathfrak{r}-\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds$ is well defined, and, using the Markovian nature of the Bessel motion, it is found that, if $t_2>t_1$, then

2.2 $$E.[e^{i\alpha b(t_2)}\,|\,\mathfrak{r}(s):\ s\leq t_1]$$
$$= e^{i\alpha b(t_1)}e^{-i\alpha\mathfrak{r}(t_1)}E.[e^{i\alpha\mathfrak{r}(t_2)-\frac{n-1}{2}\int_{t_1}^{t_2}\mathfrak{r}^{-1}ds}\,|\,\mathfrak{r}(s):\ s\leq t_1]$$
$$= e^{i\alpha b(t_1)}e^{-i\alpha\mathfrak{r}(t_1)}E_{\mathfrak{c}}[e^{i\alpha(\mathfrak{r}(t)-\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds)}]$$
$$t = t_2-t_1, \quad |\mathfrak{a}| = \mathfrak{r}(t_1).$$

2) See K. Itô [3] for the ideas behind this.

Now the evaluation of such Bessel expectations is routine:[3]

2.3 $$u(t,\mathfrak{r}) \equiv E_{\cdot}[e^{i\alpha(\mathfrak{r}(t)-\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds)}] \qquad \mathfrak{r}=|a|$$

is the bounded solution

2.4 $$e^{i\alpha\mathfrak{r}}e^{-\alpha^2 t/2}$$

of

2.5 a $$\frac{\partial u}{\partial t} = \mathfrak{G}^+u - \frac{i\alpha(n-1)}{2\mathfrak{r}}u = \frac{1}{2}\left[\frac{\partial^2}{\partial \mathfrak{r}^2}+\frac{n-1}{\mathfrak{r}}\frac{\partial}{\partial \mathfrak{r}}-\frac{i\alpha(n-1)}{\mathfrak{r}}\right]u$$

2.5 b $$u(0+,\mathfrak{r}) = e^{i\alpha\mathfrak{r}},$$

and, using the fact that $b(s): s\leq t_1$ is a Borel function of $\mathfrak{r}(s): s\leq t_1$ and inserting 2.4 into 2.2, it is seen that

2.6 a $$E.[e^{i\alpha b(t_2)}\,|\,b(s): s\leq t_1] = e^{i\alpha b(t_1)}e^{-\alpha^2 t/2} \qquad t=t_2-t_1,$$

or, what is the same,

2.6 b $$P.[b(t_2)\in db\,|\,b(s): s\leq t_1] = \frac{e^{-(b-a)^2/2t}}{\sqrt{2\pi t}} \qquad t=t_2-t_1, \quad a=b(t_1),$$

i.e., that b is Brownian.

3. $\mathfrak{r}=b+\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds$ IS SOLVABLE FOR EACH CONTINUOUS b ($b(0)\geq 0$)

Because $\mathfrak{r}=b+\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds$ is singular (at $\mathfrak{r}=0$), it is not clear that it has a non-negative solution for *each* continuous b ($b(0)\geq 0$). Consideration of the map $j: \mathfrak{r}\to b+\frac{n-1}{2}\int_0^t \mathfrak{r}^{-1}ds$ defined on the class of non-negative continuous $\mathfrak{r}$ such that $\int_0^t \mathfrak{r}^{-1}ds<+\infty$ ($t\geq 0$) converts the problem into showing that j has a fix point; the method of Leray-Schauder should be able to decide this, but I had no success with it. Here, I shall present another method, putting $n=3$ to eliminate the nuisance factor $(n-1)/2$.

$b(0)>0$ implies the existence of a positive solution for $t<t_1 = \min(t: b(t)=0)$ because, if $t\leq t_2<t_1$, if $\varepsilon=\min_{t\leq t_2} b(t)$, and if

3) See, for example, R. Hasminskii [2].

3.1 $$\mathfrak{r}_{n+1} = b+\int_0^t \mathfrak{r}_n^{-1}ds \qquad n\geq 1, \quad \mathfrak{r}_1\equiv b\,,$$

then

3.2 $$\begin{aligned} &\mathfrak{r}_1\leq \mathfrak{r}_3 < \mathfrak{r}_5 < etc. \\ &< \mathfrak{r}_- = \lim_{n\uparrow+\infty} \mathfrak{r}_{2n-1} \\ &\leq \mathfrak{r}_+ = \lim_{n\uparrow+\infty} \mathfrak{r}_{2n} \\ &< etc. < \mathfrak{r}_6 < \mathfrak{r}_4 < \mathfrak{r}_2\,, \end{aligned}$$

3.3 $$\mathfrak{r}_\pm = b+\int_0^t \mathfrak{r}_\mp^{-1}ds\,,$$

3.4 $$e = -\int_0^t (e/\mathfrak{r}_-\mathfrak{r}_+)ds \leq \varepsilon^{-2}\int_0^t e ds \qquad t\leq t_2\,, \quad e = \mathfrak{r}_+-\mathfrak{r}_-\geq 0\,,$$

and, iterating 3.4 and letting $t_2\uparrow t_1$, $\mathfrak{r}_-\equiv\mathfrak{r}_+$ is found to be the desired solution.

Given a solution $\mathfrak{r}$ that cannot be continued past $t_1>0$, it must be that

3.5 $$b^-(t_1)\equiv \lim_{t\uparrow t_1}(t_1-t)^{-1}[b(t_1)-b(t)] = -\infty\,;$$

indeed, $\int_0^t \mathfrak{r}^{-1}ds\in\uparrow(t<t_1)$ implies $\mathfrak{r}(t_1-)=b(t_1)+\int_0^{t_1}\mathfrak{r}^{-1}ds$; here, $\mathfrak{r}(t_1-)$ cannot be $=+\infty$ because $\int_0^{t_1}\mathfrak{r}^{-1}ds$ would then be $<+\infty$, and $\mathfrak{r}(t_1-)$ cannot be >0 because then the solution of $\mathfrak{r}_1=[b(t+t_1)-b(t_1)+\mathfrak{r}(t_1-)]+\int_0^t \mathfrak{r}_1^{-1}ds$ would effect a continuation of $\mathfrak{r}$ past t_1; thus, $\mathfrak{r}(t_1-)$ has to be 0, and so

3.6 $$\begin{aligned}(t_1-t)^{-1}[b(t_1)-b(t)] &= (t_1-t)^{-1}\left[-\mathfrak{r}(t)-\int_t^{t_1}\mathfrak{r}^{-1}ds\right] \\ &\leq -(t_1-t)\int_t^{t_1}\mathfrak{r}^{-1}ds \sim -\infty \qquad t\uparrow t_1\,.\end{aligned}$$

Consider the case $b(0)>0$ and let t_∞ be the supremum of positive times t_1 such that, for some continuous perturbation $h\in\uparrow(0<h\leq 1)$, $\mathfrak{r}=b+\varepsilon h+\int_0^t \mathfrak{r}^{-1}ds$ $(t<t_1)$ has a *positive* solution for *each* $0<\varepsilon\leq 1$.

$t_\infty>0$ because $b(0)>0$; in fact, $t_\infty\geq\min(t: b(t)=0)$.

$t_\infty=+\infty$ because, if $t_1<t_2<etc.\uparrow t_\infty<+\infty$, if h_1, h_2, *etc.* are the corresponding perturbations, and if $h\equiv\sum_{n\geq 1}2^{-n-1}h_n+\frac{1}{2}h_\infty$ with some continuous $h_\infty\in\uparrow(0<h_\infty\leq 1)$ such that $(b+\varepsilon h_\infty)^-(t_\infty)>$

$-\infty(0<\varepsilon\leq 1)$[4], then $\mathfrak{r}=b+\varepsilon h_\infty+\int_0^t \mathfrak{r}^{-1}ds$ is solvable up to time $t_0=\min(t: b(t)=0)$ at least; the solution $\mathfrak{r}_\varepsilon$ lies above the solution $\mathfrak{r}_n$ of $\mathfrak{r}=b+\varepsilon 2^{-n-1}h_n+\int_0^t \mathfrak{r}^{-1}ds$ because the difference $e=\mathfrak{r}_\varepsilon-\mathfrak{r}_n$ satisfies

3.7 $$\lim_{t\uparrow s}\frac{e(s)-e(t)}{s-t}\geq -e/\mathfrak{r}_\varepsilon\mathfrak{r}_n,$$

causing e to turn upwards as soon as it crosses $e=0$; $\mathfrak{r}_\varepsilon$ is then positive and continuable up to $t=t_n$, and, making $n\uparrow+\infty$ and using $(b+\varepsilon h_\infty)^-(t_\infty)>-\infty$, it is seen that $\mathfrak{r}_\varepsilon$ can be continued *up to* $t=t_\infty$ and *past*, contradicting the definition of t_∞.

But now the same proof shows that if $t_1<t_2<etc.\uparrow t_\infty=+\infty$, if h_1, h_2, *etc.* are the corresponding perturbations, and if $h=\sum_{n\geq 1}2^{-n}h_n$, then the solution $\mathfrak{r}_\varepsilon$ of $\mathfrak{r}=b+\varepsilon h+\int_0^t \mathfrak{r}^{-1}ds$ is continuable over $[0,+\infty)$ for each $0<\varepsilon\leq 1$; in addition, $\mathfrak{r}_{\varepsilon_1}<\mathfrak{r}_{\varepsilon_2}$ $(\varepsilon_1<\varepsilon_2)$, and, using monotone convergence, $\mathfrak{r}_{0+}=\lim_{\varepsilon\downarrow 0}\mathfrak{r}_\varepsilon$ is found to be a solution of $\mathfrak{r}=b+\int_0^t \mathfrak{r}^{-1}ds$ $(t<+\infty)$.

As to the case $b(0)=0$, it suffices to put $h=1$ above, to solve $\mathfrak{r}=b+\varepsilon+\int_0^t \mathfrak{r}^{-1}ds$ for each $\varepsilon>0$, and to make $\varepsilon\downarrow 0$ as before.

Given b_1, b_2 as above, if $\mathfrak{r}_1$, $\mathfrak{r}_2$ are the corresponding solutions of $\mathfrak{r}=b+\int_0^t \mathfrak{r}^{-1}ds$, then

3.8 $$|\mathfrak{r}_2(t)-\mathfrak{r}_1(t)|\leq 2\max_{s\leq t}|b_2-b_1|,$$

as will now be proved; it can be supposed that $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are positive because a small perturbation (εh) will make them so.

But, in that case, if $e=\mathfrak{r}_2-\mathfrak{r}_1$, if $\sigma=\int_0^t ds/\mathfrak{r}_1\mathfrak{r}_2$, and if $a=b_2-b_1$, then

3.9 $$e(t)=a(t)-e^{-\sigma(t)}\int_0^t ae^{\sigma}d\sigma,$$

and so

3.10 $$|e(t)|\leq[2-e^{-\sigma}]\max_{s\leq t}|a|.$$

4) h_∞ can be constructed as follows: define $h_\infty(0)=0$, let $h_\infty(t_\infty)-h_\infty(t)=\text{constant}\times\sqrt{e(t)}$, where $e(t)\equiv\max_{t_\infty\geq s\geq t}|b(t_\infty)-b(s)|$ for $t<t_\infty$, and, adjusting the constant (>0), fill in h_∞ on the rest of $[0,+\infty)$ so as to have $0<h_\infty\leq 1$ continuous and increasing; then, as $t\uparrow t_\infty$, $a=b+\varepsilon h_\infty$ satisfies

$$\frac{a(t_\infty)-a(t)}{t_\infty-t}\geq(t_\infty-t)^{-1}\sqrt{e}\,[\text{constant}\times\varepsilon-\sqrt{e}\,]\geq 0.$$

4. OTHER SINGULAR EQUATIONS

Given $\alpha>0$, the problem $\mathfrak{x}=b+\int_0^t ds/\mathfrak{x}^\alpha$ is amenable to the method of section 3, the point being that the $\mathfrak{x}$ on the left and the $1/\mathfrak{x}^\alpha$ under the integral sign balance, preventing the solution from getting too small or too big.

I do not know of any integral equation $\mathfrak{x}=b+\int_0^t k(\mathfrak{x}, s)ds$ that can be solved for *almost all* Brownian paths b but not for *all* continuous paths; it would be interesting to have such an example (see R. Cameron [1] and D. Woodward [4] for additional information on this point).

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