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On holomorphic families of fiber bundles over the Riemannian sphere[1]

By

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In a recent paper A. Grothendieck [11] classified the holomorphic fiber bundles over the Riemannian sphere P^1 whose structure group is reductive. This classification is based on the fact that each holomorphic vector bundle over P^1 splits into holomorphic line bundles, a result that is essentially due to G. D. Birkhoff [1]. In the present paper we raise the corresponding question for holomorphic families of fiber bundles $\mathcal{B} \rightarrow \mathcal{V}$ over a holomorphic family $\mathcal{V} \xrightarrow{\pi} M$ of Riemannian spheres, i.e. holomorphic fiber bundles whose base space is the total space of a holomorphic fiber bundle with fiber P^1. It turns out (Theorem 2.2) that the splitting theorem is locally still valid provided one avoids a 1-codimensional analytic subset A of the parameter space M of the family. The exceptional set A is empty if the restrictions of the vector bundle to any two fibers are isomorphic (Theorem 2. 4). The splitting theorem permits one to prove that the set of all points $t \in M$ fulfilling

$$\dim_C H^0(V_t, \Omega(B_t)) \geq j$$

for some integer j is an analytic subset of M (Theorem 2. 3). In addition one gets a counterpart to a theorem of K. Kodaira and D. C. Spencer ([14], Theorem 18. 1) stating that for any point $t_0 \in M$ there is a neighborhood U such that

$$H^0(\pi^{-1}(U), \Omega(\mathcal{B})) \rightarrow H^0(V_t, \Omega(B_t))$$

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is surjective for all points $t \in U-A$ where A is a 1-codimensional analytic subset of U. This remark leads to a short proof of a theorem of H. Grauert and R. Remmert [8] for our special case, saying that the direct image $\pi_0(\Omega(\mathscr{B}))$ is analytically coherent.

After a side remark on orthogonal vector bundles we are dealing with holomorphic families $\mathcal{F} \to \mathcal{V} \overset{\pi}{\to} M$ of fiber bundles over a holomorphic family of Riemannian spheres whose structure group G is a reductive Lie group. Let H be a Cartan subgroup of G, N its normalizer, and $W=N/H$ the Weyl group of G. The main result is that for any family $\mathcal{F} \to \mathcal{V} \overset{\pi}{\to} M$ there is a 1-codimensional analytic subset A of M such that the restriction of $\mathcal{F}$ to $\pi^{-1}(M-A)$ admits a reduction of the structure group to N; in case $M-A$ is simply connected, the structure group can be reduced to H itself. In this case it turns out that the reduction is uniquely determined up to an action of W.

The final section of the paper deals with holomorphic families of vector bundles over a holomorphic family of compact Riemann surfaces. The goal is to prove that Theorem 2.3 is still true in this more general case, a result that has been proved recently by H. Grauert [10] in a much more general situation. As a consequence one is able to describe the structure of the set of all Weierstrass points of a given type (of a holomorphic family of compact Riemann surfaces) (cf. H. E. Rauch [17]). One more application is the following. Suppose X is a compact Riemann surface (of genus $g>1$). A n-tuple $(x_1, \cdots, x_n)$ of points of X will be called a Weierstrass n-tuple if there are non-negative integers $l_1, \cdots, l_n$ with $l_1+\cdots+l_n=g$ such that the vector space of meromorphic functions f on X whose divisor (f) fulfills

$$(f)+\sum_{\nu=1}^{n} l_\nu x_\nu \geq 0$$

has a dimension (over $\boldsymbol{C}$) bigger than 1. The set of all Weierstrass n-tuples can be proved to be an analytic subset of $X\times \cdots \times X$ and thus a projective variety.

The tool used in this last section is a construction that assigns to every holomorphic vector bundle $W \to X$ over a compact Riemann surface that is realized as a covering space $X \xrightarrow{p} P^1$ a holomorphic vector bundle $p_*(W) \rightarrow P^1$ such that $H^q(X, \Omega(W))$ is naturally isomorphic to $H^q(P^1, \Omega(p_*(W)))$; this construction can be carried over to holomorphic families of compact Riemann surfaces and is applied in order to prove the required extension of Theorem 2. 4. It may be remarked that this construction has also an independent interest insofar as it yields a short proof of A. Weil's generalization of the Theorem of Riemann-Roch (an idea which can be applied to the higher dimensional case) and can be used in the construction of the space of moduli of the Riemann surface of given genus.

1. Definitions and notations.

The definitions and notations follow essentially a paper of K. Kodaira and D. C. Spencer [13]. For the sake of completeness we shall recall them as far as they are needed in this paper.

Two complex spaces $\mathcal{V}$ and M (for definition cf. [9]) together with an open holomorphic mapping π of $\mathcal{V}$ onto M are called a *holomorphic family of complex manifolds* if

(i) for any point $t \in M$ the fiber $V_t = \pi^{-1}(t)$ equipped with the induced structure is a connected complex manifold

(ii) each point of M has a neighborhood U admitting a biholomorphic mapping f of U onto $\mathbf{C}^n \times \pi(U)$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbf{C}^n \times \pi(U) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \pi(U) & = & \pi(U) \end{array}$$

commutes.

The holomorphic family $\mathcal{V} \rightarrow M$ is said to have the *total space* $\mathcal{V}$ and the *parameter space* M; for each $t \in M$, $\pi^{-1}(t)$ is denoted by V_t and called the *fiber belonging to* t. We speak of a *holomorphic family of Riemennian spheres* (*Riemann surfaces*), if each fiber of the family carries the structure of a Riemannian sphere (Riemann surface).

Suppose $\mathcal{V} \rightarrow M$ and $\mathcal{V}' \rightarrow M'$ are holomorphic families of Riemann surfaces. Then a holomorphic mapping $f: \mathcal{V} \rightarrow \mathcal{V}'$ is called a *fiber mapping* if the restriction of f to every fiber of $\mathcal{V}\to M$ is a biholomorphic mapping onto some fiber of $\mathcal{V}\to M'$. The fiber mapping f induces a mapping $f_0: M\to M'$ so that

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{f} & \mathcal{V}' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f_0} & M' \end{array}$$

commutes. f_0 is easily seen to be continuous. If in addition M and M' are normal complex spaces (for definition cf. [9]), f_0 is known to be holomorphic (R. Remmert [18]).

The fiber mapping $f: \mathcal{V}\to\mathcal{V}'$ is called a *fiber isomorphism* provided f is biholomorphic. We say that $\mathcal{V}\to M$ is (holomorphically) *trivial* if there is a fiber isomorphism $f: \mathcal{V}\to V_0\times M$ so that

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & V_0\times M \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

commutes. In a corresponding way one defines the notion of *local triviality*.

Let $\mathcal{V}'\xrightarrow{\pi'}M'$ be a holomorphic family of complex manifolds and $f_0: M\to M'$ a holomorphic mapping. The analytic subset $\{(V', m)\,|\,\pi'(V')=f_0(m)\}$ of $\mathcal{V}'\times M$ equipped with the induced structure shall be denoted by $\mathcal{V}$. Then $\mathcal{V}\xrightarrow{\pi}M$, where π is the projection of $\mathcal{V}$ onto M, is a holomorphic family of complex manifolds. $\mathcal{V}\xrightarrow{\pi}M$ is said to be *induced* from $\mathcal{V}'\xrightarrow{\pi'}M'$ by $f_0: M\to M'$. A holomorphic family is trivial if and only if it is induced by a trivial mapping, i.e. a mapping which maps M into one point.

A *holomorphic family of fiber bundles* (with structure group a complex Lie group G) is a holomorphic family $\mathcal{V}\to M$ of complex manifolds together with a holomorphic fiber bundle $\mathcal{B}\to\mathcal{V}$ (with structure group G). The restruction of $\mathcal{B}$ to the fiber V_t is denoted by B_t. We speak of a *family of fiber bundles over the Riemannian sphere* if $\mathcal{V}\to M$ is a family of Riemannian spheres.

For any subspace M' of M we get a new family of fiber bundles $\mathcal{B}|M' \to \mathcal{V}|M' \to M'$ where $\mathcal{V}|M' \to M'$ denotes the restriction of $\mathcal{V} \to M$ to M' and $\mathcal{B}|M' \to \mathcal{V}|M'$ the restriction of $\mathcal{B} \to \mathcal{V}$ to $\mathcal{V}|M'$.

Two holomorphic families $\mathcal{B} \to \mathcal{V} \to M$ and $\mathcal{B}' \to \mathcal{V}' \to M$ of fiber bundles having the same structure group and fiber are said to be *isomorphic* if there exist biholomorphic mappings $f: \mathcal{V} \to \mathcal{V}'$ and $f': \mathcal{B} \to \mathcal{B}'$ so that

$$\begin{array}{ccccc} \mathcal{B} & \longrightarrow & \mathcal{V} & \longrightarrow & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \Vert \\ \mathcal{B}' & \longrightarrow & \mathcal{V}' & \longrightarrow & M \end{array}$$

commutes.

For various reasons it is convenient to deal with holomorphic families of complex manifolds whose parameter space as well as their total space are normal complex spaces. That leads us to the following construction. Let $\mathcal{V} \xrightarrow{\pi} M$ be a holomorphic family of complex manifolds, and consider the normalizations $n: \tilde{\mathcal{V}} \to \mathcal{V}$ and $n_0: \tilde{M} \to M$ (for definitions cf. H. Rossi [20]). Then there is exactly one holomorphic mapping $\tilde{\pi}: \tilde{\mathcal{V}} \to \tilde{M}$ such that

$$\begin{array}{ccc} \tilde{\mathcal{V}} & \xrightarrow{\tilde{\pi}} & \tilde{M} \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle n_0} \\ \mathcal{V} & \xrightarrow{\pi} & M \end{array}$$

commutes. First we claim that $\tilde{\pi}$ is surjective. Indeed, $\pi \circ n$ is onto and n_0 is a homeomorphism on $\tilde{M} - n_0^{-1}(A)$ where A (=set of all reducible points of M) is a subset of M having the property that $M-A$ is dense in M as well as $\tilde{M} - n_0^{-1}(\tilde{A})$ is dense in $\tilde{M}$; consequently, $\tilde{\pi}(\tilde{\mathcal{V}})$ contains $\tilde{M} - n_0^{-1}(\tilde{A})$; i.e. $\tilde{\pi}(\tilde{\mathcal{V}})$ is dense in $\tilde{M}$. On the other hand, let $\tilde{t}_0$ be an element of $n_0^{-1}(A)$ and $t_0 = n_0(\tilde{t}_0)$. π was assumed to be an open mapping. Hence for every point v with $\pi(v) = t_0$ there is a compact neighborhood U whose image U_0 under π is a compact neighborhood of t_0. Let $\tilde{U}$ be $n^{-1}(U)$ and $\tilde{U}_0$ be $n_0^{-1}(U_0)$. n and n_0 being proper implies that both $\tilde{U}$ and $\tilde{U}_0$

are compact and $\tilde{U}_0$ is a neighborhood of $\tilde{t}_0$. As we saw, $\tilde{\pi}(\tilde{U})$ is dense in $\tilde{U}_0$; but $\tilde{U}$ being compact implies that $\tilde{\pi}(\tilde{U})$ is closed in $\tilde{U}_0$ and therefore equals $\tilde{U}_0$. This proves that $\tilde{\pi}$ is surjective.

Next we want to show that each fiber $\tilde{\pi}^{-1}(\tilde{t})$ is purely d-dimensional, where d is the dimension of the fibers in $\mathcal{V}\to M$. For that purpose it is enough to show that $n^{-1}(\pi^{-1}(t))$ is purely d-dimensional. The latter fact follows immediately from the hypothesis that V_{t_0} is a d-dimensional manifold. Consequently $\tilde{\pi}$ is an open mapping, as we see at once from a theorem of R. Remmert [18].

Now consider a connected component $\tilde{\mathcal{V}}_K$ of $\tilde{\mathcal{V}}$. Its image under $\tilde{\pi}$ is again connected and therefore contained in one of the connected components of $\tilde{M}$, say $\tilde{M}_K$. Because $\tilde{\pi}$ is an open mapping, $\tilde{\pi}(\tilde{\mathcal{V}}_K)$ is an open subset of $\tilde{M}_K$. In the case $\tilde{\pi}(\tilde{\mathcal{V}}_K)\neq\tilde{M}_K$, there is a second connected component, $\tilde{\mathcal{V}}_L$, of $\tilde{\mathcal{V}}$ such that $\tilde{\pi}(\tilde{\mathcal{V}}_K)\cap\tilde{\pi}(\tilde{\mathcal{V}}_L)$ is a non-empty open set. Denoting the set of all reducible points of $\mathcal{V}$ by A, the sets $\tilde{\mathcal{V}}_K-n^{-1}(A)\cap\tilde{\mathcal{V}}_K$ and $\tilde{\mathcal{V}}_L-n^{-1}(A)\cap\tilde{\mathcal{V}}_L$ correspond to two connected components, $\mathcal{V}_K$ respectively $\mathcal{V}_L$, of $\mathcal{V}-A$ having the property that their images under $\tilde{\pi}$ possess a non-empty intersection. That is obviously impossible because an irreducible analytic subset $(\pi^{-1}(t_0))$ contained in the union of two analytic subsets ($\overline{\mathcal{V}}_K$ respectively $\overline{\mathcal{V}}_L$)[1] is actually contained in one of them. At the same time we proved that the connected components of $\tilde{\mathcal{V}}$ and those of $\tilde{M}$ correspond to each other in a one-to-one way; corresponding connected components shall have the same subscript.

Furthermore we show that each fiber $\tilde{\pi}^{-1}(\tilde{t}_0)$ is a complex manifold that is isomorphic to V_{t_0} where $t_0=n_0(\tilde{t}_0)$. For that purpose let A_0 be the set of all reducible points of M. Then n_0 is a biholomorphic mapping of $\tilde{M}-n_0^{-1}(A_0)$ onto $M-A_0$. In case there is a reducible point v with $\pi(v)\in M-A_0$, we choose an open connected neighborhood U of v so small that $\pi(U)\subseteq M-A_0$, that $\mathcal{V}$ can be decomposed in U in $k(>1)$ irreducible components, and that for each point $t\in M$, $U\cap\pi^{-1}(t)$ is connected; then the normali-

1) $\overline{\mathcal{V}}_k$ denotes the closure of $\mathcal{V}_k$.

zation $\tilde{U}$ of U has at least k connected components, while the normalization $\widetilde{\pi(U)}$ of $\pi(U)$ has only one connected component. That contradicts our previous statement about the correspondence between the components of $\tilde{U}$ and those of $\widetilde{\pi(U)}$. Therefore there is no reducible point over $M-A_0$, and consequently $\mathcal{V}-n^{-1}\circ\pi^{-1}(A_0)$ is homeomorphic to $\mathcal{V}-\pi^{-1}(A_0)$. That proves already that $\tilde{\pi}^{-1}(\tilde{t}_0)$ is isomorphic to V_{t_0} provided $\tilde{t}_0$ is an element of $\tilde{M}-n_0^{-1}(A_0)$. In any case $\tilde{\pi}^{-1}(\tilde{t}_0)$ is a purely d-dimensional analytic subset according to a theorem of R. Remmert; furthermore it is contained in $n^{-1}\circ\pi^{-1}\circ n_0(\tilde{t}_0)$ which is easily proved to be a (possibly not connected) submanifold of $\tilde{\mathcal{V}}$. Therefore it remains to be shown that $\tilde{\pi}^{-1}(\tilde{t}_0)$ is connected. The mapping $\tilde{\pi}:\tilde{\mathcal{V}}\to\tilde{M}$ creates a situation to which a theorem of K. Stein [22] can be applied. In case, $\mathcal{V}\to M$ is a family of Riemmann surfaces[1] there is a complex space X and a holomorphic mapping $q:\tilde{\mathcal{V}}\to X$ such that for every normal complex space Y and every holomorphic mapping $\varphi:\tilde{\mathcal{V}}\to Y$ each fiber of which consists of fibers of $\tilde{\pi}$, there is exactly one holomorphic mapping $\varphi':X\to Y$ fulfilling $\varphi=\varphi'\circ q$. It turns out that the fibers of q are the connected components of the fibers of $\tilde{\pi}$. Consequently there is a uniquely determined holomorphic mapping φ^* of X onto $\tilde{M}$ such that $\tilde{\pi}=\varphi^*\circ q$. φ^* is a one-to-one mapping of $X-\varphi^{*-1}(n_0^{-1}(A_0))$ onto $\tilde{M}-n_0^{-1}(A_0)$; obviously all fibers of φ^* are discrete. If there were two points x_1, x_2 of X with $\varphi^*(x_1)=\varphi^*(x_2)$, then we could choose disjoint compact neighborhoods W_1 and W_2 of x_1 respectively x_2. Because the fibers of φ^* are discrete, φ^* is an open mapping according to a theorem of R. Remmert. Therefore $\varphi^*(W_1)\cap\varphi^*(W_2)$ is a neighborhood of $\varphi^*(x_1)$. There is an open, connected neighborhood U^* of $\varphi^*(x_1)$ contained in $\varphi^*(W_1)\cap\varphi^*(W_2)$ so that $\varphi^{*-1}(U^*)$ does not intersect the boundary of $W_1\cup W_2$: otherwise there would be for every neighborhood of $\varphi^*(x_1)$ a point on the boundary of $W_1\cup W_2$ (which is compact and does neither contain x_1 nor x_2) that is mapped by φ^* into this neighborhood, which contradicts the fact that φ^* is continuous. Denoting $W_1\cap\varphi^{*-1}(U^*)$ by W_1^* respectively $\varphi^{*-1}(U^*)\cap W_2$ by W_2^* we have now two neighborhoods of x_1 respectively x_2 so that the

1) The restriction to families of Riemann surfaces, i.e. $d=1$, is unnecessary by virtue of a theorem of H. Cartan. (Theorem 3 in "Quotients of complex analytic spaces", Contributions to function theory, Tata Inst. Fund. Research, Bombay, 1960).

restriction of φ^* to $W_1^*\cup W_2^*$ is proper. Hence a theorem of H. Cartan [6] shows that $W_1^*\cup W_2^*$ is a covering space of $\varphi^*(W_1^*\cup W_2^*)=U^*$ which is unramified and unbounded outside of an analytic subset of U^*. But $\varphi^*\,|\,W_1^*\cup W_2^*$ maps $W_1^*\cup W_2^*-\varphi^{*^{-1}}(n_0^{-1}(A_0))$ homeomorphically onto its image. Consequently the number of sheets of the covering $W_1^*\cup W_2^*\to U^*$ is one and therefore φ^* is a biholomorphic mapping of $W_1^*\cup W_2^*$ onto U^*. This means that each fiber of $\tilde\pi:\tilde{\mathcal{V}}\to\tilde M$ is connected.

Finally we want to show that $\tilde\pi:\tilde{\mathcal{V}}\to\tilde M$ fulfills also the last condition of a holomorphic family of complex manifolds. For that purpose consider the commutative diagram

$$\begin{array}{ccccc} & \tilde U & & & \boldsymbol{C}^d\times\widetilde{\pi(U)} \\ U & \swarrow\!\!\longleftarrow\!\!\!-\!\!\!\longrightarrow & \boldsymbol{C}^d\times\pi(U) & \swarrow & \downarrow \\ \downarrow & \widetilde{\pi(U)} & \| & & \widetilde{\pi(U)} \\ & \swarrow & \downarrow & \swarrow & \\ \pi(U) & = & \pi(U) & & \end{array}$$

According to the definition of the normalization there exists a holomorphic mapping $\tilde U\to\boldsymbol{C}^d\times\widetilde{\pi(U)}$ so that the above diagram remains commutative after inserting that mapping. Again according to the definition of the normalization it turns out, that the mapping $\tilde U\to\boldsymbol{C}^d\times\widetilde{\pi(U)}$ is biholomorphic.
Summarizing we get

Theorem 1.1: *Let $\mathcal{V}\xrightarrow{\pi}M$ be a holomorphic family of Riemann surfaces, $\tilde{\mathcal{V}}$ and $\tilde M$ the normalizations of $\mathcal{V}$ respectively M, and $\tilde\pi:\tilde{\mathcal{V}}\to\tilde M$ the uniquely determined holomorphic mapping so that*

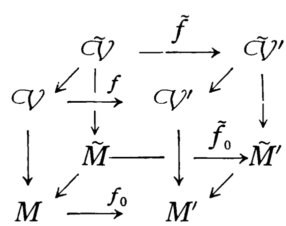
$$\begin{array}{ccc} \tilde{\mathcal{V}} & \xrightarrow{\tilde\pi} & \tilde M \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle n_0} \\ \mathcal{V} & \xrightarrow{\pi} & M \end{array}$$

commutes. Then $\tilde{\mathcal{V}}\xrightarrow{\tilde\pi}\tilde M$ is a holomorphic family (whose parameter space as well as its total space are normal) so that each fiber $\tilde V_t$ is isomorphic to the fiber $V_{n_0(\tilde t)}$.

$\tilde{\mathcal{V}}\xrightarrow{\tilde\pi}\tilde M$ will be called the normalization of the family $\mathcal{V}\xrightarrow{\pi}M$.

Without proof we shall give another result in that direction:

Proposition 1.2: *Let $\mathcal{V}\to M$ and $\mathcal{V}'\to M'$ be two holomorphic families of Riemann surfaces. Let $f: \mathcal{V}\to\mathcal{V}'$ be a fiber mapping and suppose that no irreducible component of $\mathcal{V}$ is mapped into the set of all reducible points of $\mathcal{V}'$. Then there is exactly one fiber mapping $\tilde{f}$ of the normalization $\tilde{\mathcal{V}}\to\tilde{M}$ into the normalization $\tilde{\mathcal{V}}'\to\tilde{M}'$ so that*



commutes. The induced mapping $\tilde{f}_0: \tilde{M}\to\tilde{M}'$ is holomorphic.

Finally we need in the sequel the following statement, the easy proof of which we shall omit:

Proposition 1.3: *Let $\mathcal{B}\to\mathcal{V}\to M$ be a holomorphic family of fiber bundles where the holomorphic family $\mathcal{V}\to M$ fulfillus the hypothesis of Theorem 1.1. Let $n: \tilde{\mathcal{V}}\to\mathcal{V}$ be the normalization mapping and $\tilde{\mathcal{B}}\to\tilde{\mathcal{V}}$ the fiber bundle induced from $\mathcal{B}\to\mathcal{V}$ by n.*[1] *Then for every $\tilde{t}\in\tilde{M}$, the restriction $\tilde{B}_t\to\tilde{V}_t$ is isomorphic to $B_{n_0(\tilde{t})}\to V_{n_0(\tilde{t})}$ by means of the induced fiber mapping of $\tilde{\mathcal{B}}$ into $\mathcal{B}$.*

2. Holomorphic families of vector bundles over the Riemannian sphere.

Let $\mathcal{B}\to\mathcal{V}\to M$ be a holomorphic family of vector bundles over the Riemannian sphere P^1 where the parameter space is supposed to be a complex manifold. First we study the local structure of such families. This problem can be simplified by results of K. Kodaira and D. C. Spencer ([13], Theorem 6.3, [14] Lemma 14.1 and Theorem 18.2,) which show that a holomorphic family of Riemannian spheres whose parameter space is a complex manifold is actually locally trivial. That means the above family of vector bundles is locally isomorphic to a holomorphic family

1) $\tilde{\mathcal{B}}$ coincides with the normalization in case the fiber $\mathcal{B}\to\mathcal{V}$ is a normal complex space.

$\mathcal{B} \to U \times P^1 \to U$ of vector bundles where U is a polycylinder and $U \times P^1 \to U$ the canonical projection. Because both $(P^1 - 0) \times U$ and $(P^1 - \infty) \times U$ are holomorphically complete manifolds which are of the topological type of the cell, $\mathcal{B}$ is (holomorphically) trivial over both $(P^1 - 0) \times U$ and $(P^1 - \infty) \times U$, according to a theorem of H. Grauert [7]. This implies that the vector bundle $\mathcal{B} \to U \times P^1$ can be defined by a holomorphic mapping $G(u, z)$ of $U \times (0 < |z| < \infty)$ into the general linear group $GL(q, \mathbf{C})$.

In the case we are dealing with a family of line bundles $(q=1)$, $\mathcal{B} \to U \times P^1$ is defined by a holomorphic and holomorphically invertible function $g(u, z)$ in $U \times (0 < |z| < \infty)$. The fundamental group of $U \times (0 < |z| < \infty)$ being free cyclic there exists an integer k such that $\log g(u, z) - k \log z$ is holomorphic and single valued in $U \times (0 < |z| < \infty)$. If

$$\sum_{\lambda=-\infty}^{+\infty} g_\lambda(u) z^\lambda$$

is the Hartogs series of that function, then the equation

$$\exp\left(\sum_{\lambda=-\infty}^{0} g_\lambda(u) z^\lambda\right) \cdot g(u, z) \cdot \exp\left(\sum_{\lambda=1}^{\infty} g_\lambda(u) z^\lambda\right) = z^k$$

shows the vector bundle $\mathcal{B} \to U \times P^1$ can be given by the function z^k. If we denote finally by L_k the holomorphic line bundle over P^1 which is defined by z^k, then we have

Lemma 2.1: *For each holomorphic family $\mathcal{B} \to U \times P^1 \to U$ of line bundles over the Riemannian sphere with a polycylinder as parameter space there exists an integer k and an isomorphism of families*

$$\begin{array}{ccccc} \mathcal{B} & \to & U \times P^1 & \to & U \\ \updownarrow & & \| & & \| \\ U \times L_k & \to & T \times P^1 & \to & U. \end{array}$$

If the rank of the vector bundle $\mathcal{B} \to U \times P^1$ is bigger than 1, then we consider the restriction B_u of the bundle $\mathcal{B}$ to the fiber V_u, $u \in U$. Denoting the sheaf of germs of holomorphic sections in B_u by $\Omega(B_u)$ the principle of upper semi-continuity ([13], Theorem 2.1) shows that there is a neighborhood U' of the origin 0 in U such that for all points u of U' the inequality

$$(1) \qquad \dim_C H^1(V_u, \Omega(B_u)) \leq \dim_C H^1(V_0, \Omega(B_0))$$

holds.

On the other hand, a theorem of G. D. Birkhoff [1] and A. Grothendieck [12] shows that the vector bundle B_u splits into the direct sum

$$\bigoplus_{\kappa=1}^{q} L_{k_\kappa(u)}$$

of holomorphic line bundles where $k_1(u), \cdots, k_q(u)$ are suitable integers.

An easy calculation shows

$$(2) \qquad \dim H^1(V_u, \Omega(B_u)) = \sum_{\kappa=1}^{q} \max(0, -k_\kappa(u)-1).$$

(1) and (2) imply that in U all exponents are uniformly bounded from below. Thus we can find an integer l such that the Whitney product $\mathscr{B}' \to U' \times P^1$ of the holomorphic line bundle $U' \times L_l \to U' \times P^1$ and the restriction of the bundle $\mathscr{B}$ to $U' \times P^1$ has the property:

for each point $u \in U'$ the bundle B'_u splits into the direct sum $\bigoplus_{\kappa=1}^{q} L_{k_\kappa'(u)}$ of line bundles whose degrees $k'_\kappa(u)$ are not negative.

The fact that each $k'_\kappa(u)$ is not negative implies

$$(3) \quad \dim H^0(V_u, \Omega(B'_u)) = \sum_{\kappa=1}^{q} \max(0, k'_\kappa(u)+1) = q + \sum_{\kappa=1}^{q} k'_\kappa(u).$$

The functor det on the category of vector bundles of rank q with values in the category of line bundles[1] maps the direct sum $\bigoplus_{\kappa=1}^{q} L_{k_\kappa'}(u)$ into the element $L_{\sum_{\kappa=1}^{q} k'_\kappa(u)}$. Lemma 2.1 shows that $\sum_{\kappa=1}^{q} k'_\kappa(u)$ is in fact independent of u. The functor det gives rise to a mapping det of the set $\prod_{\kappa=1}^{q} H^0(V, \Omega(B^q))$ into the set $H^0(V, \Omega(\det B^q))$ where B^q is a vector bundle of rank q[2]. Because all $k'_\kappa(u)$'s are not negative, we have enough holomorphic sections in B'_u over P^1.

1) The image of a vector bundle of rank q given by the transition functions g_{ij} is the line bundle given by the transition functions det g_{ij}.

2) The image of a q-tuple $s_1, \cdots, s_q$ of elements in $H^0(V, \Omega(B^q))$ under det is the element det $(s_1, \cdots, s_q)$ in $H^0(V, \Omega(\det B^q))$.

Hence det maps $\prod_{\kappa=1}^{q} H^0(V_u, \Omega(B'_u))$ not into the neutral element of $H^0(V_u, \Omega(\det B'_u))$.

Now we want to construct a set of meromorphic sections in $\mathscr{B}'$ over $U''\times P^1$ (U'' being a suitable neighborhood of 0) so that the restrictions of those sections to a fiber of $U''\times P^1\to U''$ form "in general" a basis of the module of global holomorphic sections in the restriction of $\mathscr{B}'$ to that fiber. For later use we also need a certain normalization of those meromorphic sections at ∞.

Because (3) does not depend on u a theorem of K. Kodaira and D. C. Spencer ([14], Theorem 18.1) establishes the existence of a neighborhood U'' of 0 and the existence of elements $s'_1, \cdots, s'_k$ $\in H^0(U''\times P^1, \Omega(\mathscr{B}'))$ such that for each point $u\in U''$ the restrictions of those sections to V_u form a basis of $H^0(V_u, \Omega(B'_u))$. Writing down those sections in terms of the above mentioned fiber coordinates we get two vectors $a'_\kappa(u, z)$ and $b'_\kappa(u, z)$ holomorphic in $U''\times(|z|<\infty)$ resp. $U''\times(0<|z|)$ which fulfill the equation

$$a'_\kappa(u, z) = G'(u, z)b'_\kappa(u, z) \quad \text{in} \quad U''\times(0<|z|<\infty). \tag{4}$$

Let

$$\sum_{\lambda=0}^{\infty} b'_\lambda(u)z^{-\lambda}$$

be the Hartogs expansion of $b'(u, z)$ in $z=\infty$. Denoting the μ-component of the vector b by $b^{(\mu)}$ we choose the integer λ_{11} such that (after a suitable rearrangement of the sequence $s'_1, \cdots, s'_{k'}$)

$$b'^{(1)}_{\kappa\lambda}(u)\equiv 0 \qquad \text{for} \quad \lambda<\lambda_{11}, \quad \kappa=1, \cdots, k', \quad \text{but}$$
$$b'^{(1)}_{1,\lambda_{11}}(u)\not\equiv 0$$

Such a choice is possible because $\det(\prod_{\kappa=1}^{q} H^0(V_0, \Omega(B'_0)))$ is not contained in the neutral element of $H^0(V_0, \Omega(\det B'_0))$. Replacing s'_1 by $s''_1=(b'^{(1)}_{1\lambda_{11}}(u))^{-1}s'_1$ and s'_κ by $s''_\kappa=s'_\kappa-b'^{(1)}_{\kappa\lambda_{11}}(u)s''_1$ for $\kappa>1$, we have new meromorphic sections $s''_1, \cdots, s''_{k'}$ in $\mathscr{B}'$ over $U''\times P^1$; their restrictions to $(U''-A')\times P^1$, A' being the support of the divisor defined by $b'^{(1)}_{1\lambda_{11}}(u)$, are holomorphic sections in $\mathscr{B}'$ and for every point $u\in U''-A'$ the restrictions of $s''_1, \cdots, s''_{k'}$ to V_u form a basis of $H^0(V_u, \Omega(B'_u))$. Applying the above process to $s''_2, \cdots, s''_{k'}$ and continuing by induction, we find integers $0\leq\lambda_{11}<\lambda_{12}<\cdots<\lambda_{1i_1}$, a 1-codimensional analytic subset A'' of U'', and meromorphic sections $s_1''', \cdots, s_{k'}'''$ in $\mathscr{B}'$ over $U''\times P^1$ so that

(i) the restrictions to $(U''-A'')\times P^1$ are holomorphic sections in $\mathscr{B}'$

(ii) the restrictions of $s_1''', \cdots, s_{k'}'''$ to V_u form a basis for $H^0(V_u, \Omega(B_u'))$ for each point $u\in U''-A''$

(iii)
$$b_{\kappa\lambda_1\mu}^{'''(1)}(u)\equiv\delta_{\kappa\mu} \qquad \kappa, \mu=1, \cdots, i_1$$
$$b_{\kappa\lambda}^{'''(1)}(u)\equiv 0 \qquad \kappa=i_1+1, \cdots, k', \ \lambda=0, 1, \cdots$$

hold. The fact that $\det(\prod_{\kappa=1}^{q} H^0(V_u, \Omega(B_u')))$ is not contained in the neutral element of $H^0(V_u, \Omega(\det B_u'))$ implies again that $i_1<k'$.

Now we carry out the same constuction for $s_{i_1+1}''', \cdots, s_{k'}'''$ and the components $b_{\kappa\lambda}^{''(2)}(u)$. If we continue by induction we finally construct a 1-codimensional analytic subset A of U'', q non-empty sets of integers $0\leq\lambda_{\kappa 1}<\lambda_{\kappa 2}<\cdots<\lambda_{\kappa i_\kappa}$, $\kappa=1, \cdots, q$, and meromorphic sections $s_1, \cdots, s_{k'}$ in $\mathscr{B}'$ over $U''\times P^1$ which fulfill the conditions (i) and (ii) with A instead of A'' and have the property that for $\mu=1, \cdots, q$

(5)
$$b_{\kappa\lambda}^{(\mu)}(u)\equiv 0 \qquad \text{for} \quad \kappa=i_1+\cdots+i_\mu+1, \cdots, k' \quad \lambda=0, 1, \cdots$$
$$b_{\kappa\lambda_{\mu\nu}}^{(\mu)}(u)\equiv\delta_{\mu\delta}\delta_{\nu\rho} \qquad \text{for} \quad \begin{matrix}\nu=1, \cdots, i_\mu & \rho=1, \cdots, i_\sigma \\ \sigma=1, \cdots, \mu & \kappa=i_1+\cdots+i_{\sigma-1}+\rho .\end{matrix}$$

Let A_0 be the smallest analytic subset of U'' so that the sections $s_{i_1}, s_{i_1+i_2}, \cdots, s_{i_1+\cdots+i_k}$ are holomorphic sections over $(U''-A_0)\times P^1$. A_0 is either empty or 1-codimensional in each of its points. Furthermore it is obvious that A_0 is uniquely determined by the matrix $G'(u, z)$ because $G'(u, z)$ and the conditions (5) determine the sections $s_1, \cdots, s_{k'}$ uniquely.

The matrices

$$A(u, z)=(a_{i_1}(u, z), \cdots, a_{i_1+\cdots+i_q}(u, z))$$
$$B(u, z)=(b_{i_1}(u, z), \cdots, b_{i_1+\cdots+i_q}(u, z))$$

are holomorphic in $(U''-A_0)\times(|z|<\infty)$ resp. $(U''-A_0)\times(0<|z|)$ and fulfill the equation

(4′) $$A(u, z)=G(u, z)B(u, z) \qquad \text{in} \quad (U''-A_0)\times(0<|z|<\infty)$$

Because of (5) $\det B(u, z)$ does not vanish identically. Thus the same holds for $\det A(u, z)$. Furthermore we derive from (5) and (4′)

$$\det A(u, z) = \det G(u, z) z^{-\sum_1^q \lambda_{\kappa i_\kappa}} f(u, z)$$

where $f(u, z)$ is holomorphic in $(U''-A_0)\times(0<|z|)$ and does not vanish for each fixed $u\in U''-A_0$ and all points z of a suitable neighborhood of ∞ which might depend on the choice of u. On the other hand, the restriction of $\det \mathscr{B}'$ to U'' is isomorphic to the line bundle $U''\times L_{k'-q}\to U''\times P^1$. Hence Lemma 2.1 assures the existence of functions $h_0(u, z)$ and $h_\infty(u, z)$ holomorphic and holomorphically invertible in $U''\times(|z|<\infty)$ resp. $U''\times(0<|z|)$ such that

$$h_0^{-1}(u, z) z^{k'-q} h_\infty(u, z) = \det G(u, z)$$

holds. Therefore we have

$$(6) \qquad h_0(u, z) \det A(u, z) = z^{k'-q-\sum_1^q \lambda_{\kappa i_\kappa}} f(u, z) h_\infty(u, z)\,.$$

Obviously the inequality

$$(7) \qquad \sum_{\kappa=1}^{q} k'_\kappa(u) = k'-q \leq \sum_{\kappa=1}^{q} \lambda_{\kappa i_\kappa}$$

holds. The two sides of (6) define a holomorphic function in $(U''-A_0)\times P^1$ which vanishes identically if (7) is no equality. Thus $\det A(u, z)\not\equiv 0$ implies that we have an equality in (7). That means that

$$\lambda_{\kappa\mu} = \mu-1 \qquad \text{for} \quad \kappa = 1, \cdots, q \quad \mu = 1, \cdots, i_\kappa\,.$$

Hence there is a matrix $C(u, z)$ holomorphic in $(U''-A_0)\times(0<|z|)$ such that

$$B(u, z) = C(u, z)\begin{pmatrix} z^{-\lambda_{1 i_1}} & & 0 \\ & \ddots & \\ 0 & & z^{-\lambda_{q i_q}} \end{pmatrix}$$

holds. One verifies easily the following property of $C(u, z)$: for each compact subset K of $U''-A_0$ there is a neighborhood D_K of $z=\infty$ such that $\det C(u, z)$ has no zero in $K\times D_K$.

One more consequence of the inequality (7) is the fact that det $A(u, z)$ does not depend on z; the remark that for each $u\in U''-A_0$ there is a complex number z with det $A(u, z)\neq 0$ shows now that $A(u, z)$ is holomorphically invertible in $(U''-A_0)\times(|z|<\infty)$.

Denoting $\lambda_{\kappa i_\kappa}$ by n'_κ the last remarks together with (4′) show that the equation

$$A^{-1}(u, z)G(u, z)C(u, z)=\begin{pmatrix} z^{n'_1} & & 0 \\ & \ddots & \\ 0 & & z^{n'_q}\end{pmatrix}$$

holds in $(U''-A_0)\times(0<|z|<\infty)$. This implies that $C(u, z)$ is holomorphically invertible in $(U''-A_0)\times(0<|z|<\infty)$; in addition this propery holds for $K\times D_K$ where K is a compact subset of $U''-A_0$ and D_K some neighborhood of $z=\infty$. Hence $C(u, z)$ is holomorphically invertible in $(U''-A_0)\times(0<|z|)$. This means that the restriction of $\mathscr{B}'$ to $(U''-A_0)\times P^1$ is isomorphic to the vector bundle $(U''-A_0)\times\bigoplus_{\kappa=1}^{q}L_{n'_\kappa}$. Tensoring $\mathscr{B}'$ by $U'\times L_{-l}\to U'\times P^1$ leads back to the original bundle $\mathscr{B}$ and gives therefore

Theorem 2.2: *Let $\mathscr{B}\to U\times P^1\to U$ be a holomorphic family of vector bundles of rank q whose parameter space is a polycylinder. Then there is a neighborhood U_0 of the origin in U, a uniquely determined smallest analytic subset A_0 of U_0, and a sequence of q integers $n_1\leq n_2\leq\cdots\leq n_q$ such that there is an isomorphism of families*

$$\begin{array}{ccccc} \mathscr{B}|U_0-A_0 & \to & (U_0-A_0)\times P^1 & \to & U_0-A_0 \\ \updownarrow & & \| & & \| \\ (U_0-A_0)\times\bigoplus_{\kappa=1}^{q}L_{n_\kappa} & \to & (U_0-A_0)\times P^1 & \to & U_0-A_0. \end{array}$$

In the case of line bundles ($q=1$) A_0 is empty. In case $q>1$, A_0 is either empty or 1-codimensional in each of its points.

The only statement which remains to be proved is the fact that A_0 is uniquely determined. As we already remarked, the above used sections $s_1,\cdots,s_{k'}$ are uniquey defined by matrix $G'(u, z)$ and the properties (5). Suppose that $\tilde{U}$ is a neighborhood of the origin in U and $\tilde{A}$ an analytic subset of $\tilde{U}$ such that the statement of Theorem 2.2 holds for $\tilde{U}-\tilde{A}$ instead of U_0-A_0. Then there are matrices $H_1(u, z)$ and $H_2(u, z)$ holomorphic and holomorphically invertible in $(|z|<\infty)\times(\tilde{U}-\tilde{A})$ respectively $(0<|z|)\times(\tilde{U}-\tilde{A})$ such that

$$H_1^{-1}(u, z)G(u, z)H_2(u, z) = \begin{pmatrix} z^{n_1} & & 0 \\ & \ddots & \\ 0 & & z^{n_q} \end{pmatrix}.$$

Without loss of generality we may assume that we are dealing with $\mathscr{B}'\to U''\times P^1$ instead of $\mathscr{B}\to U''\times P^1$; therefore we may assume that none of the exponents $n_1, \cdots, n_q$ is negative. Denoting the columns of $H_1(u, z)$ by $\hat{a}_\kappa(u, z)$ and the columns of $H_2(u, z)$ by $\hat{b}_\kappa(u, z)$ we see that

$$z^i\hat{a}_\kappa(u, z) = G(u, z)\cdot z^{i-n_\kappa}\hat{b}_\kappa(u, z)$$

holds. Consequently, the pairs of vectors

$$(z^i\hat{a}_\kappa(u, z),\ z^{i-n_\kappa}\hat{b}_\kappa(u, z)) \qquad i = 0, \cdots, n_\kappa, \quad \kappa = 1, \cdots, q$$

form holomorphic sections in $\mathscr{B}'$ over $(\tilde{U}-\tilde{A})\times P^1$ having the property that their restrictions to each fiber V_u form a basis for the module of global holomorphic sections in $B'_u\to V_u$ for every $u\in\tilde{U}-\tilde{A}$. Obviously we retain the same conditions if we change both $H_1(u, z)$ and $H_2(u, z)$ to $H_1(u, z)Q$ respectively $H_2(u, z)Q$ where Q is a permutation matrix. Because $H_2(u, \infty)$ is non-singular in $U-A$ we may assume (we only have to pick an appropriate permutation matrix Q) that the product

$$\hat{b}_1^{(1)}(u, \infty)\cdot \cdots \cdot\hat{b}_q^{(q)}(u, \infty)$$

does not vanish for a given point $u=u_0$ of $\tilde{U}-\tilde{A}$ and therefore for a full neighborhood $\tilde{\tilde{U}}$ of u_0. Hence the sections

$$\hat{s}_{i\kappa} = (z^i\hat{b}_\kappa^{(\kappa)^{-1}}(u, \infty)\hat{a}_\kappa(u, z),\ z^{i-n_\kappa}\hat{b}_\kappa^{(\kappa)^{-1}}(u, \infty)\hat{b}_\kappa(u, z))$$
$$i = 0, \cdots, n_\kappa \quad \kappa = 1, \cdots, q$$

still form a basis for the module of holomorphic sections in $\mathscr{B}'$ over $\tilde{\tilde{U}}\times P^1$. Forming suitable linear combinations of the sections

$$\hat{s}_{0,1}, \cdots, \hat{s}_{n_1 1}$$

we get sections $s'_1, \cdots, s'_{n_1+1}$ fulfilling the conditions

$$b^{(1)}_{\kappa\lambda}(u) \equiv \delta_{\kappa,\lambda+1} \qquad \kappa = 1, \cdots, n_1+1 \quad \lambda = 0, \cdots, n_1 .$$

That means we are able to normalize the rest of the sections s_{i_κ} $(\kappa>1)$ in such a way that first components of them have a zero at least of order n_1+1 in ∞. Proceeding by induction we end up with a new basis $\tilde{s}_1, \cdots, \tilde{s}_{k'}$ of the vector space of holomorphic sections in $\mathcal{B}'$ over $\tilde{\tilde{\tilde{U}}}\times P^1$ fulfilling the second set of equations (5) without any restrictions and the first set of equations (5) for $\lambda=0, \cdots, i_\mu$ and $\mu=1, \cdots, q$. In case the first set of equations (5) does not hold for $\mu=\mu_0$ and $\lambda=0, 1, \cdots$, we could proceed as in the construction leading to the basis fulfilling (5) and would consequently come up with a different representation of $\mathcal{B}'$ as a sum of line bundles. Thus the basis $\tilde{s}_1, \cdots, \tilde{s}_{k'}$ fulfills all equations (5) and hence coincides with the basis $s_1, \cdots, s_{k'}$. That implies, according to our previous construction, that $H_2(u, z)$ equals $C(u, z)$ up to a right factor Q. Hence we have $A_0\cap U_0\cap \tilde{U}\subseteq \tilde{A}\cap U_0\cap \tilde{U}$ which proves that A_0 is uniquely determined.

Corollary 1: *Let $\mathcal{B}\rightarrow \mathcal{V}\overset{\pi}{\rightarrow} M$ be a holomorphic family of vector bundles over the Riemannian sphere. Then there is a 1-codimensional analytic subset A in M such that $\mathcal{B}|M-A\rightarrow \mathcal{V}|M-A\rightarrow M-A$ is locally trivial and therefore* $\dim_C H^s(V_t, \Omega(B_t))$ *is constant on every connected component of $M-A$.*

Proof: It is sufficient to consider the case $s=0$: $s=1$ follows imediately from $s=0$, Lemma 2.1, and the Theorem of Riemann-Roch. Furthermore, it is sufficient to consider the case where both $\mathcal{V}$ and M are normal complex spaces: Let $\tilde{\mathcal{V}}\rightarrow \tilde{M}$ be the normalization of $\mathcal{V}\rightarrow M$ and A_1 the set of all non-normal points of M; then the set of all non-normal points of $\mathcal{V}$ is contained in $\pi^{-1}(A_1)$ (according to the definition of a holomorphic family of complex manifolds) and therefore we have an isomorphism of families

$$\begin{array}{ccc} \tilde{\mathcal{V}}\,|\,\tilde{M}-n_0^{-1}(A_1) & \leftrightarrow & \mathcal{V}|M-A_1 \\ \downarrow & & \downarrow \\ \tilde{M}-n_0^{-1}(A_1) & \leftrightarrow & M-A_1 \end{array} .$$

If in addition $\tilde{A}_2$ is a 1-codimensional analytic set in $\tilde{M}$ so that

$$\tilde{\mathcal{B}}|\tilde{M}-\tilde{A}_2 \to \mathcal{CV}|\tilde{M}-\tilde{A}_2 \to \tilde{M}-\tilde{A}_2$$

is locally trivial, then

$$\mathcal{B}|(M-A_1\cup n_0(\tilde{A}_2)| \to \mathcal{CV}|(M-A_1\cup n_0(\tilde{A}_2)) \to M-A_1\cup n_0(\tilde{A}_2)$$

is locally trivial and $A=A_1\cup n_0(\tilde{A}_2)$ is a 1-codimensional analytic subset of M according to a theorem due to R. Remmert [18]. Therefore we may restrict ourselves to the case where $\mathcal{CV}$ and M are normal. Denoting the singular locus of M by A', Theorem 2.2 states that for each point $t\in M-A'$ there is a neighborhood U_t, a uniquely determined smallest analytic subset A_t of U_t, and a sequence of integers $n_{1t}, \cdots, n_{qt}$ such that the above isomorphy holds. Because A_t is uniquely determined, $A_t\cap U_t\cap U_{t'}=A_{t'}\cap U_t\cap U_{t'}$ for all pairs of points $t, t'\in M-A'$. Therefore the collection of analytic subsets A_t defines an analytic subset $\tilde{A}''$ of $M-A'$. Because A' is of codimension two, while $\tilde{A}''$ is of codimension one in each of its points, a theorem of R. Remmert and K. Stein [19] shows that the closure A'' of $\tilde{A}''$ in M is an analytic subset of M. If we finally put $A=A'\cup A''$, the statements of the corollary are certainly fulfilled.

The above corollary tells that the set of points in which the dimension of the cohomology module jumps can be included in some analytic subset. Actually we get

Theorem 2.3 (cf. H. Grauert [10]): *Let $\mathcal{B}\to\mathcal{CV}\to M$ be a holomorphic family of vector bundles over the Riemannian sphere and j be an integer. Then the set of all points $t\in M$ fulfilling the inequality*

$$\dim_C H^s(V_t, \Omega(B_t))\geq j$$

is an analytic subset of M.

Proof: It is again sufficient to consider the case $s=0$. According to the corollary of Theorem 2.2 there is an analytic subset A_1 of at least codimension one such that $\dim_C H^0(V_t, \Omega(B_t))$ is constant in each connected component of $M-A_1$. Because A_1 is nowhere dense in M, the principle of upper semi-continuity (K. Kodaira and D. C. Spencer [14]) shows that this dimension actually equals the minimal $\dim_C H^0(V_t, \Omega(B_t))$ on the closure of that connected component of $M-A_1$. But the closure of a connected component of $M-A_1$ is an analytic subset of M. We denote the union of A_1 and the closures of those connected components of $M-A_1$, where the minimal $\dim_C H^0(V_t, \Omega(B_t))$ is greater or equal j, by M_1. M_1 is an analytic subset of M which we equip with the induced structure. Now we consider the family of vector bundles

$$\mathcal{B}|M_1 \to \mathcal{V}|M_1 \to M_1$$

and continue by induction. Then we get a descending chain $M_1, M_2, \cdots$ of analytic subsets of M whose intersection is again an analytic subset which obviously equals the set of all points of M for which $\dim_C H^0(V_t, \Omega(B_t)) \geq j$ holds.

Now we consider holomorphic families $\mathcal{B} \to \mathcal{V} \to M$ of vector bundles of rank q over the Riemann sphere having the property that all vector bundles $B_t \to V_t$ are isomorphic to $\bigoplus_{\kappa=1}^{q} L_{n_\kappa} \to P^1$. Furthermore we assume that the parameter space M is a complex manifold. Without loss of generality (cf. proof of Theorem 2.2) we may assume $n_1 \geq \cdots \geq n_q \geq 0$. Let t_0 be a point of M. Then we can find a basis $\bar{s}_1, \cdots, \bar{s}_k$ of $H^0(V_{t_0}, \Omega(B_{t_0}))$ fulfilling (5) for $t=t_0$. Theorem 18.1 of [14] shows that there is a neighborhood U of t_0 and elements $s'_1, \cdots, s'_k$ in $H^0(\mathcal{V}|U, \Omega(\mathcal{B}|U))$ whose restrictions to t_0 are the sections $\bar{s}_1, \cdots, \bar{s}_k$. After a suitable rearrangement of $s'_1, \cdots, s'_k$, the matrix

$$((b'^{(1)}_{\mu\nu}(t)))_{\substack{\mu=1, \cdots, n_1+1 \\ \nu=0, \cdots, n_1}}$$

is holomorphically invertible in some neighborhood U' of t_0 because it reduces itself to the identity matrix for $t=t_0$. That means we are able to find elements $s_1, \cdots, s_{n_1+1}$ in $H^0(\mathcal{V}|U', \Omega(\mathcal{B}|U'))$ so that the corresponding set of equations (5) is fulfilled. If there were any element s in $H^0(\mathcal{V}|U', \Omega(\mathcal{B}|U'))$ with $b^{(1)}_\nu(t) \equiv 0$ for $\nu=0, \cdots, n_1$, but $b^{(1)}(t, z) \not\equiv 0$, then the construction leading to Theorem 2.2 would show that the highest order of the line bundles in which B_{t^*} (t^* a suitable point in U') splits, is actually bigger than n_1. Therefore we can normalize the rest of the sections $s_{n_1+2}, \cdots, s_k$ in such a way that all equations (5) regarding the first component are satisfied. Proceeding by induction, we find a neighborhood U_0 of t_0 and elements $s_1, \cdots, s_k$ in $H^0(\mathcal{V}|U_0, \Omega(\mathcal{B}|U_0))$ so that the equations (5) are fulfilled with $i_\mu = n_\mu + 1$. Hence we have

Theorem 2.4: *Let $\mathcal{B} \to \mathcal{V} \to M$ be a holomorphic family of vector bundles over Riemannian sphere whose parameter space is a complex manifold. Assume that any two vector bundles $B_t \to V_t$ are isomorphic. Then the family is locally trivial.*

From the proof of Theorem 2.4 one concludes furthermore

Corollary 1: *Hypothesis as is Theorem 2.2. Then no bundle $B_\tau \to V_\tau$, $\tau \in A_0$, is isomorphic to the "general bundle of the family" $B_t \to V_t$ where $t \notin A_0$.*

One more consequence of the construction leading to Theorem 2.2 is

Lemma 2.5: *Suppose $\mathcal{B} \to \mathcal{V} \to M$ is a holomorphic family of vector bundles over the Riemannian sphere whose parameter space is a complex manifold. Then there is a 1-codimensional analytic subset A of M such that for each point $t_0 \in M$ there is a neighborhood U fulfilling the property:*

the bundle $\mathcal{B}|U \to \mathcal{V}|U$ admits holomorphic sections $s_1, \cdots, s_k$ whose restrictions to V_t form a basis for $H^0(V_t, \Omega(B_t))$ for each $t \in U - A \cap U$.

Proof: Going back to the proof of Theorem 2.2, we see that the sections $s_1, \cdots, s_{k'}$ fulfilling (5) are meromorphic sections in $\mathcal{B}'$ over $U'' \times P^1$; tensoring by the section σ in L_{-l} which is holomorphic and different from zero in $|z| < \infty$ carries those sections into meromorphic sections $\tilde{s}_1, \cdots, \tilde{s}_{k'}$ in $\mathcal{B}$ over $U'' \times P^1$. According to the construction there are holomorphic functions $f_1(t), \cdots, f_{k'}(t)$ in U'' such that the coefficients in the power series of $f_1 \cdot s_1, \cdots, f_{k'} \cdot s_{k'}$ around $z=0$ and $z=\infty$ are holomorphic in t. Then it is easy to verify that the sections

$$f_{l+1}s_{l+1}, \cdots, f_{i_1}s_{i_1}, f_{i_1+l+1}s_{i_1+l+1}, \cdots$$

fulfill the requirements of Lemma 2. 5.

Lemma 2. 5. is a kind of counterpart to Theorem 18. 1 of K. Kodaira and D. C. Spencer [14] for particular families of complex manifolds.

Given a family $\mathcal{B} \to \mathcal{V} \xrightarrow{\pi} M$ of holomorphic vector bundles we define the direct image $\pi_q(\Omega(\mathcal{B}))$ of the sheaf $\Omega(\mathcal{B})$ as follows (cf. [8]): the base space of the direct image is the parameter space M, the sheaf itself is defined by means of the presheaf which associates to each open subset U of M the module $H^q(\pi^{-1}(U), \Omega(\mathcal{B}))$; it is easy to see (cf. [8]) that this presheaf is canonical for $q=0$. We get

Theorem 2. 6: *Let $\mathcal{B} \to \mathcal{V} \xrightarrow{\pi} M$ be a holomorphic family of vector bundles over the Riemannian sphere whose parameter space is a complex manifold. Then the direct image $\pi_0(\Omega(\mathcal{B}))$ is an analytically coherent sheaf.*

This theorem has been proved by H. Grauert and R. Remmert [8] in a much more general case. But we shall give here a much simpler proof resting upon Lemma 2. 5. The property of being coherent is a local property. Thus we may consider a family $\mathcal{B} \to U \times P^1 \xrightarrow{\pi} U$ where U is a polycylinder. First we have to show that $\pi_0(\Omega(\mathcal{B}))$ is locally finitely generated. For that purpose we consider the sections $s_1, \cdots, s_k$ of Lemma 2. 5 (assuming that $t_0=0$). Let s be another section in $\mathcal{B}$ which is holomorphic in some neighborhood of $0 \times P^1$. Then we have obviously an equality

$$s = h_1(u) \cdot f_1^{-1} \cdot s_1 + \cdots + h_k(u) \cdot f_k^{-1} \cdot s_k$$

where $h_1, \cdots, h_k$ are holomorphic functions defined in some neighborhood of 0. In the given fiber coordinates we have the power series development

$$s_\kappa = \sum_{\lambda=0}^{\infty} \bar{b}_{\kappa\lambda}(u) z^{-\lambda}$$

where the vectors $\bar{b}_{\kappa\lambda}(u)$ are holomorphic in some fixed neighborhood $\overline{\overline{\overline{U}}}$ of 0. Hence the holomorphic function $h_1, \cdots, h_k$ define a holomorphic section if and only if

$$(8) \qquad \frac{h_1}{f_1}\bar{b}_{1\lambda}(u)+ \cdots + \frac{h_k}{f_k}\bar{b}_{k\lambda}(u) \qquad \lambda = 0, 1, \cdots$$

are holomorphic functions $\bar{b}_{0\lambda}(u)$ for all integers $\lambda=0, 1, \cdots$. Rewriting the equations (8) in terms of their components we get

$$(8') \qquad f_1 \cdots f_k \bar{b}_{0\lambda}^{(\mu)} = h_1 \cdot f_2 \cdots f_k \bar{b}_{1\lambda}^{(\mu)} + \cdots + h_k \cdot f_1 \cdots f_{k-1} \bar{b}_{k\lambda}^{(\mu)}, \quad \lambda = 0, 1, \cdots, \quad \mu = 1, \cdots, q.$$

Let $\overline{\overline{U}}$ be a relatively compact, open subset of $\overline{\overline{\overline{U}}}$ which contains 0. Then a theorem due to H. Cartan [4] states that there are finitely many, in $\overline{\overline{U}}$ holomorphic vectors $(g_{1\nu}, \cdots, g_{k\nu})$, $\nu=1, \cdots, N$, so that the germs of each of the vectors

$$(f_2 \cdots f_k b_{1\lambda}^{(\mu)}, \cdots, f_1 \cdots f_{k-1} b_{k\lambda}^{(\mu)}) \qquad \lambda = 0, 1, \cdots \quad \mu = 1, \cdots, q$$

are linear combinations of the germs of those vectors with holomorphic germs as coefficients. That implies that we have to consider only the equations

$$(9) \qquad f_1 \cdots f_k h_0 = h_1 g_{1\nu} + \cdots + h_k g_{k\nu} \qquad \nu = 1, \cdots, N$$

Therefore the sheaf of germs of holomorphic functions over $\overline{\overline{U}}$ fulfilling the equations (8) is isomorphic to the sheaf of germs of holomorphic functions $h_1, \cdots, h_k$ over $\overline{\overline{U}}$ fulfilling (9). The first sheaf is obviously isomorphic to the sheaf $\pi_0(\Omega(\mathcal{B}))$ over $\overline{\overline{U}}$; the latter is analytically coherent because it is the image of an analytically coherent sheaf (namely the intersection of finitely many relation sheaves which are coherent (cf. [8])) and therefore locally finitely generated. What remains to be proved is the fact that each relation sheaf of $\pi_0(\Omega(\mathcal{B}))$ is locally finitely generated. But this is an immediate consequence of the fact that $\pi_0(\Omega(\mathcal{B}))$ is a subsheaf of the sheaf of germs of k tuples of holomorphic functions over $\overline{\overline{U}}$.

Corollary: *Let $\mathcal{B} \to \mathcal{V} \to M$ be a holomorphic family of vector bundles over the Riemannian sphere whose parameter space is either* (i) *holomorphically complete or* (ii) *a normal projective* *variety. Then there is a* 1-*codimensional analytic subset* A *of* M *such that*

(i) *for* $t \in M-A$ *the restriction map* $H^0(\mathcal{V}, \Omega(\mathcal{B})) \to H^0(V_t, \Omega(B_t))$ *is surjective*

(ii) *for* $t \in M-A$ *the restriction to* V_t *maps the vector space of all meromorphic sections in* $\mathcal{B}$ *over* $\mathcal{V}$ *which are holomorphic over* $\mathcal{V}|M-A$ *onto* $H^0(V_t, \Omega(B_t))$.

Proof: (i) Let A' be the singular locus of M; A' is known to be an at least 2-codimensional analytic subset of M (M is normal!). According to Theorem 2.2 the set of all points of $M-A'$ in which $\mathcal{B} \to \mathcal{V}$ ceases to be locally trivial is 1-codimensional in each of its points; therefore its closure A'' in M is an analytic subset according to a theorem of R. Remmert and K. Stein [19]. If A is the union of A' and A'' it is easy to see that for each point $t \in M-A$ the stalk of $\pi_0(\Omega(\mathcal{B}))$ is isomorphic to the module $\mathcal{O}_t \cdot H^0(V_t, \Omega(B_t))$ where $\mathcal{O}_t$ is the ring of germs of holomorphic functions in t. Theorem A of H. Cartan and J. P. Serre [5], Exposé XVIII shows that the statement is true.

(ii) In this case, A is supposed to be the union of a hyperplane section and the set in (i). Again Theorem A of J. P. Serre [5], Exposé XVIII concludes the proof.

Finally it may be remarked that the corollary of Theorem 2.2 is useful for the classification of holomorphic families of vector bundles over the Riemannian sphere provided the parameter space M is holomorphically complete space. In this case a family of vector bundles can be given by a 1-codimensional analytic subset A of M, a locally trivial holomorphic family of vector bundles over $M-A$, and extension of the corresponding mapping of $M-A$ into the universal base space to a continuous mapping of M into the universal base space; a theorem of H. Grauert [7] states that the homotopy classes of those extensions are in a 1-1 correspondence to those holomorphic families of vector bundles over M which extend the given one over $M-A$. That means after characterizing the holomorphic family over $M-A$ one has only a topological problem to solve (extension of a given map and computing the homotopy classes of all possible extensions). Therefore it remains to characterize the locally trivial holomorphic family over $M-A$. Because the group of fiber preserving automorphisms of the bundle $L_1 \to P^1$ is the multiplicative group $\boldsymbol{C}^*$ of complex numbers, the family will be given by a sequence of integers $n_1 = \cdots = n_{q_1} < n_{q_1+1} = \cdots = n_{q_1+q_2} < \cdots$ and certain elements of

$$H^1(M-A, \mathcal{O}_{M-A}(GL(q_1, \boldsymbol{C}))) \times \cdots \times H^1(M-A, \mathcal{O}_{M-A}(GL(q_s, \boldsymbol{C})))$$

where $q_1 + \cdots + q_s = q$. Especially in the case that $M-A$ itself is holomorphically complete and $n_1 < n_2 < \cdots < n_q$ $(q_1 = q_2 = \cdots = 1)$ the family over $M-A$ is given by $n_1, \cdots, n_q$ and a certain element in

$$H^2(M-A, \boldsymbol{Z}) \times \cdots \times H^2(M-A, \boldsymbol{Z}) \qquad (q\text{-times});$$

this is an immediate consequence of a well known isomorphism [23].

3. Orthogonal fiber spaces.

We shall deal with the question under which circumstances the structure group of a holomorphic family of vector bundles over P^1 can be reduced to the complex orthogonal group $O(q, \boldsymbol{C})$. A necessary condition for the possibility of reducing the structure group is, that the given bundle $\mathcal{B} \to \mathcal{V}$ is isomorphic to its dual bundle $\mathcal{B}^* \to \mathcal{V}$. In the case where M consists of a single point A. Grothendieck [12] showed that this condition is also sufficient and that the reduction is uniquely determined (up to an isomorphism).

Using A. Grothendieck's method and his Lemma stating that

$$H^1(X, \mathcal{O}_X(O(q, \boldsymbol{C}))) \to H^1(X, \mathcal{O}_X(GL(q, \boldsymbol{C})))$$

is injective for compact complex spaces we have as an immediate consequence of Theorem 2.2

Thenrem 3.1: *Let $\mathcal{B} \to \mathcal{V} \to M$ be a holomorphic family of vector bundles over the Riemannian sphere whose parameter space is normal such that $\mathcal{B} \to \mathcal{V}$ and $\mathcal{B}^* \to \mathcal{V}$ are isomorphic. Then there is an analytic subset A of M (which is different from M) such that the restriction $\mathcal{B}|M-A \to \mathcal{V}|M-A \to M-A$ of the* *original family to $M-A$ is a family of holomorphic vector bundles admitting the complex orthogonal group $O(q, \boldsymbol{C})$ as structure group.*

This result cannot be improved in general in so far as one cannot get a reduction of the structure group over the whole parameter space. To show that we consider the following example.

Example: Let $\mathscr{B}\to \boldsymbol{C}^{l+1}\times P^1\to \boldsymbol{C}^{l+1}$ be the family of vector bundles which is defined by the transition function

$$\begin{pmatrix} z^{-l} & f(z) \\ 0 & z^{+l} \end{pmatrix}$$

where $f(z)$ is the polynomial $c_0+\cdots+c_l z^l$. Applying the method used in proving Theorem 2.2 we have to construct holomorphic sections in the Whitney product $\mathscr{B}'\to \boldsymbol{C}^{l+1}\times P^1$ of $\mathscr{B}\to \boldsymbol{C}^{l+1}\times P^1$ and $\boldsymbol{C}^{l+1}\times L_l \to \boldsymbol{C}^{l+1}\times P^1$. Such sections are given by vectors (a_1, a_2), (b_1, b_2) holomorphic in $\boldsymbol{C}^{l+1}$ $(|z|<\infty)$ resp. $\boldsymbol{C}^{l+1}$ $(0<|z|)$ such that

$$a_1 = b_1+z^l f(z)b_2, \qquad a_2 = z^{2l}b_2 \quad \text{hold.}$$

Therefore a basis for the holomorphic sections in $\mathscr{B}'\to \boldsymbol{C}^{l+1}\times P^1$ is given by

$$(a_1, a_2)=(1,0), \qquad (b_1, b_2)=(1,0)$$
$$(a_1, a_2)=(z^{l+i-\kappa}(c_i+\cdots+c_1 z^{l-i}),\ z^{2l-\kappa}), \quad (b_1, b_2)=(0, z^{-\kappa}), \quad \kappa=0, \cdots, l+i$$
$$(a_1, a_2)=(z^{l+i-\kappa}(c_{\kappa-1}z^{\kappa-l-i}+\cdots+c_1 z^{l-i}),\ z^{2l-\kappa}),$$
$$(b_1, b_2)=(-z^{l+i-\kappa}(c_i+\cdots+c_{\kappa-l-1}z^{\kappa-l-i-1}),\ z^{-\kappa}), \quad \kappa=l+i+1, \cdots, 2l$$

provided $c_0=\cdots=c_{i-1}=0$, $c_i\neq 0$. That implies that for all points c of $\boldsymbol{C}^{l+1}$ with $c_0=\cdots=c_{i-1}=0$, $c_i\neq 0$ the bundle $B'_c\to V_c$ can be described by the matrix

$$\begin{pmatrix} z^{l-i} & 0 \\ 0 & z^{l+i} \end{pmatrix}.$$

Hence the bundle $B_c\to V_c$ is actually isomorphic to $L_i\oplus L_{-i}\to P^1$. Now it is a straight-forward calculation to show for instance for $l=2$ that the structure group of the corresponding bundle $\mathscr{B}\to \boldsymbol{C}^3\times P^1$ cannot be reduced to $O(2, \boldsymbol{C})$.

4. Holomorphic families of fiber bundles over the Riemannian sphere.

We summarize a couple of definitions concerning complex Lie groups. A complex Lie group G is called *reductive* in the case its Lie algebra $\mathcal{G}$ is reductive, i.e. direct sum of its center and a semi-simple algebra. Furthermore a *Cartan subgroup* H of G is a connected holomorphic subgroup of G whose Lie algebra is a Cartan subagebra of $\mathcal{G}$; because any two Cartan subalgebras of $\mathcal{G}$ are conjugate the same holds for any two Cartan subgroups of G. If N denotes the normalizer of H the quotient group N/H turns out to be discrete (resp. finite provided G has only finitely many connected components); $N/H=W$ is called the *Weyl group* of G.

In the sequel we need a result which is essentially due to A. Grothendieck [12]:

Lemma 4.1: *Let $\mathcal{B}\rightarrow\mathcal{V}\rightarrow M$ be a holomorphic family of fiber bundles whose structure group is the complex Lie group G and whose fiber is the Lie algebra of G on which G operates by means of the adjoint representatian. Suppose that each fiber of $\mathcal{V}\rightarrow M$ is compact. Suppose there is a meromorphic section s in $\mathcal{B}$ over $\mathcal{V}$ whose restriction to some fiber V_{t_0} is holomorphic and which maps a point $v_0\in\mathcal{V}$ into a regular element $s(v_0)$ of the fiber of $\mathcal{B}$ over v_0. Then there is a 1-codimensional analytic subset A of M such that for each point $v\in\mathcal{V}|M-A$ the image $s(v)$ is regular. A does not contain $\pi(v_0)$.*

Proof: The coefficients $c_i(s(v))$ of the characteristic polynomial of $ad(s(v))$ are meromorphic functions on $\mathcal{V}$ which are not identically ∞ because the restriction of s to V_{t_0} is holomorphic. In addition the functions $c_i(s(v))$ are holomorphic in a full neighborhood of V_{t_0} and therefore constant on each fiber V_t. $\mathcal{V}\rightarrow M$ admits local holomorphic cross sections according to the definition of a family of complex manifolds. Hence each function $c_i(s(v))$ may be regarded as a meromorphic function in M. Because $s(v_0)$ is regular the highest coefficient $c_r(s(v))$ cannot vanish identically;

thus $s(v)$ is regular for all points of $\mathcal{V}|M-A$ where A is the set of zeros and poles of $c_r(s(v))$ as function on M.

Corollary 1 (Grothendieck [12]): *Let* $\mathcal{F}\to\mathcal{V}\overset{\pi}{\to}M$ *be a holomorphic family of fiber bundles whose structure group is* G. *If the associated holomorphic family* $\mathcal{B}_{\mathcal{F}}\to\mathcal{V}\overset{\pi}{\to}M$ *of fiber bundles whose fiber is the Lie algebra* $\mathcal{G}$ *of* G *admits a meromorphic section* s *in* $\mathcal{B}_{\mathcal{F}}$ *over* $\mathcal{V}$ *fulfilling the hypotheses of Lemma* 4.1, *then there is a* 1-*codimensional analytic subset* A *of* M *such that the structure group* G *of the restriction* $\mathcal{F}|M-A\to\mathcal{V}|M-A$ *can be reduced to the normalizer* N *of a Cartan subgroup* H *of* G. A *does not contain* $\pi(v_0)$.

Proof: A. Grothendieck [12].

Now we restrict ourselves to the case of holomorphic families $\mathcal{F}\to\mathcal{V}\to M$ of fiber bundles over the Riemannian sphere. Corollary of Theorem 2.2 states the existence of a 1-codimensional analytic subset A of M such that the restriction $\mathcal{B}_{\mathcal{F}}|M-A\to\mathcal{V}|M-A\to M-A$ is locally trivial. It has been proved in [12] that for each point $t\in M-A$ the vector bundle $(\mathcal{B}_{\mathcal{F}})_t\to V_t$ admits a holomorphic section s such that all elements $s(v)$, $v\in V_t$, are regular elements of the corresponding fiber provided the structure group G is reductive. Because of the local triviality of $\mathcal{B}_{\mathcal{F}}|M-A\to\mathcal{V}|M-A\to M-A$ one can extend s to a holomorphic section over a neighborhood of t. Obviously the extension has also the property that it maps all points of a certain neighborhood of V_t into regular points of the fiber. Corollary 1 of Lemma 4.1 shows now that the structure group G can be reduced to the normalizer N of the Cartan subgroup H provided we restrict the initial family to some neighborhood U of t. If U is properly chosen, then $\mathcal{V}|U$ is simply connected (because $\mathcal{V}\to M$ is locally trivial and $M-A$ may be assumed to be a complex manifold). Therefore Corollary 2 to Lemma 4.1 in [12] concludes the proof of

Corollary 2: *Suppose* $\mathcal{F}\to\mathcal{V}\to M$ *is a holomorphic family of fiber bundles over the Riemannian sphere whose parameter space is normal; assume furthermore that the structure group* G *is* *reductive. Then there is a* 1-*codimensional analytic subset* A *of* M *such that the family* $\mathcal{F}|M-A\to\mathcal{V}|M-A\to M-A$ *admits locally a reduction of the structure group* G *to the Cartan subgroup* H *of* G.

It is easy to see that the rest of A. Grothendieck's [12] statements as to reduction of the structure group holds in our case locally in the sense described in the above Corollary 2.

Next we want to study global reductions of the structure group. For that purpose we must restrict ourselves to parameter spaces M which are either holomorphically complete spaces or normal projective spaces. In this case the corollary of Theorem 2.6 shows the existence of a point $t\in M$ and a meromorphic section s in $\mathcal{B}_{\mathcal{F}}$ over $\mathcal{V}$ whose restriction to V_t coincides with a given holomorphic section in $(\mathcal{B}_{\mathcal{F}})_t\to V_t$. It has been shown in [12] that there is always a holomorphic section in $(\mathcal{B}_{\mathcal{F}})_t\to V_t$ which maps every point into a regular point of the corresponding fiber. Hence we are in the position to apply Lemma 4.1 which tells that there is an analytic subset A of M such that the fiber bundle $\mathcal{F}|M-A\to\mathcal{V}|M-A$ admits a reduction of the structure group G to the normalizer N of H. Hence we get

Theorem 4.2: *Let* $\mathcal{F}\to\mathcal{V}\to M$ *be a holomorphic family of fiber bundles over the Riemannian sphere whose structure group* G *is reductive and whose parameter space* M *is either holomorphically complete or a normal projective space. Then there is a* 1-*codimensional analytic subset* A *of* M *such that the fiber bundle* $\mathcal{F}|M-A\to\mathcal{V}|M-A$ *admits a reduction of the structure group to the normalizer* N *of the Cartan subgroup* H.

The group N operates on G by inner automorphisms; the subgroup H is stable under that action. Therefore N operates on both $H^1(X, \mathcal{O}_X(H))$ and $H^1(X, \mathcal{O}_X(G))$. The action of N on $H^1(X, \mathcal{O}_X(G))$ is obviously trivial. Hence $W=N/H$ operates on $H^1(X, \mathcal{O}_X(H))$ and one has a natural mapping

$$H^1(X, \mathcal{O}_X(H))/W \xrightarrow{\alpha} H^1(X, \mathcal{O}_X(G)). \tag{10}$$

Suppose now that the hypothesis of Theorem 4.2 is fulfilled and that in addition $M-A$ is simply connected. $\mathcal{V}|M-A\rightarrow M-A$ is a fiber bundle whose base space and whose fiber are both simply connected. Therefore $\mathcal{V}|M-A$ is simply connected and the fibering with fiber W which is associated to the reduction of the structure group to N is trivial. That proves that the element in $H^1(M-A, \mathcal{O}_{M-A}(G))$ defining $\mathcal{F}|M-A\rightarrow \mathcal{V}|M-A$ is in the image of α. A. Grothendieck [12] proved that for each fiber in $\mathcal{V}\rightarrow M$ the reduction to H is uniquely determined up to an action of W. This together with the finiteness of W shows that α is injective. Therefore we have

Corollary 1: *Under the hypothesis of Theorem* 4.2 *and the assumption that* $M-A$ *is simply connected, the fiber bundle* $\mathcal{F}|M-A\rightarrow \mathcal{V}|M-A$ *admits a reduction of the structure group to the Cartan subgroup* H; *the reduction is uniquely determined up to an action of* W.

Corollary 1 shows that it is of some interest to calculate $H^1(X, \mathcal{O}_X(H))$. Following [12] we denote the Lie algebra of H by $\mathcal{H}$. G reductive implies that H is abelian, and we have an exact sequence $0\rightarrow \pi_1(H)\rightarrow \mathcal{H}\rightarrow H\rightarrow 0$ where the mapping $\mathcal{H}\rightarrow H$ is given by $\mathcal{H}\ni h\rightarrow \exp(2\pi ih)\in H$ and $\pi_1(H)$ is the fundamental group of H based in the neutral element. From this we derive the exact sequence

$$(11)\quad H^1(X, \mathcal{O}_X(\mathcal{H}))\rightarrow H^1(X, \mathcal{O}_X(H))\rightarrow H^2(X, \pi_1(H))\rightarrow H^2(X, \mathcal{O}_X(\mathcal{H})).$$

$\mathcal{O}_X(\mathcal{H})$ is the direct sum of finitely many copies of $\mathcal{O}_X$; hence it is enough to calculate $H^q(X, \mathcal{O}_X)$.

Let X be the total space of a holomorphic fiber bundle whose base space Y is holomorphically complete and whose fiber is P^1. Consider the category C of abelian sheaves over X, the category C' of abelian sheaves over Y, and the category C'' of abelian groups. Then we have the covariant left exact functors $\pi_0: C\rightarrow C'$ and $H^0(Y, \ ): C'\rightarrow C''$. It is well known that each sheaf is subsheaf of some injective sheaf and that π_0 maps injective sheaves into injective sheaves (cf. A. Grothendieck [11], Lemma 3.7.1); therefore we get $H^q(Y, \pi_0(\mathcal{G}))=0$, $q\geqq 1$, for each injective sheaf $\mathcal{G}\in C$.

Consequently there is a spectral sequence whose initial term is

$$E_2^{p,q}(\mathcal{G}) = H^p(Y, \pi_q(\mathcal{G}))$$

and which terminates at $H^*(X, \mathcal{G})$ equipped with a suitable filtration (A. Grothendieck [11], Theorem 2. 4. 1). Applying everything for $\mathcal{G}=\mathcal{O}_X=\Omega(1)$, Theorem 2. 6 shows that $\pi_0(\mathcal{O}_X)$ is analytically coherent. We claim that $\pi_q(\mathcal{O}_X)=0$ for $q\geq 1$. In the case $q\geq 2$ we show that there are arbitrarily small open subsets U of Y such that $H^q(\pi^{-1}(U), \mathcal{O}_X)=0$. For any open subset U' of Y there are open subsets U which are holomorphically complete and have the property that $X\rightarrow Y$ is trivial over U, i.e. that $\pi^{-1}(U)$ is isomorphic to $U\times P^1$. But $U\times(|z|<\infty)$, $U\times(0<|z|)$ forms an open covering $\mathcal{U}$ of $U\times P^1$ whose elements are holomorphically complete. Therefore a theorem of J. Leray [14] together with Theorem B of H. Cartan and J. P. Serre [4] implies $H^q(\pi^{-1}(U), \mathcal{O}_X)\simeq H^q(\mathcal{U}, \mathcal{O}_X)$ for $q\geq 1$; but $H^q(\mathcal{U}, \mathcal{O}_X)$ vanishes for $q\geq 2$, because $\mathcal{U}$ consists of only two elements. An element of $H^1(\mathcal{U}, \mathcal{O}_X)$ is a holomorphic function $f(u, z)$ in $U\times(0<|z|<\infty)$; Cauchy's integral formula

$$f(u, z) = \frac{1}{2\pi i}\int_{|z|=2} f(u, \zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{|z|=1} f(u, \zeta)\frac{d\zeta}{\zeta - z}$$

shows that the cocycle $f(u, z)$ is cohomologous to zero which means that $H^1(\mathcal{U}, \mathcal{O}_X)=0$. Thus we get finally $E_2^{p,q}(\mathcal{O}_X)=0$ for $(p, q)\neq(0, 0)$. A trivial spetral sequence argument leads to

Lemma 4. 3: *Let $X\rightarrow Y$ be a holomorphic fiber bundle whose base space is holomorphically complete and whose fiber is P^1. Then $H^q(X, \mathcal{O}_X)=0$ for $q\geq 1$.*

Applying Lemma 4. 3 to (11) we find

Theorem 4. 4: *Let G be a reductive complex Lie group and H a Cartan subgroup of G. Suppose X is the total space of a holomorphic fiber bundle whose base space is holomorphically complete and whose fiber is P^1. Then there is a natural isomorphism*

$$H^1(X, \mathcal{O}_X(H)) \simeq H^2(X, \pi_1(H)).$$

It may be remarked that it is not difficult to calculate

$H^2(X, \pi_1(H))$ in concrete cases either by means of the spectral sequence of the fiber bundle with total space X or by means of the homotopy sequence of the fiber bundle and the universal coefficient theorem.

5. Holomorphic families of vector bundles over holomorphic families of compact Riemann surfaces.

Let $X \xrightarrow{p} Y$ be a compact Riemann surface realized as covering space of a compact Riemann surface Y; the covering is supposed to have q sheets. Let $z_1, \cdots, z_k$ be the projections of the ramification points of $X \to Y$, z_0 a point in $Y - \{z_1, \cdots, z_k\}$, and $\{x_1, \cdots, x_q\} = p^{-1}(z_0)$. Any closed curve α in $Y - \{z_1, \cdots, z_k\}$ which is based in z_0 can be lifted into x_κ; denote the end point $\alpha_{x_\kappa(1)}$ of the lifted curve by $x_{\alpha(\kappa)}$. $\kappa \to \alpha(\kappa)$ is a permutation of $\{1, \cdots, q\}$ which depends only on the homotopy class of α; that establishes an anti-homomorphism of the fundamental group $\pi_1(Y - \{z_1, \cdots, z_k\}, z_0)$ into the symmetric group S_q. For each permutation γ of S_q we define the matrix $\mu(\gamma) = ((a_{ij}))$ by $a_{ij} = \delta_{i\gamma^{-1}(j)}$ which gives rise to an anti-isomorphism of S_q into $GL(q, \boldsymbol{C})$. Altogether we get a homomorphism $\mu: \pi_1(Y - \{z_1, \cdots, z_k\}, z_0) \to GL(q, \boldsymbol{C})$. Such a homomorphism defines (cf. [3], [23]) a holomorphic vector bundle over $Y - \{z_1, \cdots, z_k\}$ whose holomorphic (meromorphic) sections over the open set $U \subset Y - \{z_1, \cdots, z_k\}$ correspond in a one to one way to the set of holomorphic (meromorphic) functions in $p^{-1}(U)$. This vector bundle is given as follows: choose an open covering U_i, $i \in I$, of $Y - \{z_1, \cdots, z_k\}$ by discs, choose in each U_i a point y_i and assign to it a curve α_i in $Y - \{z_1, \cdots, z_k\}$ with $\alpha_i(0) = z_0$ and $\alpha_i(1) = y_i$; for $z \in U_i \cap U_j$ choose curves β_i resp. β_j joining y_i resp. y_j and z in U_i resp. U_j and define the transition functions by

$$h_{ij}(z) = \mu(\alpha_i \beta_i \beta_j^{-1} \alpha_j^{-1}).$$

Next we extend that holomorphic vector bundle over $Y - \{z_1, \cdots, z_k\}$ to a holomorphic vector bundle $W_{X,Y}$ over Y by means of the following construction. Choose an open disc D_κ around z_κ such that any two of those discs are disjoint, pick a point d_κ in

$D_\kappa - \{z_\kappa\}$ and a curve γ_κ in $Y - \{z_1, \cdots, z_k\}$ with $\gamma_\kappa(0) = z_0$ and $\gamma_\kappa(1) = d_\kappa$; given a local coordinate t_κ in D_κ which vanishes in z_κ, there is a uniquely determined generator δ_κ of $\pi_1(D_\kappa - \{z_\kappa\}, d_\kappa)$ such that the analytic continuation $\delta_\kappa \log t_\kappa$ of $\log t_\kappa$ along δ_κ is $\log t_\kappa + 2\pi i$.[1] The matrix $\mu(\gamma_\kappa \delta_\kappa \gamma_\kappa^{-1})$ can be written as $P_\kappa Q_\kappa P_\kappa^{-1}$ where P_κ is a matrix having in each row exactly one non zero element which equals 1, while Q_κ splits into blocks $B_{\kappa 1}, \cdots, B_{\kappa m_\kappa}$ each of which has the form

$$((1)) \qquad \text{resp.} \left(\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & 1 \\ 1 & 0 & & 0 \end{pmatrix}\right).$$

Suppose $B_{\kappa\lambda}$ has $q_{\kappa\lambda}$ rows ($q_{\kappa 1} + \cdots + q_{\kappa m_\kappa} = q$). Let $\tau_{\kappa\lambda}$ be a certain branch of $t_\kappa^{1/q_{\kappa\lambda}}$ in d_κ and $\varepsilon_{\kappa\lambda} = \exp 2\pi i \frac{1}{q_{\kappa\lambda}}$. Then the matrix

$$M_{\kappa\lambda} = \left(\begin{pmatrix} 1, & \tau_{\kappa\lambda} & , & \tau_{\kappa\lambda}^2, & \cdots & \tau_{\kappa\lambda}^{q_{\kappa\lambda}-1} \\ \vdots & \varepsilon_{\kappa\lambda}\tau_{\kappa\lambda} & , & \vdots & & \vdots \\ 1, & \varepsilon_{\kappa\lambda}^{q_{\kappa\lambda}-1} q\tau_{\kappa\lambda}, & & \vdots & & \vdots \end{pmatrix}\right)$$

has the properties

$$\delta_\kappa M_{\kappa\lambda} = B_{\kappa\lambda} M_{\kappa\lambda}, \qquad (\det M_{\kappa\lambda})^2 = \text{const} \cdot t_\kappa^{q_{\kappa\lambda}-1}.$$

Therefore the matrix

$$M_\kappa = P_\kappa \left(\begin{pmatrix} M_{\lambda_1} & 0 \\ 0 & M_{\kappa m_\kappa} \end{pmatrix}\right) P_\kappa^{-1}$$

is not singular for $t_\kappa \neq 0$ and fulfills

$$\delta_\kappa M_\kappa = \mu(\gamma_\kappa \delta_\kappa \gamma_\kappa^{-1}) \cdot M_\kappa .$$

In order to extend the above constructed vector bundle we have to define transition functions $h_{i\kappa}(z)$ for $z \in U_i \cap D_\kappa$. For this purpose we choose a curve ε_κ in $D_\kappa - \{z_\kappa\}$ joining d_κ and z and set

$$h_{i\kappa}(z) = \mu(\alpha_i \beta_i \varepsilon_\kappa^{-1} \gamma_\kappa^{-1}) \cdot (\varepsilon_\kappa \cdot M_\kappa).$$

This definition is independent of the choice of ε_κ and fulfills the

1) For the purpose we fix some branch of $\log t_\kappa$ in d_κ.

compatibility relations; hence it gives rise to a holomorphic vector bundle $W_{X,Y}\to Y$.

Lemma 5.1: *Suppose U is an open subset of Y. Then the module of holomorphic sections in $W_{X,Y}$ over U (over the ring of holomorphic functions in U) is naturally isomorphic to the module of holomorphic functions on $p^{-1}(U)$. The module of meromorphic sections in $W_{X,Y}$ over U (over the ring of meromorphic functions in U) is naturally isomorphic to the module of meromorphic functions on $p^{-1}(U)$.*

Proof: Cf. [23]

We will use the stated property of $W_{X,Y}\to Y$ in the following construction. Let $G_r(X)$ be the set of equivalence classes of holomorphic vector bundles of rank r over the compact Riemann surface X; $G(X)=\bigcup G_r(X)$ is eqipped with an additive (Whitney sum) and a multiplicative (Whitney product) structure. Let $X\xrightarrow{p}Y$ be a realization of X which has q sheets. Then we construct (cf. [2]) a natural mapping

$$p_*: G(X)\to G(Y)$$

as follows. It is well known ([7], [23]) that for each open subset $U\subsetneqq Y$ the restriction of $W\in G_r(X)$ to $p^{-1}(U)$ is holomorphically trivial. That means W is determined by certain transition functions $g_{ij}(x): p^{-1}(U_i)\cap p^{-1}(U_j)\to GL(r, \mathbf{C})$ resp. $g_{i\kappa}(x): p^{-1}(U_i)\cap p^{-1}(D_\kappa)\to GL(r, \mathbf{C})$. By means of these transition functions we define for $z\in U_i\cap U_j$ [1)]

$$(12)\qquad G_{ij}(z)=\left(\begin{pmatrix} g_{ij}(\alpha_i\beta_i x_1) & & 0 \\ & g_{ij}(\alpha_i\beta_i x_2) & \\ 0 & & \ddots \end{pmatrix}\right)\cdot(1_r\otimes h_{ij}(z))^{1)}$$

and for $z\in U_i\cap D_\kappa$

$$(12')\qquad G_{i\kappa}(z)=\left(\begin{pmatrix} g_{i\kappa}(\alpha_i\beta_i x_i) & & 0 \\ & g_{i\kappa}(\alpha_i\beta_i x_2) & \\ 0 & & \ddots \end{pmatrix}\right)\cdot(1_r\otimes h_i(z)).$$

1) By $A\otimes B$ we denote the matrix $\left(\begin{pmatrix} Ab_{11} & \cdots & Ab_{1m} \\ Ab_{m1} & \cdots & Ab_{mm}\end{pmatrix}\right)$. By 1_r we denote the unit matrix with r rows.

The matrices (12) and (12′) are holomorphic functions with values in $GL(qr, \boldsymbol{C})$; it is easy to see that they fulfill the compatibility relations and thus define a holomorphic vector bundle $p_*(W)$ over Y. Denoting the lifting map $G(Y) \rightarrow G(X)$ by p^* and the canonical line bundle over X by K_X, one verifies immediately (cf. [2]) the formulas

$$
\begin{aligned}
& p_*(1) = W_{X,Y},\ p_*(G_r(X)) \subset G_{qr}(Y),\ (p_1 \circ p_2)_* = p_{1*} \circ p_{2*}, \\
(13)\quad & p_*(W \oplus W') = p_*(W) \oplus p_*(W'),\ p_*(W \otimes p^*(W')) = p_*(W) \otimes W' \\
& p_*(W^* \otimes K_X) = (p_*(W))^* \otimes K_Y .
\end{aligned}
$$

Theorem 5.2: *Suppose U is an open subset of Y. Then the module of holomorphic sections in $p_*(W)$ over U (over the ring of holomorphic functions in U) is naturally isomorphic to the module of holomorphic sections in W over $p^{-1}(U)$. The module of meromorphic sections in $p_*(W)$ over U (over the ring of meromorphic functions in U) is naturally isomorphic to the module of meromorphic sections in W over $p^{-1}(U)$.*

Proof: A holomorphic section in $p(W)$ over U is given by holomorphic vectors $G_i(z)$ resp. $G_\kappa(z)$ such that $G_i(z) = G_{ij}(z)G_j(z)$ resp. $G_i(z) = G_{i\kappa}(z)G_\kappa(z)$ in $U_i \cap U_j \cap U$ resp. $U_i \cap D_\kappa \cap U$ holds. Denoting the vector consisting of the components $r(n-1)+1, \cdots, rn$ of $G_i(z)$ by $G_i^n(z)$ we define

$$g_i(\alpha_i\beta_i x_\nu) = G_i^\nu(p(\alpha_i\beta_i x_\nu)) .$$

Then we get from (12)

$$
\begin{aligned}
g_i(\alpha_i\beta_i x_\nu) &= g_{ij}(\alpha_i\beta_i x_\nu) g_j(\alpha_j\beta_j x_{\alpha_i\beta_i\beta_j^{-1}\alpha_j^{-1}(\nu)}) \\
&= g_{ij}(\alpha_i\beta_i x_\nu) g_j(\alpha_i\beta_i\beta_j^{-1}\alpha_j^{-1}\alpha_j\beta_j x_\nu) = g_{ij}(\alpha_i\beta_i x_\nu) g_j(\alpha_i\beta_i x_\nu).
\end{aligned}
$$

That means that the collection of vectors $g_i(\alpha_i\beta_i x_\nu)$ forms a holomorphic section in W over $p^{-1}(U - U \cap \{z_1, \cdots, z_k\})$. Now we have to check the behavior of this section over the ramification points. The equation $G_i(z) = G_{i\kappa}(z)G_\kappa(z)$ can be written in the form

$$G_i(z) = \left[\left(\left(\begin{array}{c:c} g_{i\kappa}(\alpha_i\beta_i x) & 0 \\ \cdots\cdots & \cdots \\ 0 & \end{array}\right)\right)(1_r \otimes \mu(\alpha_i\beta_i\varepsilon_\kappa^{-1}\gamma_\kappa^{-1}))\right] \cdot \left[(1_r \otimes \varepsilon_\kappa M_\kappa) \cdot G_\kappa(z)\right]$$

Defining $g_\kappa(\gamma_\kappa\varepsilon_\kappa x_n)$ by the components $r(n-1)+1, \cdots, rn$ of $(1_r\otimes\varepsilon_\kappa M_\kappa)G_\kappa(z)$ we see that again

$$g_i(\alpha_i\beta_i x_\nu) = g_{i\kappa}(\alpha_i\beta_i x_\nu)\, g_\kappa(\gamma_\kappa\varepsilon_\kappa x_{\alpha_i\beta_i\varepsilon_\kappa{}^{-1}\gamma_\kappa{}^{-1}(\nu)})$$
$$= g_{i\kappa}(\alpha_i\beta_i x_\nu)\, g_\kappa(\alpha_i\beta_i x_\nu)$$

holds. The fact that the matrix $\varepsilon_\kappa M_\kappa$ may be interpreted as a holomorphic matrix in $p^{-1}(D_\kappa)$ shows that every component of g_κ is a holomorphic function in $p^{-1}(D_\kappa)$. In addition we have

$$\delta_\kappa(1_r\otimes\varepsilon_\kappa M_\kappa)G_\kappa(z) = (1_r\otimes\mu(\gamma_\kappa\delta_\kappa\gamma_\kappa^{-1}))(1_r\otimes\varepsilon_\kappa M_\kappa)G_\kappa(z)$$

which shows that the vector $g_\kappa(\gamma_\kappa\varepsilon_\kappa x_{\gamma_\kappa\delta_\kappa\gamma_\kappa{}^{-1}(\nu)})$ is the analytic continuation of $g_\kappa(\gamma_\kappa\varepsilon_\kappa x_\nu)$. Hence we have a natural mapping of $H^0(U, \Omega(p_*(W)))$ into $H^0(p^{-1}(U), \Omega(W))$. This mapping is obviously injective. The fact that it is surjective follows immediately from Lemma 5. 1. The second part of Theorem 5. 2 can be proved in a similar way.

Corollary: $\Omega(p_*(W))=p_0(\Omega(W))$.

Proof: According to an earlier remark $H^0(U, p_0(\Omega(W)))$ is isomorphic to $H^0(p^{-1}(U), \Omega(W))$ which in turn is isomorphic to $H^0(U, \Omega(p_*(W)))$.

This corollary shows that $p_*(W)$ depends only on W and the realization $X\to Y$, and not on the special construction we used.

It may be remarked that Theorem 5. 2 together with the last of the formulas (13) and the theorem of A. Grothendieck [12] and G. D. Birkhoff [1] on the splitting of holomorphic vector bundles over the Riemannian sphere furnishes an elementary proof of the Theorem of Riemann-Roch for holomorphic vector bundles over compact Riemann surfaces. This idea will be carried over to higher dimensional spaces in a subsequent paper.

Theorem 5. 3 (Cf. [24]): *Let $\mathcal{B}\to\mathcal{V}\to M$ be a holomorphic family of vector bundles over a holomorphic family of compact Riemann surfaces. Then the set of points $\{t: \dim H^0(V_t, \Omega(B_t))\geq j\}$ is an analytic subset of M for any natural number j.*

In order to prove Theorem 5. 3 we need

Lemma 5.4: *Let $\mathcal{V}\to M$ be a holomorphic family of compact Riemann surfaces of genus $g>1$ whose parameter space is normal. Then the subset of $\mathcal{V}$ consisting of all Weierstrass points of fibers of $\mathcal{V}\to M$ is an analytic subset of $\mathcal{V}$.*

Proof: Let A_1 be the singular locus of M. $p^{-1}(A_1)$ as well as A_1 is a 2-codimensional analytic subset [16], [9]. First we are dealing with $\mathcal{V}|M-A_1\to M-A_1$ and prove that the set of all Weierstrass points is a 1-codimensional analytic subset of $\mathcal{V}|M-A_1$. It is sufficient to show that this is true locally. Hence we have to consider a holomorphic family $\mathcal{V}'\to U$ of compact Riemann surfaces whose parameter space is a polycylinder. Let $\mathcal{B}'\to\mathcal{V}'$ be the bundle of contravariant holomorphic vectors tangent to the fibers. Then we have the equality $\dim_C H^0(V_u, \Omega(B'_u))=g$ and therefore Theorem 18.1 of K. Kodaira and D. C. Spencer [14] gives the existence of holomorphic differential forms of degree 1 $\omega_1(v), \cdots, \omega_g(v)$ along the fibers whose restrictions to the fiber V_u form a basis of $H^0(V_u, \Omega(B'_u))$ for any point u of a suitable neighborhood U_0 of $0\in U$. Let v_0 be a point of V_0 and h a fiber preserving biholomorphic mapping of some neighborhood $\bar{U}$ of v_0 into $p(\bar{U})\times \boldsymbol{C}$. Then there are holomorphic functions $f_1(u, z), \cdots, f_g(u, z)$ in $h(\bar{U})$ such that

$$\omega_1(v) = h^*(f_1(u, z)dz), \cdots, \omega_g(v) = h^*(f_g(u, z)dz) .$$

The determinant of $\left(\left(\frac{d^\beta f_\gamma(u, z)}{dz^\beta}\right)\right)_{\beta,\gamma=1,\cdots,g}$ is obviously holomorphic in $h(\bar{U})$ and the set of its zeros is exactly the set of all Weierstrass points on fibers passing through $h(\bar{U})$. This shows immediately that the set of all Weierstrass points of $\mathcal{V}'\to U$ forms an analytic subset of $\mathcal{V}'$ which is 1-codimensional in each of its points. This proves that the set A' of all Weierstrass points of $\mathcal{V}|M-A_1$ is a 1-codimensional analytic subset of $\mathcal{V}|M-A_1$. According to a theorem of R. Remmert and K. Stein [19], the closure A of A' in $\mathcal{V}$ is also an analytic subset of $\mathcal{V}$. It remains to be proved that A is the set of all Weierstrass points of $\mathcal{V}$. Let t be a point of the singular locus A_1. Because A_1 is of positive codimension, there is a 1-dimensional analytic subset of some neighborhood of t passing through t and hitting A_1 in this neighborhood only in t; without loss of generality we may assume that this 1-dimensional subset B is a manifold in the induced structure. Applying the above result to the restriction $\mathcal{V}|B \to B$ we find that each Weierstrass point of the fiber V_t belongs to the closure A of A'. In order to prove that a non-Weierstrass point does not belong to A we proceed in the same way using the fact that the set of all Weierstrass points in $\mathcal{V}|M - A_1$ has only finitely may irreducible components. After this preparation we come to the

Proof of Theorem 5.3: Because of Theorem 1.1 it is again sufficient to assume that M is normal and to prove our theorem only locally. Let V_t be the fiber belonging to the point $t_0 \in M$ and v_0 a point in V_{t_0} which is not a Weierstrass point. Suppose h is a fiber preserving biholomorphic mapping of some neighborhood $\bar{U}$ of v_0 into $p(\bar{U}) \times \boldsymbol{C}$; let $h(v_0)$ be (t_0, z_0). Then there is a neighborhood U' of t_0 such that none of the points $h^{-1}(t, z_0)$ is a Weierstrass point of the corresponding fiber: according to Lemma 5.4 the set of all Weierstrass points is closed. The set A of points $h^{-1}(t, z_0)$, $t \in U'$, is a 1-codimensional analytic subset of U'. The sheaf of germs of holomorphic functions in $\mathcal{V}|U'$ which vanish on A is obviously the sheaf of germs of holomorphic sections in a holomorphic line bundle $\boldsymbol{C} \to \mathcal{V}|U'$. According to the construction we get for the line bundle $\boldsymbol{C} = \boldsymbol{C}^{-g-1}$ the relation

$$\dim_C H^0(V_t, \Omega(C_t)) = 2\,, \qquad t \in U'\,.$$

Denoting the singular locus again by A_1, Theorem 18.1 of K. Kodaira and D. C. Spencer [14] shows that for each point t_1 in $U' - U' \cap A_1$ there is a neighborhood U_{t_1} and a meromorphic function $f(v)$ in $\mathcal{V}|U_{t_1}$ whose restriction to each fiber V_t, $t \in U_{t_1}$, is a meromorphic function of (exact) degree $g+1$. This means that the mapping

$$\mathcal{V}|U_{t_1} \ni v \to (p(v), f(v)) \in U_{t_1} \times P^1$$

realizes $\mathcal{V}|U_{t_1}$ as a covering space of $U_{t_1} \times P^1$. The covering $\mathcal{V}|U_{t_1} \xrightarrow{P_t} U_{t_1} \times P^1$ has exactly $g+1$ sheets and is unbounded (but ramified). In order to study the set of points in $U_{t_1}\times P^1$ over which $\mathcal{V}|U_{t_1}$ is ramified we have to consider the meromorphic function $F(t, z)=f(h^{-1}(t, z))$ in $\bar{U}\cap p^{-1}(U_{t_1})$. The support of the divisor of $\frac{\partial F(u, z)}{\partial z}$ is exactly the set of ramification points. Therefore the set of points in $\mathcal{V}|U_{t_1}$ in which $\mathcal{V}|U_{t_1}\xrightarrow{P_{t_1}}U_{t_1}\times P^1$ is ramified is a 1-codimensional analytic subset of $\mathcal{V}|U_{t_1}$. Its projection A into $U_{t_1}\times P^1$ is also a 1-codimensional analytic subset of $U_{t_1}\times P^1$, according to a theorem of R. Remmert [18]. From the geometrical properties of the covering $\mathcal{V}|U_{t_1}\rightarrow U_{t_1}\times P^1$ it is easy to see that $P_{t_1}^{-1}(A)$ does separate nowhere in $\mathcal{V}|U_{t_1}$; hence the covering in discussion is an analytic covering [9]. A being an analytic subset of $U_{t_1}\times P^1$ implies the existence of a neighborhood U'_{t_1} of t_1 and of pairwise disjoint discs $D_1, \cdots, D_k$ such that

(i) $\tilde{A}\cap(U'_{t_1}\times P^1)\subset U'_{t_1}\times\bigcup_{\kappa=1}^{k}D_\kappa$

(ii) for each connected component $V_{\kappa,\lambda}$ of $\mathcal{V}|U'_{t_1}$ over $U'_{t_1}\times D_\kappa$, $\kappa=1, \cdots, k$, there is a biholomorphic mapping $h_{\kappa\lambda}$ of that component into $U'_{t_1}\times \boldsymbol{C}$.

Using the functions $V_{\kappa\lambda}\xrightarrow{h_{\kappa\lambda}}U'_{t_1}\times\boldsymbol{C}\rightarrow\boldsymbol{C}$ instead of $\tau_{\kappa\lambda}$ we construct as at the beginning of this section a holomorphic vector bundle W_{t_1} over $U'_{t_1}\times P^1$ having the property (cf. Lemma 5.1) that the set of holomorphic sections (meromorphic sections) in W_{t_1} over an open subset U_0 of $U'_{t_1}\times P^1$ corresponds in a one to one way to the set of holomorphic functions (meromorphic functions) in $P_{t_1}^{-1}(U_0)$. Going back to the family $\mathcal{B}|U'\rightarrow\mathcal{V}|U'\rightarrow U'$ we found for each point $t_1\in U'-U'\cap A_1$ a neighborhood U'_{t_1} and the bundle $W_{t_1}\rightarrow U'_{t_1}\times P^1$. Using this bundle and the covering mapping P_{t_1} we can construct the holomorphic vector bundle $P_{t_1*}(\mathcal{B}|U'_{t_1})\rightarrow U'_{t_1}\times P^1$ in an analogous fashion as we did before (for a single fiber). This vector bundle has the property

$$(14)\quad \dim_C H^0(V_t, \Omega(B_t)) = \dim_C H^0(t\times P^1, \Omega(P_{t_1*}(\mathcal{B}|U'_{t_1}))), \quad t\in U'_{t_1}.$$

Applying Theorem 2.2 to the vector bundle $P_{t_1*}(\mathcal{B}|U'_{t_1})\rightarrow U'_{t_1}\times P^1$ we find that there is a minimal 1-codimensional analytic subset A_{t_1} in U'_{t_1} such that the bundle $P_{t_1*}(\mathcal{B}|U'_{t_1})$ is locally trivial over

$U'_{t_1} - A_{t_1}$. If we are able to show that for any choice of t_1, t_2 in $U' - U' \cap A_1$ the relation $A_{t_1} \cap U_{t_1} \cap U_{t_2} = A_{t_2} \cap U_{t_1} \cap U_{t_2}$ holds, then we have an analytic subset $B_{t_0} = \bigcup \{A_t : t \in U' - U' \cap A_1\}$ in $U' - U' \cap A_1$ which is of codimension 1 in each of its points and has the property that the vector bundles $P_{t_1 *}(\mathcal{B} | U'_{t_1})$ are locally trivial over the complement of B_{t_0}. Hence we have a similar situation as in the proof of Theorem 2. 3; proceeding as in the proof of Theorem 2. 3 and using the equality (14) we see that Theorem 5. 3 holds locally and thus globally. What remains to be shown is the equation $A_{t_1} \cap U_{t_1} \cap U_{t_2} = A_{t_2} \cap U_{t_1} \cap U_{t_2}$. For this purpose we have to study the projection mappings P_{t_1}. From the definition it is immediate that the functions defining P_{t_1} and P_{t_2} differ in $U'_{t_1} \cap U'_{t_2}$ only by a linear transformation (for each fiber) resp. by a holomorphic mapping of that intersection into the 1-dimensional affine group (over the complex numbers). But such a mapping of the base space $(U'_{t_1} \cap U'_{t_2}) \times P^1$ of the fiberings $P_{t_1 *}(\mathcal{B} | U'_{t_1})$ resp. $P_{t_2 *}(\mathcal{B} | U'_{t_2})$ does obviously not disturb the local triviality; hence the uniqueness of A_{t_1} (cf. Theorem 2. 2) shows that the required equation is true.

6. Applications.

Let X be a compact Riemann surface of genus $g > 1$. A Weierstrass point x in X is a point such that the vector space of those meromorphic functions f on X whose divisor (f) fulfills $(f) + k \cdot x \geq 0$ for some integer $1 \leq k \leq g$ has a dimension bigger than 1. Correspondingly we say that an n-tuple $(x_1, \cdots, x_n)$ of points of X is a *Weierstrass n-tuple* if there are non-negative integers $k_1, \cdots, k_n$ with $k_1 + \cdots + k_n \leq g$ such that the vector space of those meromorphic functions f on X whose divisor fulfills

$$(f) + k_1 \cdot x_1 + \cdots + k_n \cdot x_n \geq 0 \tag{15}$$

has a dimension bigger than 1. We consider the set of all Weierstrass n-tuples as a subset of X^n, and prove

Theorem 6. 1: *Let X be a compact Riemann surface* (*of genus* >1). *Then the set of all Weierstrass n-tuples is an analytic* *subset of X^n and therefore a complete projective variety.*

Proof: We consider the trivial family $X^n \times X \to X^n$. Let $v=(x_1, \cdots, x_n, x_0)$ be a point in $X^n \times X$ and choose simply connected (and connected) neighborhoods D_i of x_i, $i=0, \cdots, n$, such that $D_i \cap D_j = \phi$ provided $x_i \neq x_j$ and $D_i = D_j$ in case $x_i = x_j$. Denoting a local coordinate in D_i which is centered in x_i by $t_i(x)$ we define

$$f_v(y_1, \cdots, y_n, y) = \prod_{j=1}^{n} (t_0(y) - t_i(y_i))^{k_i e_i}, \quad (y_1, \cdots, y) \in D_1 \times \cdots \times D_0$$

where e_i equals 1 in case $x_1 = x_0$ and is zero otherwise. Considering two points v and v', we define $g_{vv'}(y_1, \cdots, y) = f_v(y_1, \cdots, y) f_{v'}^{-1}(y_1, \cdots, y)$ in the intersection $(D_1 \times \cdots \times D_0) \cap (D_1' \times \cdots \times D_0')$. Fulfilling the compatibility relations, the functions $g_{vv'}$ may serve as transition functions for a holomorphic line bundle $\mathscr{B}_{k_1, \cdots, k_n} \to X^n \times X$. Therefore we have a holomorphic family of line bundles $\mathscr{B}_{k_1, \cdots, k_n} \to X^n \times X \to X^n$. The restriction of this family to the fiber over $(x_1, \cdots, x_n)$ has the property that its vector space of global holomorphic sections is naturally isomorphic to the vector space of meromorphic functions fulfilling (15). Therefore $(x_1, \cdots, x_n)$ is a Weierstrass n-tuple if and only if for some $k_1, \cdots, k_n$

$$\dim_C H^0(V_{(x_1, \cdots, x_n)}, \Omega(B_{k_1, \cdots, k_n(x_1, \cdots, x_n)})) \geq 2$$

holds. Because there are only finitely many possibilities for $k_1, \cdots, k_n$, Theorem 5.3 proves the statement of Theorem 6.1.

Let X be a compact Riemann surface of genus $g > 1$. For each point $x \in X$ there are g natural numbers $1 = l_1(x) < l_2(x) \cdots < l_g(x)$ (gap numbers) such that the vector space of meromorphic functions f on X with $(f) + (l_\gamma(x) - 1) \cdot x \geq 0$ has the same dimension as the vector space of meromorphic functions f on X fulfilling $(f) + l_\gamma(x) \cdot x \geq 0$. A point for which $l_g(x) \neq g$ is a Weierstrass point; the sequence $(l_1(x), \cdots, l_g(x))$ is called its *type*.

Suppose $\mathcal{V} \xrightarrow{p} M$ is a holomorphic family of Riemann surfaces (of genus $g > 1$). Then we want to determine the structure of the set of all Weierstrass point (of fibers in $\mathcal{V} \to M$) of given type $(l_1, \cdots, l_g)$. For that purpose we consider the family $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ given by the projection mapping onto the second factor. For each point (v_1, v_2) in $\mathcal{V}\times\mathcal{V}$ we construct neighborhoods U_1 of v_1 and U_2 of v_2 such that

(i) $U_1\cap U_2=\phi$ provided $v_1\neq v_2$, $U_1=U_2$ provided $v_1=v_2$

(ii) there is a fiber preserving biholomorphic mapping h_i of U_i into $p(U_i)\times \boldsymbol{C}$, $i=1, 2$.

For each point $(v_1, v_2)\in\mathcal{V}\times\mathcal{V}$ and each integer $l\geq 0$ we define a holomorphic function

$$f_{(v_1,v_2)}(y_1, y_2) = (p_1(h_1(y_1))-p_1(h_2(y_2)))^{l\varepsilon(v_1,v_2)}, \quad (y_1, y_2)\in U_1\times U_2$$

where p_1 denotes the projection $p(U_i)\times\boldsymbol{C}\to\boldsymbol{C}$, and $\varepsilon(v_1, v_2)=1$ in case $v_1=v_2$ and is zero otherwise. The functions

$$g_{(v_1,v_2),(v_1',v_2')}(y_1, y_2) = f_{(v_1,v_2)}(y_1, y_2)f^{-1}_{(v_1',v_2')}(y_1, y_2)$$

define a holomorphic line bundle $\tilde{\mathcal{B}}_l\to\mathcal{V}\times\mathcal{V}$ and therefore a holomorphic family of line bundles $\tilde{\mathcal{B}}_l\to\mathcal{V}\times\mathcal{V}\to\mathcal{V}$. Now we consider the subset $\mathcal{W}$ of $\mathcal{V}\times\mathcal{V}$ consisting of the points $(p^{-1}(p(v)), v)$. $\mathcal{W}$ is the counterimage of the diagonal under the mapping $p\times p$: $\mathcal{V}\times\mathcal{V}\to M\times M$ and therefore an analytic subset of $\mathcal{V}\times\mathcal{V}$. Equipped with the induced structure, $\mathcal{W}$ is a complex space. It is easy to see that $\mathcal{W}\to\mathcal{V}$ (projection onto the second factor) is a holomorphic family of complex spaces (with singularities). Denoting the restriction of $\tilde{\mathcal{B}}_l\to\mathcal{V}\times\mathcal{V}$ to $\mathcal{W}$ by $\mathcal{B}_l\to\mathcal{W}$ we have a holomorphic family of holomorphic line bundles $\mathcal{B}_l\to\mathcal{W}\to\mathcal{V}$. The restriction of this family to the fiber in $\mathcal{W}\to\mathcal{V}$ which belongs to $v\in\mathcal{V}$ is a line bundle $B_{l,v}\to V_{p(v)}\times v$ whose vector space of global holomorphic sections is naturally isomorphic to the set of all meromorphic functions f on $V_{p(v)}$ fulfilling $(f)+l\cdot v\geq 0$. That gives us the opportunity of applying Theorem 5.3. Consequently the set of all Weierstrass points in $\mathcal{V}$ whose type has first component l_k is given by the set of all points v fulfilling

$$\dim H^0(V_{p(v)}\times v, \Omega(B_{l_k,v})) = \dim H^0(V_{p(v)}\times v, \Omega(B_{l_k-1,v})) = l_k+1-k$$

and hence according to Theorem 5.3 equal $(A_k-B_k)\cap(A_k'-B_k')$ where A_k, A_k', B_k, B_k' are analytic subsets of $\mathcal{V}$ with $B_k\subseteq A_k$, $B_k'\subseteq A_k'$. Therefore an easy set theoretic argument proves

Theorem 6.2: *Let $\mathcal{V} \to M$ be a holomorphic family of compact Riemann surfaces (of genus$>$1). Then the set of all Weierstrass points in $\mathcal{V}$ of given type is of the form $A-B$ where A and B are analytic subsets of $\mathcal{V}$ and $B \subseteq A$.*

Corollary (cf. H. E. Rauch [17]): *Let $\mathcal{V} \to M$ be a holomorphic family of compact Riemann surfaces (of genus$>$1) whose parameter space is normal. Then the set of all Weierstrass points in $\mathcal{V}$ for which $l_n > n$ is an analytic subset of $\mathcal{V}$ for any choice of n.*

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