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On a class of multiplicity-free nilpotent $K_{\mathbb{C}}$-orbits

By

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Abstract

Let G be a real, connected, noncompact, semisimple Lie group, let $K_{\mathbb{C}}$ be the complexification of a maximal compact subgroup K of G, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the complexified Lie algebra of G. Sequences of strongly orthogonal noncompact weights are constructed and classified for each real noncompact simple Lie group of classical type. We show that for each partial subsequence $\{\gamma_1, \dots, \gamma_i\}$ there is a corresponding family of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$, ordered by inclusion and such that the representation of K on the ring of regular functions on each orbit is multiplicity-free. The K-types of regular functions on the orbits and the regular functions on their closures are both explicitly identified and demonstrated to coincide, with one exception in the Hermitian symmetric case. The classification presented also includes the specification of a base point for each orbit and exhibits a corresponding system of restricted roots with multiplicities. A formula for the leading term of the Hilbert polynomials corresponding to these orbits is given. This formula, together with the restricted root data, allows the determination of the dimensions of these orbits and the algebraic-geometric degree of their closures. In an appendix, the location of these orbits within D. King's classification of spherical nilpotent orbits in complex symmetric spaces is depicted via signed partitions and Hasse diagrams.

1. Introduction

An action of an algebraic reductive group G on an affine variety M is called *multiplicity-free* if the multiplicity of any particular irreducible representation of G in the space $\mathbb{C}[M]$ of regular functions on M is at most one. In [Ka], Kac provides a complete list of multiplicity-free actions for the situation where G

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is a connected reductive algebraic group and M is a finite-dimensional vector space upon which G acts by an irreducible representation. We remark that Kac initiated this classification in order to understand the possibilities for the G_0-orbits in $\mathfrak{g}_i$, where $\mathfrak{g}_i$ is i^{th} homogeneous component of a $\mathbb{Z}$-graded semisimple Lie algebra and G_0 is the adjoint group of $\mathfrak{g}_0$.

In [KO], Kato and Ochiai develop a formula for the algebraic-geometric degree of a multiplicity-free G-variety Y in the situation where G is a connected reductive complex algebraic group, and Y is a closed G-stable subset of a finite-dimensional vector space V carrying an irreducible, multiplicity-free representation of G and such that the image of G in $GL(V)$ contains all nonzero scalar matrices. Kato and Ochiai then proceed to explicitly evaluate their formula for the case when V is the holomorphic tangent space of a Hermitian symmetric space G/K regarded as a representation of the complexification $K_{\mathbb{C}}$ of K. In this last situation, there exists a set of linearly independent dominant weights $\{\varphi_1, \ldots, \varphi_i\}$ so that

$$\mathbb{C}[Y] \cong \bigoplus_{m \in \mathbb{N}^i} V_{m_1\varphi_1 + \cdots + m_i\varphi_i}$$

where $V_{m_1\varphi_1 + \cdots + m_i\varphi_i}$ denotes the irreducible representation of K of highest weight $m_1\varphi_1 + \cdots + m_i\varphi_i$ and the sum is over all m-tuples of non-negative integers $(m_1, \ldots, m_i)$. Moreover, in the Hermitian symmetric situation, there is a natural way of constructing the weights φ_j, $j = 1, \ldots, i$ from a subsequence $\{\gamma_1, \ldots, \gamma_i\}$ of a Harish-Chandra sequence $\{\gamma_1, \ldots, \gamma_n\}$ of strongly orthogonal non-compact roots, as well as an explicit accounting of the roots that contribute, via the Weyl dimension formula, to the degree of the orbit. It happens that the contributing (restrictions of) positive roots break up into two disjoint subsets

$$\Delta^+_{short} = \left\{ \frac{1}{2}\gamma_j \mid 1 \le j \le i \right\} \text{ with a common multiplicity } r$$
$$\Delta^+_{long} = \left\{ \frac{1}{2}\gamma_j - \frac{1}{2}\gamma_k \mid 1 \le j < k \le i \right\} \text{ with a common multiplicity } k$$

These circumstances allow Kato and Ochiai to reduce the problem of determining the algebraic-geometric degree of Y to an application of the Selberg integral formula ([Se]).

From the "orbit philosophy" point of view in representation theory, there are two other especially important, general cases of multiplicity-free actions: the case when M is nilpotent $Ad(\mathfrak{g})$-orbit in the Lie algebra of a complex semisimple Lie algebra $\mathfrak{g}$ for which a Borel subgroup of $Ad(\mathfrak{g})$ has a dense orbit, and the case when M is an irreducible component of the associated variety of a multiplicity-free $(\mathfrak{g}, K_{\mathbb{C}})$-module. The orbits in the first case are called *spherical* nilpotent orbits and these have been studied and classified by Panyushev [Pa]. (See also [KY], where spherical nilpotent orbits for a complex Lie algebra are realized within the secant variety attached to the adjoint variety of a simple complex Lie algebra.)

The associated varieties in the second case correspond to multiplicity-free $K_{\mathbb{C}}$-orbits in $\mathcal{N}_{\mathfrak{p}}$, the nilpotent cone in $(\mathfrak{g}\backslash\mathfrak{k})^* \cong \mathfrak{p}$. Such orbits are referred to as spherical nilpotent orbits for the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. These have been classified by D. King ([Ki]). We remark that the varieties Y studied by Kato and Ochiai can be viewed as a special cases (the Hermitian symmetric cases) of a spherical nilpotent orbit for a symmetric pair. We note further the papers [N1], [N], [NO], [NOT], [NOZ]; wherein the associated varieties of singular unitary representations attached to certain dual pairs are shown to be multiplicity free. In the last three papers, integral formulas for the Bernstein degrees of the representations are also developed and in some cases explicitly computed. In particular, in [NOT] it is observed that the explicit formulas for Bernstein degrees so obtained coincide with the classical Giambelli formulas for the degrees of determinantal varieties. In fact, such integral formulas for degrees are common to spherical varieties in general ([Br1], [Br2]).

In this paper, we reverse-engineer the results of Kato and Ochiai to obtain a construction of a family of multiplicity-free $K_{\mathbb{C}}$-orbits in $\mathcal{N}_{\mathfrak{p}}$ that is applicable for any noncompact semisimple Lie algebra $\mathfrak{g}$. However, instead of starting with $K_{\mathbb{C}}$-orbits known to be multiplicity free, and looking for an associated sequence of strongly orthogonal noncompact roots; we proceed as follows:

1. In the context of an arbitrary connected noncompact real semisimple Lie group G we introduce an algorithm for constructing sequences $\{\gamma_1, \ldots, \gamma_n\}$ of strongly orthogonal noncompact weights.

2. We then attach to each subsequence $\{\gamma_1, \ldots, \gamma_i\}$ a certain nilpotent element Y_i of $\mathfrak{p}$, and set $\mathcal{O}_i = K_{\mathbb{C}} \cdot Y_i$. We show that the closure $\overline{\mathcal{O}_i}$ of each such orbit is multiplicity-free, and we explicitly identify the K-types of regular functions on the closure $\overline{\mathcal{O}_i}$ of $\mathcal{O}_i$ as

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] \cong \bigoplus V_{a_1\gamma_1+\cdots+a_i\gamma_i}$$

where the sum is over the $a_i \in \mathbb{N}$ such that $a_1 \geq a_2 \geq \cdots \geq a_i \geq 0$. (However, when the restricted root system is type D_n, and $\{\gamma_1, \ldots, \gamma_n\}$ is a sequence of maximal length, the bound on the last coefficient is actually $|a_n| \geq 0$.)

3. We observe that the degree of homogeneity of a polynomial in $V_{a_1\gamma_1+\cdots+a_i\gamma_i}$ is $\sum_{j=1}^{i} a_j$, and thereby reproduce the canonical filtration of $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ by degree by setting

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right]_\ell \cong \bigoplus_{\substack{a_1\geq a_2\geq\cdots\geq a_i\geq 0\\ \sum a_j\leq \ell}} V_{a_1\gamma_1+\cdots+a_i\gamma_i} \tag{1.1}$$

Using the restricted root data obtained in Section 2 and the Weyl dimension formula, we are then able to calculate the leading term of the corresponding Hilbert polynomial and thereby obtain formulas for the dimension and algebraic-geometric degree of $\overline{\mathcal{O}_i}$ in the classical cases.

The organization of this paper is as follows. In Section 2 we define certain sequences of strongly orthogonal noncompact weights. These sequences will provide the basic substratum upon which everything else is pinned. Table 1 in that section lists, for each real classical Lie group, the sequences of strongly orthogonal noncompact weights of maximal length and the form of their restricted root systems (as defined in that section).

In Section 3 we attach to each sequence of strongly orthogonal noncompact weights $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ a corresponding sequence $\{x_i, h_i, y_i\}$, $i = 1, \ldots, n$ of mutually centralizing normal S-triples. These in turn allow us to construct the "telescoping" sequences of $K_{\mathbb{C}}$-orbits $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset \mathcal{O}_n$ which will be the principal objects of study for the rest of the paper. We show that each orbit $\mathcal{O}_i$ is multiplicity-free and determine the K-type decompositions of the rings of regular functions on $\mathcal{O}_i$ and its closure.

We conclude Section 3 with two remarks; the first indicating where our family of nilpotent $K_{\mathbb{C}}$-orbits sits within D. King's [Ki] classification of nilpotent orbits for classical symmetric pairs. The second remark sketches our plan to attach to such a family of $K_{\mathbb{C}}$-orbits a corresponding family of unipotent representations. We note that effectively this has already been achieved by Sahi in the situation where G is the conformal group of a Euclidean [Sa1] or non-Euclidean [Sa2] real simple Jordan algebra. We show in Section 3.1.2 how one can recover, in the context of an arbitrary connected semisimple Lie group, nearly all of the structural niceties employed by Sahi in [Sa2] to bring to light families of unitarizable unipotent representations residing within families of degenerate principal series representations attached to corresponding families of nilpotent $K_{\mathbb{C}}$-orbits.

In Section 4 we utilize the K-type decompositions determined in Section 3 and along with the forms of the restricted root systems given in Table 1 to obtain closed formulas for the dimension and algebraic-geometric degrees of the closures of the orbits. We thereby produce analogs of the formulas of Kato and Ochiai in the general setting of noncompact classical Lie groups.

The multiplicity-free $K_{\mathbb{C}}$-orbits of real classical noncompact groups that we obtain in this paper all lie within the King's classification [Ki] of spherical nilpotent orbits for symmetric pairs (and we hereby apologize for adopting a nomenclature that might suggest otherwise). In an appendix, we illustrate via Hasse diagrams how our orbits are situated amongst the other orbits in King's classification. We remark that the K-type decompositions for *all* the spherical orbits of the symmetric pairs $(U(p,p)/U(p) \times U(p))$ have recently been obtained by K. Nishiyama ([N]) using dual pair methods, while for the same symmetric pairs, our method yields only the spherical orbits that reside along the outer edges of the corresponding Hasse diagram.

2. Sequences of strongly orthogonal noncompact weights

Let G be a connected noncompact real semisimple Lie group. Let K be a maximal compact subgroup, θ the corresponding Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the corresponding Cartan decomposition of the complexification of the Lie algebra of G. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and extend it to a θ-stable Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ of $\mathfrak{g}$. Choose a positive system $\Delta^+(\mathfrak{t};\mathfrak{k})$ for $\Delta(\mathfrak{t};\mathfrak{k})$ and extend it to a positive system $\Delta^+(\mathfrak{h};\mathfrak{g})$ of $\Delta(\mathfrak{h};\mathfrak{g})$ in such a way that

$$\alpha|_{\mathfrak{t}} \in \Delta^+(\mathfrak{t};\mathfrak{k}) \quad \Longrightarrow \quad \alpha \in \Delta^+(\mathfrak{h};\mathfrak{g}) .$$

Table 1.

G	K	Σ	Γ
$SL(n,\mathbb{R})$	$SO(n)$	$(d_{[n/2]})^1$ if n is even $(b_{[n/2]})^1 (d_{[n/2]})^1$ if n is odd	$2\sigma_{[n/2],\pm}$
$SL(n,\mathbb{H})$	$Sp(n)$	$(C_{[n/2]})^3 (d_{[n/2]})^4$	$\gamma_1 = \omega_2$ $\gamma_i = \omega_{2i} - \omega_{2i-2}$ $\gamma_{[n/2]} = \begin{cases} -\omega_{n-2} + \omega_n & \text{if } n \text{ is even} \\ -\omega_{n-3} + \omega_{n-1} & \text{if } n \text{ is odd} \end{cases}$
$SU(p,q)$ $(2 \le p \le q)$	$S(U(p) \times U(q))$	$(a_p)^2 (b_p)^{q-p}$	$\gamma_1 = \omega_1 + \omega_{p+q-1}$ $\gamma_i = -\omega_{i-1} + \omega_i + \omega_{p+q-i-1} - \omega_{p+q-i}$ $\gamma_p = -\omega_{p-1} + \omega_q - \omega_{q-1}$
$SO(2,q)$ $(q > 2)$	$S(O(2) \times O(q))$	$(A_2)^{q-2}$	$\gamma_1 = \omega_1$ $\gamma_2 = -\omega_1$
$SO(p,q), I$ $(2 < p \le q)$	$S(O(p) \times O(q))$	$\left(b_{[\frac{p}{2}]}\right)^{q-p+2\delta_p} \left(d_{[\frac{p}{2}]}\right)^2$	$\sigma_{p,q,\pm,\pm}$
$SO(p,q), II$ $(2 < p \le q)$	$S(O(p) \times O(q))$	$(a_{11,\pm})^{p-2} (a_{11,\mp})^{q-2}$	$\tau_{p,q,\pm}$
$SO^*(2n)$	$U(n)$	$\left(b_{[\frac{n}{2}]}\right)^2 \left(d_{[\frac{n}{2}]}\right)^4$	$\gamma_1 = \omega_1$ $\gamma_i = -\omega_{2(i-1)} + \omega_{2i}$ $\gamma_{[\frac{n}{2}]} = \begin{cases} -\omega_{n-2} + \omega_n & \text{if } n \text{ is even} \\ -\omega_{n-3} + \omega_{n-1} & \text{if } n \text{ is odd} \end{cases}$
$Sp(n,\mathbb{R})$	$U(n)$	$(a_n)^1$	$\gamma_1 = 2\omega_1$ $\gamma_i = -2\omega_{i-1} + 2\omega_i$ $\gamma_n = -2\omega_{n-1}$
$Sp(p,q)$ $(p \le q)$	$Sp(p) \times Sp(q)$	$(b_p)^{2(q-p)} (C_p)^2 (d_p)^2$	$\gamma_1 = \omega_1 + \omega_{p+1}$ $\gamma_i = -\omega_{i-1} + \omega_i - \omega_{p+i} + \omega_{p+i+1}$ $\gamma_p = -\omega_{p-1} + \omega_p - \omega_{2p} + \omega_{2p+1}$

(The term δ_p that appears in the exponent of $b_{[p/2]}$ for type $SO(p,q)$ is integer remainder of p when divided by 2.)

Let $\widetilde{\beta}$ be a highest weight of an irreducible representation of K on $\mathfrak{p}$. We remark that β is unique when $\mathfrak{g}$ is simple not of Hermitian type. In the simple Hermitian symmetric case, where $\mathfrak{p}$ decomposes into a sum of two irreducibles, $\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$, and one can take β to be the highest weight of the representation of K on $\mathfrak{p}_+$ or $\mathfrak{p}_-$. We now construct sequences $\{\gamma_1, \ldots, \gamma_n\}$ of strongly orthogonal noncompact weights as follows.

- We set $\gamma_1 = \widetilde{\beta}$;
- γ_{i+1} is determined from its predecessors $\{\gamma_1, \ldots, \gamma_i\}$ by the requirements
  - (i) γ_{i+1} is in the orbit of $\widetilde{\beta}$ under the action of the Weyl group of K.
  - (ii) γ_{i+1} is strongly orthogonal in $\mathfrak{g}$ to each γ_j for $j = 1, , \ldots, i$ (meaning there is no compact or noncompact weight vector of weight $\gamma_{i+1} \pm \gamma_j$ for $j = 1, \ldots, i$.)
  - (iii) $\omega_{i+1} = \sum_{j=1}^{i+1} \gamma_j \in \mathfrak{t}^*$ is dominant.

Of course, since $\dim \mathfrak{p}$ is finite, this constructive process will eventually terminate. It turns out that, almost always, the maximal length of such a sequence is equal to the lesser of the rank of K and the real rank of G. (See the remarks following Table 1.)

In Table 1 below we tabulate, for each real classical noncompact Lie group of real rank ≥ 2, sequences $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of maximal length. (The real rank one cases are excluded simply by virtue of their triviality: in these cases $\Gamma = \left\{\widetilde{\beta}\right\}$.)

We also provide in the table the form of the *restricted root systems* for Γ. This restricted root system is defined as follows. For each noncompact weight $\gamma_i \in \Gamma$ we can choose a representative nilpotent element x_i in $\mathfrak{p}_{\gamma_i}$, and then via a standard construction, a normal S-triple $\{x_i, h_i, y_i\}$ where $y_i \in \mathfrak{p}_{-\gamma_i}$, $h_i \in \mathfrak{t}$ and

$$[x_i, y_i] = h_i, \quad [h_i, x_i] = 2x_i, \quad [h_i, y_i] = -2y_i.$$

Set $\mathfrak{t}_1 = span_{\mathbb{C}}(h_1, \ldots, h_n) \subset \mathfrak{k}$. The restricted root system Σ corresponding to Γ is the set of $\mathfrak{t}_1$-roots in $\mathfrak{k}$. In the table, the form of a restricted root system Σ is indicated follows:

$$\Sigma = (a_n)^{m_a} (A_n)^{m_A} (b_n)^{m_b} (C_n)^{m_C} (d_n)^{m_d} (a_{11,+})^{m_+} (a_{11,-})^{m_-} \tag{2.1}$$

means that the set of positive roots in Σ consists of roots of the form

- $a_n = \left\{\frac{1}{2}\gamma_i - \frac{1}{2}\gamma_j \mid 1 \leq i < j \leq n\right\}$, each occurring with multiplicity m_a;
- $A_n = \{\gamma_i - \gamma_j \mid 1 \leq i < j \leq n\}$, each occurring with multiplicity m_A;
- $b_n = \left\{\frac{1}{2}\gamma_i \mid 1 \leq i \leq n\right\}$, each occurring with multiplicity m_b;
- $C_n = \{\gamma_i \mid 1 \leq i \leq n\}$, each occurring with multiplicity m_C;
- $d_n = \left\{\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j \mid 1 \leq i < j \leq n\right\}$, each occurring with multiplicity m_d;
- $a_{11,+} = \left\{\pm\left(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right)\right\}$, each occurring with multiplicity m_+; and
- $a_{11,-} = \left\{\pm\left(\frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_2\right)\right\}$, each occurring with multiplicity m_+.

We remark that $m_a \neq 0$ or $m_A \neq 0$ only in the Hermitian symmetric case, and in this case $m_d = 0$.[*1] We specify in Table 1 the non-compact weights γ_i in terms of a basis of fundamental weights of the semisimple part $[K, K]$ of K and the conventions of Bourbaki ([Bour]). When $[K, K]$ has two factors, say for rank r and s, we denote by $\omega_1, \ldots, \omega_r$ a basis (à la Bourbaki) for fundamental weights for the first factor, and $\omega_{r+1}, \ldots, \omega_{r+s}$ a basis of fundamental weights for the second factor.

When $[K, K]$ has an $SO(n)$ factor several idiosyncrasies occur which we shall now describe in detail. First of all, we have to deal with the fact that $SO(n) \sim D_{[\frac{n}{2}]}$ when n is even and $SO(n) \sim B_{[\frac{n}{2}]}$ when n is odd. It also turns out that, for even n, we have two different ways of terminating maximal sequences of strongly orthogonal noncompact weights (corresponding to the outer automorphism of D_n). We shall employ the following shorthand to deal efficiently these variations. Let $\sigma_{n,\pm}$ denote the following sequences of weights (of $SO(n)$).

$$\sigma_{n,\pm} = \begin{cases} \omega_1 + \omega_2 & \text{if } n = 4, \\ \omega_1,\ \omega_2 - \omega_1,\ \ldots,\ \omega_{k-1} + \omega_k - \omega_{k-2},\ \pm\omega_k \mp \omega_{k-1} & \text{if } n = 2k > 4, \\ \omega_1,\ \omega_2 - \omega_1,\ \ldots,\ -\omega_{k-2} + \omega_{k-1},\ 2\omega_k - \omega_{k-1} & \text{if } n = 2k+1. \end{cases}$$

We indicate by $2\sigma_{n,\pm}$, the sequences $2\omega_1,\ 2\omega_2 - 2\omega_1,\ \ldots$, etc which occur in the case of $SL(n, \mathbb{R})$.

To describe the sequences for $SO(p, q)$, $p \leq q$, we first denote by $\sigma_{p,q,\pm,\pm}$ the sequence of $\left[\frac{p}{2}\right]$ weights for $SO(q) \times SO(q)$ obtained by adding to each element of the sequence $\sigma_{p,\pm}$ the corresponding element in the sequence $\sigma_{q,\pm}$. Secondly, we denote by $\tau_{p,q,\pm}$ the two-element sequences

$$\tau_{p,q,\pm} = \omega_1 + \omega_{[\frac{p}{2}]+1},\ \pm\omega_i \mp \omega_{[\frac{p}{2}]+1}.$$

The sequences of noncompact weights for $SO(p, q)$ will then consist of the sequences $\sigma_{p,q,\pm,\pm}$ and $\tau_{p,q,,\pm}$. Depending on the parities of p and q, in the $SO(p, q)$ case, $3 < p \leq q$, there can be as many as six different sequences of noncompact weights, or as few as three.

2.1. Remarks

2.1.1. One could consider relaxing the requirement that each γ_i lie in the K-Weyl orbit of the highest noncompact weight by instead stipulating that each γ_i is a weight of the representation of K on $\mathfrak{p}$. This leads to more sequences of strongly orthogonal noncompact weights, but it seems that the sequences don't get any longer and, moreover, our method of identifying the K-types supported on the closures of the orbits is not applicable for such sequences. In the Hermitian symmetric case, where $\mathfrak{p}$ is a direct sum of two irreducible representations of K, one could consider utilizing weights from both summands to form strongly orthogonal sequences of noncompact weights. This does lead to additional long sequences of strongly orthogonal noncompact weights and, as K.

[*1] Note also that our mnemonic notation for restricted root systems is a bit misleading for the types a_n and A_n since these are actually root systems of Cartan type A and rank $n - 1$.

Nishiyama has pointed out to us, the corresponding sequences of $K_{\mathbb{C}}$-orbits may actually exhaust the spherical nilpotent orbits for Hermitian symmetric pairs. However, for such sequences it is also difficult to identify exactly which K-types appear in the ring of regular functions on the closures of the corresponding $K_{\mathbb{C}}$-orbits.

2.1.2. In all but the case of $SU^*(2n)$ the maximal number of elements in a sequence of strongly orthogonal noncompact weights is equal to $\min(rank(G/K), rank(K))$. This suggests a connection with the maximal number of commuting $\mathfrak{sl}(2,\mathbb{R})$ subalgebras of $\mathfrak{g}_{\mathbb{R}} = Lie_{\mathbb{R}}(G)$.

Indeed, to each $\gamma_i \in [\gamma_1, \ldots, \gamma_n]$ we have an associated normal triple $\{x_i, h_i, y_i\}$. In fact, one can arrange matters so that $y_i = \overline{x_i}$ and $\overline{h_i} = -h_i$. In this case, the real span of the Cayley transform

$$\mathfrak{c}_j : \{x_j, h_j, y_j\} \to \left\{\frac{1}{2}(x_j + y_j - ih_j), -i(x_j - y_j), \frac{1}{2}(x_j + y_j + ih_j)\right\} \tag{2.2}$$

will be a subalgebra $\mathfrak{s}_i$ of $\mathfrak{g}_{\mathbb{R}}$ that is isomorphic to $\mathfrak{s}l(2,\mathbb{R})$ and moreover

$$[\mathfrak{s}_i, \mathfrak{s}_j] = 0, \qquad i \neq j.$$

Since the semisimple element h_i of the original triple $\{x_i, h_i, y_i\}$ lies in $i\mathfrak{k}_{\mathbb{R}}$ and these all commute we must have $n \leq rank(K)$. On the other hand, since the semisimple element h_i' of the Cayley transform of $\{x_i, h_i, y_i\}$ is a semisimple element of $\mathfrak{p}_{\mathbb{R}}$, we must have $n \leq rank(G/K)$. And so it's rather interesting that in all cases except $SU^*(2n)$ we're getting the maximal possible (from this simple argument) number of commuting triples in $\mathfrak{g}_{\mathbb{R}}$. In the case of $SU^*(2n)$, however, the number of γ_i is $\left[\frac{n}{2}\right]$, while $rank(K) = n-1$ and $rank(G/K) = n$. We note that there is another circumstance that distinguishes $SU^*(2n)$ from the other simple noncompact Lie groups of classical type: in the case of $SU^*(2n)$ and only in the case of $SU^*(2n)$, there are actually two weights in $\Delta(\mathfrak{h};\mathfrak{g})$ that restrict to $\widetilde{\beta} \in \mathfrak{t}^*$; that is to say, for $SU^*(2n)$, and only $SU^*(2n)$, there is a pair of complex roots $\beta, \theta^*\beta \in \Delta(\mathfrak{h};\mathfrak{g})$ such that $\beta|_{\mathfrak{t}} = \widetilde{\beta} = \theta^*\beta|_{\mathfrak{t}}$. Although, by and large, it is rare that $\widetilde{\beta} \in \mathfrak{t}^*$ corresponds to a pair of θ-conjugate roots in $\Delta(\mathfrak{h};\mathfrak{g})$ rather than a single imaginary root, in Section 3 we posit both possibilities on an equal footing.

3. Families of multiplicity-free $K_{\mathbb{C}}$-orbits

Let G be a noncompact real semisimple Lie group and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a sequence of strongly orthogonal noncompact weights as constructed in the preceding section. We'll now associate to Γ a corresponding sequence $\{\mathcal{O}_1, \ldots, \mathcal{O}_n\}$ of $K_{\mathbb{C}}$-orbits in $\mathcal{N}_{\mathfrak{p}}$.

We begin by choosing representative elements $x_i \in \mathfrak{p}_{\gamma_i}$. As these are nilpotent elements of $\mathfrak{g}$, via a standard construction we can associate an S-triple; that is to say, we can find elements $h_i, y_i \in \mathfrak{g}$ so that for the triple $\{x_i, h_i, y_i\}$ the following commutation relations are satisfied:

$$[h_i, x_i] = 2x_i, \quad [h_i, y_i] = -2y_i, \quad [x_i, y_i] = h_i. \tag{3.1}$$

In fact, we can choose $y_i \in \mathfrak{p}_{-\gamma_i}$ and $h_i \in \mathfrak{t} \subset \mathfrak{k}$ so that $\{x_i, h_i, y_i\}$ is a normal triple in $\mathfrak{g}$; that is to say, $\{x_i, h_i, y_i\}$ satisfy both (3.1) and

$$\theta(x_i) = -x_i, \quad \theta(y_i) = -y_i, \quad \theta(h_i) = h_i \tag{3.2}$$

and $\mathfrak{s}_i = span_{\mathbb{R}}(x_i, h_i, y_i)$ is a θ-stable $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Moreover, since the γ_i's are strongly orthogonal, the corresponding $\mathfrak{s}_i$'s will be mutually centralizing; i.e., $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ if $i \neq j$.

We now set

$$\begin{aligned} X_i &= x_1 + x_2 + \cdots + x_i, \\ H_i &= h_1 + h_2 + \cdots + h_i, \\ Y_i &= y_1 + y_2 + \cdots + y_i \end{aligned} \tag{3.3}$$

and

$$\mathcal{O}_i = K_{\mathbb{C}} \cdot Y_i \subset \mathcal{N}_p.$$

$\{X_i, H_i, Y_i\}$ is easily seen to be another normal S-triple in $\mathfrak{g}$.*2

We'll now show that the orbits $\mathcal{O}_i$, $i = 1, \ldots, n$ are all multiplicity-free.

To accomplish this, we need to first elaborate a bit more on the setup in Section 2. Let G be a connected noncompact real semisimple Lie group. Let K be a maximal compact subgroup, θ a corresponding Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the corresponding Cartan decomposition of the complexification of the Lie algebra of G. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and extend it to a θ-stable Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ of $\mathfrak{g}$. Choose a positive system $\Delta^+(\mathfrak{t}; \mathfrak{k})$ for $\Delta(\mathfrak{t}; \mathfrak{k})$ and extend it to a positive system $\Delta^+(\mathfrak{h}; \mathfrak{g})$ of $\Delta(\mathfrak{h}; \mathfrak{g})$ in such a way that

$$\alpha|_{\mathfrak{t}} \in \Delta^+(\mathfrak{t}; \mathfrak{k}) \quad \Longrightarrow \quad \alpha \in \Delta^+(\mathfrak{h}; \mathfrak{g}).$$

We adopt a Chevalley basis $\{E_\alpha \mid \alpha \in \Delta(\mathfrak{h}; \mathfrak{g})\}$ so that

$$\begin{aligned} [E_\alpha, E_{-\alpha}] &= H_\alpha, \\ [E_\alpha, E_\gamma] &= \begin{cases} N_{\alpha,\gamma} E_{\alpha+\gamma} & \text{if } \alpha + \gamma \in \Delta(\mathfrak{h}; \mathfrak{g}), \\ 0 & \text{if } \alpha + \gamma \notin \Delta(\mathfrak{h}; \mathfrak{g}), \end{cases} \\ [H_\alpha, E_\gamma] &= \langle \alpha, \gamma \rangle E_\gamma. \end{aligned} \tag{3.4}$$

and define the induced mapping $\theta^* : \Delta(\mathfrak{h}; \mathfrak{g}) \to \Delta(\mathfrak{h}; \mathfrak{g})$ and numbers ρ_α by means of the formulas

$$\begin{aligned} \theta H_\alpha &= H_{\theta^* \alpha}, \\ \theta E_\alpha &= \rho_\alpha E_{\theta^* \alpha}. \end{aligned}$$

*2 *Note added in proof.* After submitting this article, the author learned of a paper by Muller, Rubenthaler and Schiffmann [MRS] wherein a similar construction of orbits is made. The situation studied in that paper is where $\mathfrak{g}$ is a complex semisimple Lie algebra with a $\mathbb{Z}$ grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and the orbits of the adjoint group of $\mathfrak{g}_0$ in $\mathfrak{g}_1$. As in the Kato-Ochiai [KO] paper, however, sequences of strongly orthogonal weights appear there as an auxiliary device arising from an underlying Hermitian-symmetric structure. By way of contrast, we remark that in the present paper the sequences of strongly orthogonal roots are employed as a constructive principle.

We set

$$\Delta_0 = \{\alpha \in \Delta \mid \theta^*\alpha = \alpha\} = \text{ the set of pure imaginary roots,}$$
$$\Delta_1 = \{\alpha \in \Delta \mid \alpha \notin \Delta_0\} = \text{ the set of complex roots,}$$

and

$$\Delta_{0,\pm} = \{\alpha \in \Delta_0 \mid \theta E_\alpha = \pm E_\alpha\}\,.$$

$\Delta_{0,+}$ and $\Delta_{0,-}$ are, respectively, the sets of, compact imaginary roots and noncompact imaginary roots. We remark that since we have set up $\mathfrak{h}$ as a maximally compact Cartan subalgebra, there are no real roots in $\Delta(\mathfrak{h};\mathfrak{g})$.

Let $\beta \in \Delta(\mathfrak{h};\mathfrak{g})$ be a root such that

(*)
$$(1-\theta)\,\mathfrak{g}_\beta = \mathfrak{p}_{\widetilde{\beta}} \equiv \text{ the highest weight space of the representation of } K \text{ on } \mathfrak{p}.$$

There are several situations that we shall reduce to two basic cases:

(i) β is a complex root;

(ii) β is a pure imaginary noncompact root.

When $rank(\mathfrak{g}) = rank(\mathfrak{k})$, all roots in $\Delta(\mathfrak{h};\mathfrak{g}) = \Delta(\mathfrak{t};\mathfrak{g})$ will be pure imaginary and so β will be a non-compact imaginary root. When $rank(\mathfrak{g}) > rank(\mathfrak{k})$, β will either be a non-compact pure imaginary root or there will be a θ^*-conjugate pair of complex roots $\{\beta, \theta^*\beta\}$ sharing the property (*). In the latter case, even though we make an initial choice for β, subsequent developments will be manifestly independent of that choice.

Lemma 3.1. *Suppose β is a complex root and $\widetilde{\beta} = \beta|_{\mathfrak{t}}$ is the highest weight of the representation of K on $\mathfrak{p}$. Then both*

$$[\theta E_{\pm\beta}, E_{\mp\beta}] = 0$$

and

$$[\theta E_{\pm\beta}, E_{\pm\beta}] = 0.$$

Proof. To prove the first, we note that

$$[\theta E_{\pm\beta}, E_{\mp\beta}] \in \mathfrak{g}_{\pm(\theta^*\beta-\beta)}.$$

However, $\pm(\theta^*\beta - \beta)$ will be real root, but for our choice of $\mathfrak{h}$ there are no real roots, and so $\theta E_{\pm\beta}$ and $E_{\mp\beta}$ must commute.

To prove the second relation, we set

$$k_\beta = (1+\theta)\,E_\beta \in \mathfrak{k}_{\widetilde{\beta}},$$
$$p_\beta = (1-\theta)\,E_\beta \in \mathfrak{p}_{\widetilde{\beta}}.$$

Since k_β is positive root vector for K and p_β is the highest weight of the representation of K on $\mathfrak{p}$ we must have

$$\begin{aligned} 0 = [k_\beta, p_\beta] &= [(1+\theta)\,E_\beta, (1-\theta)\,E_\beta] \\ &= [E_\beta, E_\beta] + [\theta E_\beta, E_\beta] - [E_\beta, \theta E_\beta] - [\theta E_\beta, \theta E_\beta] \\ &= 2\,[\theta E_\beta, E_\beta]\,. \end{aligned}$$

Similarly, $[k_{-\beta}, p_{-\beta}]$ implies $[\theta E_{-\beta}, E_{-\beta}] = 0$. □

Lemma 3.2. *Suppose $\beta \in \Delta(\mathfrak{h};\mathfrak{g})$ is such that $\widetilde{\beta} = \beta|_{\mathfrak{t}}$ is a highest weight of the representation of K on $\mathfrak{p}$. Let $x_\beta, h_\beta, y_\beta$ be defined by*

$$\begin{aligned} x_\beta &= E_\beta, \\ h_\beta &= \frac{2}{\langle \beta, \beta \rangle} H_\beta, \\ y_\beta &= E_{-\beta}, \end{aligned}$$

if β is pure imaginary, or

$$\begin{aligned} x_\beta &= (1-\theta) E_\beta, \\ h_\beta &= \frac{2}{\langle \beta, \beta \rangle} (1+\theta) H_\beta, \\ y_\beta &= \frac{2}{\langle \beta, \beta \rangle} (1-\theta) E_\beta \end{aligned}$$

if β is complex. Then $\{x_\beta, h_\beta, y_\beta\}$ is a normal S-triple in $\mathfrak{g}$ and

(**) $$z \in \mathfrak{p} \text{ and } [h_\beta, z] = -2z \quad \Longrightarrow \quad z \in span_{\mathbb{C}}(y_\beta).$$

Proof. An obvious calculation using the commutation relations (3.4), and Lemma 3.1 in the case when β is complex, confirms that

$$[h_\beta, x_\beta] = 2x_\beta, \quad [h_\beta, y_\beta] = -2y_\beta, \quad [x_\beta, y_\beta] = h_\beta$$

and so $\{x_\beta, h_\beta, y_\beta\}$ is an S-triple. It is also obvious that

$$\theta x_\beta = -x_\beta, \quad \theta y_\beta = -y_\beta, \quad \theta h_\beta = h_\beta$$

and so $x_\beta, y_\beta \in \mathfrak{p}$, $h_\beta \in \mathfrak{k}$; hence $\{x_\beta, h_\beta, y_\beta\}$ is a normal S-triple. Of course, x_β and y_β live, respectively, in the $+2$ and -2 eigenspaces of h_β.

To prove (**) we have to show that no other weight vector in $\mathfrak{p}$ can live in the -2-eigenspace of $\mathfrak{h}_\beta$. We shall handle the cases β is a complex root or a noncompact imaginary root separately.

Case (i). Assume β is a non-compact complex root and set

$$k_\beta = (1+\theta) E_\beta, \qquad k_{-\beta} = \frac{2}{\langle \beta, \beta \rangle} (1+\theta) E_{-\beta}.$$

Then it is easy to check that $\{k_\beta, h_\beta, k_{-\beta}\}$ is a θ-stable S-triple in $\mathfrak{g}$ with the same semisimple element as that of $\{x_\beta, h_\beta, y_\beta\}$. Suppose $\widetilde{\alpha} \in \mathfrak{t}^*$ is a weight of the representation of $\mathfrak{k}$ on $\mathfrak{p}$, and $\widetilde{\alpha} \neq \pm\widetilde{\beta} = \pm\,\beta|_{\mathfrak{t}}$. Having chosen a positive system for $\Delta(\mathfrak{h};\mathfrak{g})$ subordinate to that of $\Delta(\mathfrak{t};\mathfrak{k})$, we can regard $\widetilde{\alpha}$ as, respectively, a "positive" or "negative" weight of $\mathfrak{p}$, depending on whether $\widetilde{\alpha} = \alpha|_{\mathfrak{t}}$ for some $\alpha \in \Delta^{\pm}(\mathfrak{h};\mathfrak{g})$. Assume $\widetilde{\alpha}$ is "positive"; then $[k_\beta, z] \in \mathfrak{p}_{\widetilde{\alpha}+\widetilde{\beta}} = \{0\}$ since $\widetilde{\beta}$ is the highest weight of the representation of K on $\mathfrak{p}$, and $[k_{-\beta}, [k_{-\beta}, z]] \in \mathfrak{p}_{\widetilde{\alpha}-2\widetilde{\beta}} = \{0\}$ since $-\beta$ is the lowest weight of $\mathfrak{p}$.

A similar (albeit up-side-down) argument shows that if $\widetilde{\alpha}$ is "negative", then $[k_{-\beta}, z] = 0 = [k_\beta, [k_\beta, z]]$. And so, in either case, the representation of $\mathfrak{sl}(2, \mathbb{R})$ generated by the action of $k_{\pm\beta}$ on $z \in \mathfrak{p}_{\widetilde{\alpha}}$ is at most 2-dimensional, and so the lowest possible eigenvalue of h_β is -1.

Case (ii). Assume β is a pure imaginary noncompact root. In this case, the S-triple $\{x_\beta, h_\beta, y_\beta\}$ is just a renormalization of the Weyl triple $\{E_\beta, H_\beta, E_{-\beta}\}$. Since each weight $\widetilde{\alpha}$ of $\mathfrak{p}$ either comes from a pair of θ^*-conjugate complex roots $\alpha, \theta^*\alpha \in \Delta_1(\mathfrak{h}; \mathfrak{g})$ or corresponds more or less directly to a unique pure imaginary non-compact root, it will suffice to show that for any root $\alpha \in \Delta(\mathfrak{h}; \mathfrak{g})$ such that $\alpha \neq \pm\beta$, the maximal length of a β-string through α is 2.

Suppose α is root in $\Delta_1^+(\mathfrak{h}; \mathfrak{g})$ then $[E_\alpha, E_\beta] = 0$, since otherwise $\alpha + \beta$ would be a complex root and we'd have a root vector with a non-zero projection to $\mathfrak{p}$ and a weight higher than β. Therefore, for any root $\alpha \in \Delta_1^+(\mathfrak{h}; \mathfrak{g})$, α would have to be at the top of a (perhaps trivial) β-string. And so the situation we have to worry about when $\alpha \in \Delta_1^+(\mathfrak{h}; \mathfrak{g})$ is when the string is $\alpha, \alpha - \beta, \alpha - 2\beta$ or longer. If $\alpha - 2\beta$ is a root, it must be a complex root and so $(\alpha - 2\beta)|_{\mathfrak{t}}$ must be a $\mathfrak{t}$-weight of $\mathfrak{p}$. But $(\alpha - 2\beta)|_{\mathfrak{t}}$ is a weight lower than $-\beta$, the lowest weight of $\mathfrak{p}$. Hence, we have a contradiction if $\alpha - 2\beta \in \Delta(\mathfrak{h}, \mathfrak{g})$. If $\alpha \in \Delta_1^-(\mathfrak{h}; \mathfrak{g})$, an analogous (albeit upside-down) arguments show that neither $\alpha - \beta$ or $\alpha + 2\beta$ can be roots in $\Delta(\mathfrak{h}; \mathfrak{g})$.

Our last concern then would be the possible existence of a β-string $\ldots, \alpha - \beta, \alpha, \alpha + \beta, \ldots$ through a non-compact pure imaginary root $\alpha \neq -\beta$ such that $[h_\beta, E_\alpha] = -2E_\alpha$. In this case we'd have $[x_\beta, [x_\beta, E_\alpha]] \in \mathfrak{p}_{2\beta+\alpha}$, which is impossible since β is the highest weight of $\mathfrak{p}$. $\square$

Corollary 3.3. *If $z \in \mathfrak{p}$ and $[H_i, z] = -2z$, then $z \in span_{\mathbb{C}}(y_1, \ldots, y_i)$.*

Proof. Since the weights γ_i are all in the Weyl group orbit of the highest weight $\widetilde{\beta}$ of $\mathfrak{p}$, for each i we can choose a positive systems so that x_i is a highest weight vector in $\mathfrak{p}$. It then follows from the preceding lemma, that for each $i = 1, \ldots, n$, the lowest eigenvalue of h_i will be -2 and $[h_i, z] = -2z$ will imply that $z \in \mathbb{C}y_i$. Since the h_i are simultaneously diagonalizable, we can conclude that the smallest eigenvalue of $H_i = h_1 + \cdots + h_i$ will be -2 and that if $[H_i, z] = -2z$ then we must have $z \in span_{\mathbb{C}}(y_1, \ldots, y_i)$. $\square$

Let $\mathfrak{t}_1 = span_{\mathbb{C}}(h_1, \ldots, h_i)$ and let $\mathfrak{t}_0$ be the orthogonal complement of $\mathfrak{t}_1$ in $\mathfrak{t}$ (with respect to the Killing form). Let $\mathfrak{m}_i$ be the subalgebra of $\mathfrak{k}$ generated by root spaces $\mathfrak{k}_\alpha$ such that $\alpha|_{\mathfrak{t}_1} = 0$.

Lemma 3.4. *Let $\overline{\mathfrak{n}}_i$ be the direct sum of the negative eigenspaces of $ad(H_i)$ in $\mathfrak{k}$. Then $\mathfrak{m}_i + \overline{\mathfrak{n}}_i \subseteq \mathfrak{k}^{Y_i}$.*

Proof. Since -2 is the lowest eigenvalue of H_i, certainly $\overline{\mathfrak{n}}_i \subset \mathfrak{k}^{Y_i}$. Suppose k_α is a root vector corresponding to a root space $\mathfrak{t}_\alpha \subset \mathfrak{m}_i$. We then have $[H_i, k_\alpha] = 0$ and so k_α will preserve the (-2)-eigenspace of H_i. By the preceding lemma

$$(-2)\text{-eigenspace of } H_i = span_{\mathbb{C}}(y_1, \ldots, y_i).$$

Because the y_j, $j = 1, \dots, i$ are weight vectors corresponding to multiplicity-free weights of $\mathfrak{p}$, we must have

$$[k_\alpha, y_j] = c y_k \qquad \text{for some } k \in \{1, \dots, i\} - j \text{ and some } c \in \mathbb{C}.$$

But if $c \neq 0$, γ_k will not be strongly orthogonal to γ_j; for otherwise we would have $\gamma_k - \gamma_j = \alpha \in \Delta(\mathfrak{t}; \mathfrak{k})$. We conclude that Y_i commutes with every root vector in $\mathfrak{m}_i$ and so, since $\mathfrak{m}_i$ is semisimple, $[Y_i, \mathfrak{m}_i] = 0$. □

It now follows readily from results of Servedio [Sev] and Kimel'fel'd-Vinberg [KV] that the $K_{\mathbb{C}}$-orbit through Y_i is multiplicity free.[*3] However, we shall instead apply algebraic Frobenius reciprocity; so that we can not only demonstrate that $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ is multiplicity-free, but we can also identify the K-types.

Theorem 3.5 (Kostant, [Ko])**.** *Suppose G is a reductive algebraic group and V is an irreducible G-module. For $x \in V$, let G^x denote the stabilizer of x in G. If we denote by O_x the G-orbit through x, and by $\mathbb{C}[\mathcal{O}_x]$ the ring of everywhere-defined rational functions on $\mathcal{O}_x$, by V_λ, $\lambda \in \widehat{G}$, the representation space of an irreducible finite-dimensional representation of G and by $\widetilde{V_\lambda}$ the dual module of V_λ, then*

$$\text{multiplicity of } \lambda \text{ in } \mathbb{C}[\mathcal{O}_x] = \dim \widetilde{V_\lambda}^{G^x}$$

where $\widetilde{V_\lambda}^{G^x}$ is the space of vectors in $\widetilde{V_\lambda}$ that are fixed by G^x.

(In the preceding theorem the stabilizer G^x of x need not be reductive.)

Corollary 3.6. *$\mathbb{C}[\mathcal{O}_i]$ is multiplicity-free and V_λ is a K-type in $\mathbb{C}[\mathcal{O}_i]$ then $\lambda \in span_{\mathbb{R}}(\gamma_1, \dots, \gamma_i)$.*

Proof. By Lemma 3.4 the stabilizer of Y_i in $\mathfrak{k}$ contains $\mathfrak{m}_i + \overline{\mathfrak{n}}_i$. It is easy to see that the semisimple element h_j in the normal S-triple $\{x_j, h_j, y_j\}$ is an element of $\mathfrak{t}$ such that $[h_j, k_\alpha] = \langle \alpha, \gamma_j \rangle k_\alpha$ for any root vector $k_\alpha \in \mathfrak{k}_\alpha$, $\alpha \in \Delta(\mathfrak{t}; \mathfrak{k})$. Since the γ_i are chosen such that each $\mathfrak{t}$-weight $\gamma_1 + \cdots + \gamma_i$ is dominant and since $H_i \equiv h_1 + \cdots + h_i$, it follows that all the negative root vectors of $\mathfrak{k}$ will be contained in $\mathfrak{m}_i + \overline{\mathfrak{n}_i}$. Hence, a nonzero element of $\widetilde{V_\lambda}^{K^{Y_i}}$ will be a lowest weight vector that is also $\mathfrak{m}_\mathfrak{i}$-invariant. Algebraic Frobenius reciprocity then implies that if a K-type V_λ appears in $\mathbb{C}[\mathcal{O}_i]$ then the lowest

[*3] A little more explicitly, the argument would proceed as follows. It is easy to see that the tangent space to Y_i is generated by a parabolic subalgebra of $\mathfrak{k}$ corresponding to the non-negative eigenspaces of $ad_{\mathfrak{k}} H_i$. The preceding lemma then allows one to whittle this parabolic down to a Borel subalgebra. One then applies

Theorem ([Sev], [KV]) Let K be a connected reductive algebraic group acting on an irreducible affine variety M. Then $\mathbb{C}[M]$ is multiplicity-free if and only if there exists a Borel subgroup $B \subset K$ admitting an open orbit in M.

weight vector of $\widetilde{V_\lambda}$ must be $\mathfrak{m}_i$-invariant. This in turn implies that $-\lambda$ (and so λ) is not supported on $\mathfrak{t}_0$. Thus, we must have

$$\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i \in \mathfrak{t}_1^*.$$

And, of course, since the space of lowest weight vectors in $\widetilde{V_\lambda}$ will be 1-dimensional, algebraic Frobenius reciprocity also tells us that $\mathbb{C}[\mathcal{O}_i]$ is multiplicity-free. □

Proposition 3.7. *Let $n = |\Gamma|$.* (i) *If $i < n$, then V_λ is a K-type in $\mathbb{C}[\mathcal{O}_i]$, if and only if its highest weight is of the form*

$$\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$$

with $a_j \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_i \geq 0$. (ii) *$V_\lambda$ is a K-type in $\mathbb{C}[\mathcal{O}_n]$, if and only if λ is of the form*

$$\lambda = a_1\gamma_1 + \cdots + a_n\gamma_n$$

with $a_j \in \mathbb{Z}$ and

$$\begin{aligned} &a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n, \quad \textit{if } \Sigma = (a_n)^{m_a} \quad \textit{or} \quad (A_n)^{m_A}\,; \\ &a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n| \geq 0, \quad \textit{if } \Sigma = \ (d_n)^{m_d}\,; \\ &a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n \geq 0, \quad \textit{otherwise.} \end{aligned}$$

Proof. From the Corollary above, we know that if V_λ is a K-type in $\mathbb{C}[\mathcal{O}_i]$ then its highest weight must be of the form

$$\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$$

We first show that coefficients a_k must be integers. Note that

$$\exp(i\pi h_j) \cdot Y_i = \exp(-2i\pi)\, Y_i = Y_i \qquad \text{for } 1 \leq j \leq i$$

and so $k_j \equiv \exp(i\pi h_j) \in K^{Y_i}$. On the other hand, $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$ and $v_{-\lambda}$ is the lowest weight vector of $\widetilde{V_\lambda}$ we have

$$k_j \cdot v_{-\lambda} = \exp(-2i\pi a_j)\, v_{-\lambda}.$$

Thus, the lowest weight vector of $\widetilde{V_\lambda}$ will not be stabilized by $k_i \in K^{Y_i}$ unless $a_j \in \mathbb{Z}$ for $j = 1, \ldots, i$.

We now to prove the necessity of the ordering of the coefficients a_i. This is just a consequence of requirement that the highest weight λ be dominant. Since, in all cases, for all $1 \leq j < k \leq n$,

$$\gamma_j - \gamma_k \text{ or } \frac{1}{2}\gamma_i - \frac{1}{2}\gamma_j \in \Sigma^+$$

The dominance condition on $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$, leads to

$$a_j \geq a_{j+1} \qquad i = j = 1, \ldots, n-1$$

If the restricted root system Σ contains a d_n factor, then we must have in addition

$$\langle \lambda, \gamma_j + \gamma_k \rangle \geq 0$$

which, together with $\langle \lambda, \gamma_{n-1} - \gamma_n \rangle \geq 0$, implies in particular that

$$a_{n-1} \geq |a_n| \, .$$

In all other cases, Σ contains either a b_n or C_n factor, and this leads to the requirement that

$$\langle \lambda, \gamma_i \rangle \geq 0 \qquad \text{for all } i = 1, \ldots, n$$

which forces all the coefficients a_i to be non-negative.

At this point we have seen that if λ is the highest weight of a K-type in $\mathbb{C}\,[\mathcal{O}_i]$, then (1) λ must lie in the span of the γ_j, $j \leq i$, (2) the coefficients $a_1, \ldots, a_i$ of λ with respect to $\gamma_1, \ldots, \gamma_i$ must satisfy certain integrality conditions so that each $\exp(2\pi h_j) \in K^{Y_i}$ acts trivially on the corresponding highest weight vector and (3) the coefficients must be ordered in such a way that λ is dominant. What is not yet clear is that these restrictions on λ are sufficient to place V_λ in $\mathbb{C}\,[\mathcal{O}_i]$. However, it is easy to see that the integrality conditions we have imposed on λ are actually stronger than those needed to guarantee that λ is a weight of a representation of K. Thus, we have enumerated all possible finite dimensional representations of K with a K^{Y_i}-fixed vector. Sufficiency now follows from Theorem 3.5. □

The preceding proposition tells us exactly which K-types occur in the ring $\mathbb{C}[\mathcal{O}_i]$. However, we are actually most interested in the ring of regular functions $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ on the closure of the orbit. Clearly,

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] \subset \mathbb{C}[\mathcal{O}_i].$$

We will now show that each K-type V_λ occurring in $\mathbb{C}[\mathcal{O}_i]$ also occurs in $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$.

Theorem 3.8 (Kumar, [Ku]). *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let V_λ denote the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ. For any weight $\lambda \in \mathfrak{h}^*$, let $\overline{\lambda}$ denote the unique dominant weight in the Weyl group orbit of λ. Then, for any pair λ, μ of dominant weights and any w in the Weyl group of $\mathfrak{g}$, the irreducible $\mathfrak{g}$-module $V_{\overline{\lambda + w\mu}}$ occurs with multiplicity exactly one in the $\mathfrak{g}$-submodule $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ of $V_\lambda \otimes V_\mu$; where e_λ and $e_{w\mu}$ are, respectively, weight vectors in the λ-weight space of V_λ and the $w\mu$-weight space of V_μ.*

Remark 3.9. The statement of the theorem is known as Kostant's strengthened Parthasarathy-Ranga Rao-Varadarajan conjecture.

Lemma 3.10. *Let $\omega_j = \gamma_1 + \gamma_2 + \cdots + \gamma_j$, $1 \leq j \leq i$. Then the K-type V_{ω_j} occurs in $S^j(\mathfrak{p})$ and the monomial $x_1 \cdots x_j \in S^j(\mathfrak{p})$ has a non-trivial projection onto the highest weight space of V_{ω_j}.*

Proof. The case when $j = 1$ is trivial, since x_i is a highest weight vector of $\mathfrak{p}$. We now proceed by induction on i. And all that this requires is the preceding theorem with the identification of e_λ with the projection of $x_1 \cdots x_j$ onto the highest weight vector of V_{ω_j} (the inductive hypothesis) and the identification of $e_{w\mu}$ with x_{j+1} which, by our construction, is always extremal weight vector of $\mathfrak{p}$. □

Theorem 3.11. *The K-type decomposition of $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ is exactly*

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] = \bigoplus_{\lambda \in \Lambda_i} V_\lambda$$

where

$$\Lambda_i = \{\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i \mid a_j \in \mathbb{N}, \quad a_1 \geq a_2 \geq \cdots \geq a_i \geq 0\}$$

if $i < n$ or $\Sigma \neq (d_n)^{m_d}$. If $i = n$ and $\Sigma = (d_n)^{m_a}$, then

$$\Lambda = \{\lambda = a_1\gamma_1 + \cdots + a_n\gamma_n \mid a_j \in \mathbb{Z}, \quad a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n| \geq 0\}.$$

Proof. We have already seen that $\lambda \in \Lambda_i$ is a necessary condition for a K-type to be in $\mathbb{C}\left[\mathcal{O}_i\right] \supset \mathbb{C}\left[\overline{\mathcal{O}_i}\right]$. What we must prohibit, or otherwise take into account, is the existence of a rational function that is defined everywhere on $\mathcal{O}_i$ but does not extend to the boundary of $\mathcal{O}_i$; in particular, rational functions that are not defined at 0.

On the other hand, if ϕ is a polynomial function on $\mathfrak{p}$ that is supported at some point on $\mathcal{O}_i$ then $0 \neq \phi|_{\overline{\mathcal{O}_i}} \in \mathbb{C}\left[\overline{\mathcal{O}_i}\right]$. We now note that the monomials $x_1 \cdots x_j$ are supported at Y_i; for

$$\begin{aligned}(x_1 \cdots x_j)(Y_i) &= \langle x_1, Y_i\rangle \cdots \langle x_j, Y_i\rangle = \langle x_1, y_1 + \cdots + y_i\rangle \cdots \langle x_j, y_1 + \cdots + y_i\rangle \\ &= \langle x_1, y_1\rangle \cdots \langle x_j, y_j\rangle \\ &= 1.\end{aligned}$$

In fact, we can choose an orthogonal basis for $\mathfrak{p}$ such that $x_1, \ldots, x_j$, $j \leq i$, are the only coordinate functions supported at Y_i. By the preceding lemma, for each $j = 1, \ldots, i$, there exists a (homogeneous) highest weight vector ϕ_{ω_j} of $V_{\omega_j} \subset S^j(\mathfrak{p})$ of the form

$$\phi_{\omega_j} = x_1 \cdots x_j + (\text{other terms with at least one factor not among } x_1, \ldots, x_i).$$

(Note that by the homogeneity of ϕ_{ω_i} and the linear independence of the weights $\gamma_1, \ldots, \gamma_i$, we cannot have a factor x_k, $j < k \leq i$ occurring in one of the "other terms" without an accompanying coordinate outside of $\{x_1, \ldots, x_i\}$.) The "other terms" will thus die upon evaluation at Y_i and so we must have

$$\phi_{\omega_j}(Y_i) = 1 \qquad \text{for} \quad 1 \leq j \leq i.$$

But once we have the highest weight vectors of each V_{ω_j} supported at Y_i it is trivial to show that the products of these highest weight vectors will remain highest weight vectors and continue to be supported at $Y_i \in \mathcal{O}_i$. Thus, each of the K-types V_λ with

$$\lambda \in \Lambda_i' \equiv \{\alpha_1\omega_1 + \cdots + \alpha_i\omega_i \mid \alpha_i \in \mathbb{N}\}$$

will be supported at Y_i. We now observe $\Lambda_i = \Lambda_i'$ So each of the K-types specified in the statement of the theorem definitely appears in $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$. Comparing with Proposition 3.7, we can conclude that in fact so long as $i \neq n$ and $\Sigma \neq A_n$ or a_n, we have

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] = \mathbb{C}\left[\mathcal{O}_i\right] = \bigoplus_{\lambda\in\Lambda_i} V_\lambda$$

However, when $i = n$ and $\Sigma = A_n$ or a_n we seem to have K-types $V_{\alpha_1\gamma_1+\cdots+a_n\gamma_n}$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_n < 0$ appearing in $\mathbb{C}\left[\mathcal{O}_n\right]$ that have not been accounted for in $\mathbb{C}\left[\overline{\mathcal{O}_n}\right]$. In fact, they do not occur in $\mathbb{C}\left[\overline{\mathcal{O}_n}\right]$; yet they can nevertheless be easily taken into account.

Note that the restricted root system Σ is of type A_n or a_n only when G/K is Hermitian symmetric (see also [Sa1]). When $Sigma$ is of this form it is easy to see that $\omega_n = \gamma_1 + \cdots + \gamma_n \in \mathfrak{t}^*$ is perpendicular to every root in $\Delta(\mathfrak{t};\mathfrak{k})$, being supported only on the center of $\mathfrak{k}$. The corresponding highest weight vector $\phi_{\omega_n} \in S^n(\mathfrak{p})$ thus corresponds to a one-dimensional K-type which, as shown above, does not vanish at Y_n. Put another way, ϕ_{ω_n} corresponds to a K-semi-invariant polynomial that does not vanish at $Y_n \in \mathcal{O}_n$. But then it vanishes nowhere on $\mathcal{O}_n$ (otherwise $\mathcal{O}_n$ would have a proper K-invariant subset). Yet being a homogeneous polynomial of degree n, it certainly vanishes at $0 \in \overline{\mathcal{O}_n}$. Now observe that if $\lambda = a_1\gamma_1 + \cdots + a_n\gamma_n \in \Lambda_n$, $a_n \geq 0$, and ϕ_λ is a polynomial corresponding the highest weight vector of $V_\lambda \subset \mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ then

$$\psi = \left(\phi_{\omega_n}\right)^{-s}\left(\phi_\lambda\right)$$

will be a rational function of highest weight $(a_1 - s)\gamma_1 + \cdots + (a_n - s)\gamma_n$ that is everywhere defined on $\mathcal{O}_n$ but undefined at 0 if $a_n - s < 0$. Evidently, such functions account for the highest weight vectors of all the K-types in $\mathbb{C}\left[\mathcal{O}_n\right]$ that have not already been shown to appear in $\mathbb{C}\left[\overline{\mathcal{O}_n}\right]$. The statement of the theorem thus provides a complete account of the K-types in $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$, even when $i = n$ and $\Sigma = A_n$ or a_n. □

3.1. Remarks

3.1.1. Connection with spherical orbits for symmetric pairs While our construction always yields a spherical nilpotent $K_{\mathfrak{C}}$-orbit in $\mathfrak{p}$ in the sense of [Ki], our construction falls short of producing all such orbits. For example, we cannot obtain any spherical orbit whose closure is not a normal variety; for in our construction the ring of regular functions on an orbit coincides with the ring of regular functions on its closure (well, except in the Hermitian symmetric case where they nevertheless coincide up to a character of the center of K). To indicate which spherical nilpotent orbits are obtainable by our construction, we display in Appendix A how the orbits we have constructed are situated within the Hasse diagrams depicting the closure relations ([D]) among the entire family of spherical nilpotent orbits as classified by King ([Ki]).

From these diagrams one can understand the "range" of our construction in the following way: if $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a strongly orthogonal sequence of noncompact weights, the corresponding orbit closures $\overline{\mathcal{O}_i}$ are totally ordered by inclusion

$$\{0\} = \overline{\mathcal{O}_0} \subset \overline{\mathcal{O}_1} \subset \overline{\mathcal{O}_2} \subset \cdots \subset \overline{\mathcal{O}_n}.$$

and are such that $\overline{\mathcal{O}_i} = \overline{\mathcal{O}_{i+1}} - \mathcal{O}_{i+1}$. Such a sequence of orbits would correspond to a chain in the Hasse diagram in which there are no branchings are encountered as one descends from the top of the chain to the trivial orbit. Nevertheless, in several cases our construction exhausts or nearly exhausts the set of spherical nilpotent orbits: for example, we get all the spherical orbits of $SL(n, \mathbb{R})$, $SL(n, \mathbb{H})$, and $SO(3, p)$; and all but two orbits for $SO(2, p)$ and $Sp(p, q)$.

3.1.2. Connection with unipotent representations We can now describe in a little more detail how we hope to attach unipotent (in the sense of [Vo], Conjecture 12.1) representations to these orbits. In [Sa2], Siddhartha Sahi shows the existence of a certain family of small unitary irreducible representations of the conformal groups of simple non-Euclidean Jordan algebras. Deliberately putting aside the Jordan theoretical underpinnings of these representations, one can say that the essential representation-theoretical ingredients of Sahi's construction and analysis are:

(i) the circumstance that each representation is realized as a constituent of a non-unitary, degenerate, spherical principal series representation

$$I(s) = Ind_{P=MAN}^{G} \left(1 \otimes e^{s\nu} \otimes 1\right)$$

whose associated variety is the closure of a single, multiplicity-free $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$; and

(ii) the fact that there exists a $w \in N_K(\mathfrak{a})$ such that both $wPw^{-1} = \overline{P}$ and $ad^*(w)\,\nu = -\nu$.

We remark that the second condition ensures that the underlying $(\mathfrak{g}, K)$-module of $I(s)$ can be endowed with a $\mathfrak{g}$-invariant (but possibly indefinite or degenerate) Hermitian form ([KZ]), while the first condition allows Sahi to carry out an explicit analysis of the action of $\mathfrak{p}$ on K-types from which both signature characters and reducibility conditions can be derived.

Using the results of Section 2 and Section 3, we can formulate a similar setup for any connected semisimple Lie group, subsuming the situation of [Sa2] in a uniform manner. Let G be such a group, $\Gamma = [\gamma_1, \ldots, \gamma_n]$ a maximal sequence of strongly orthogonal noncompact weights. As in Remark 2.1.2 we construct for each $i = 1, \ldots, n$, a normal S-triple $\{x_i, h_i, y_i\}$ such that $x_i \in \mathfrak{p}_{\gamma_i}$, $y_i = -\overline{x_i} \in \mathfrak{p}_{-\gamma_i}, h_i \in \mathfrak{k}$. We set

$$X_n = x_1 + \cdots + x_n, \quad H_n = h_1 + \cdots + h_n, \quad Y_n = y_1 + \cdots + y_n$$

as in (3.3) and apply each of the (commuting) Cayley transforms (2.2) successively to $\{X_n, H_n, Y_n\}$ to obtain a standard triple

$$\{X_n', H_n', Y_n'\} = \left\{\frac{1}{2}(X_n + Y_n - iH_n),\ -i(X_n - Y_n),\ \frac{1}{2}(X_n + Y_n + iH_n)\right\}$$

in real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of G such that the semisimple element H_n' is in $\mathfrak{p}_{\mathbb{R}}$ and $\theta X_n' = -Y_n'$. Let

$$\begin{aligned}
\mathfrak{n} &= \text{direct sum of positive eigenspaces of } ad(H_n') \text{ in } \mathfrak{g}_{\mathbb{R}},\\
\mathfrak{l} &= \text{0-eigenspace of } ad(H_n') \text{ in } \mathfrak{g}_{\mathbb{R}},\\
\mathfrak{a} &= (\text{center of } \mathfrak{l}) \cap \mathfrak{p}_{\mathbb{R}},\\
\mathfrak{m} &= \text{orthogonal complement of } \mathfrak{a} \text{ in } \mathfrak{l}
\end{aligned}$$

and set

$$\begin{aligned}
M &= Z_K(\mathfrak{a}) \exp(\mathfrak{m}),\\
A &= \exp(\mathfrak{a}),\\
N &= \exp(\mathfrak{n}).
\end{aligned}$$

Then $P = MAN$ is a (Langlands decomposition of a) parabolic subgroup of G. Moreover, it happens that

$$span_{\mathbb{R}}(h_1', \ldots, h_n') \subseteq \mathfrak{a}.$$

Now let ν be the element of the real dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ such that

$$\nu(H) = B_0(H_n', H) \qquad \forall\ H \in \mathfrak{a},$$

where $B_0(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}_{\mathbb{R}}$ restricted to $\mathfrak{a}$. Finally, we set

$$w = \exp\left(\frac{\pi}{2}(X_n' - Y_n')\right) \in K.$$

It then happens that

$$w \in N_K(\mathfrak{a}), \quad wPw^{-1} = \overline{P}, \quad Ad^*(w)\nu = -\nu.$$

Thus, we have a natural means of attaching to our families of multiplicity-free $K_{\mathbb{C}}$-orbits families of possibly unitarizable, degenerate principal series representations.

However, there is one more Jordan-theoretic device at play in Sahi's paper; and that is a generalized Capelli operator D_1 that, for a particular value of the parameter $s \in \mathbb{R}$, intertwines $I(s)$ with its Hermitian dual $I(-s)$. And yet, even this Capelli operator should be characterizable in purely representation-theoretic terms as per [Bo]. In fact, we conjecture here there is there is a quasi-invariant differential operator on $C^\infty(\overline{\mathfrak{n}})$ corresponding to the Cayley transform of a lowest weight vector of the irreducible representation homogeneous summand of $S^n(\mathfrak{p})$ of highest weight $\gamma_1 + \cdots + \gamma_n$ that intertwines a $I(s)$

with its Hermitian dual. In a subsequent paper, we hope to confirm (or correct) this conjecture and to extend the methods of [KS] and [Sa2] to study of the reducibility and signature characters of the subrepresentations $I_P(s)$ associated with the orbits $K \cdot Y_i$.

4. Dimension and degree of $\overline{\mathcal{O}_i}$

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ regarded as a graded commutative ring (the grading by degree of homogeneity). If $I \subset R$ is homogeneous ideal then $M = R/I$ is a graded R-module:

$$M = \bigoplus_{n=0}^{\infty} M_n \quad \text{where} \quad M_n \equiv R_n / (R_n \cap I).$$

Let Y be the corresponding affine variety. By a theorem of Hilbert and Serre, there is a unique polynomial $p_Y(t)$ such that

$$p_Y(t) = \sum_{k=0}^{t} \dim_{\mathbb{C}} M_k \qquad \text{for all sufficiently large } t.$$

$p_Y(t)$ is called the Hilbert polynomial of Y. When one writes

$$p_Y(t) = ct^d + \left(\text{terms of order } t^{d-1}\right)$$

the degree of the leading term is the *dimension* (often by definition) of Y and the number

$$D = \frac{c}{d!}$$

is the *degree* of Y. It turns out that D is always an integer and corresponds to the number of points where the projectivization of Y meets a generic $(n-d-1)$-dimensional linear subspace of $\mathbb{P}^{n-1}$.

Now consider a $K_{\mathbb{C}}$-orbit $\mathcal{O}_i$ associated to a sequence $\Gamma_i = \{\gamma_1, \ldots, \gamma_i\}$ of strongly orthogonal noncompact weights and the ring of regular functions $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ on its closure. We shall assume for ease of exposition that $i < n = |\Gamma|$. The cases when $i = n$ can be handled similarly; but not so uniformly. It follows from the proof of Theorem 3.11 that a given K-type $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$ in $\mathbb{C}[\mathcal{O}_i]$ is generated by the action of $\mathfrak{k}$ on

$$\phi_\lambda = (\phi_{\omega_1})^{a_1 - a_2} (\phi_{\omega_2})^{a_2 - a_3} \cdots (\phi_{\omega_i})^{a_i}$$

which is a homogeneous polynomial of degree (taking $a_j = 0$ for $j > i$)

$$\sum_{j=1}^{i} j(a_j - a_{j+1}) = \sum_{j=1}^{i} a_j.$$

It follows that

$$p_{\overline{\mathcal{O}_i}}(t) = \sum_{\ell=0}^{t} \dim M_\ell = \sum_{\lambda \in \Lambda_t} \dim V_\lambda$$

where

$$\Lambda_t = \left\{ a_1\gamma_1 + \cdots + a_i\gamma_i \mid a_1, \ldots, a_i \in \mathbb{N}\ ,\ a_1 \geq a_2 \geq \cdots \geq a_i \geq 0\ ,\ \sum_{j=1}^{i} a_i \leq t \right\}\ .$$

Applying the Weyl dimension formula, we obtain

$$p_{\overline{\mathcal{O}_i}}(t) = \sum_{\lambda \in \Lambda_t} \left(\prod_{\alpha \in \Delta^+(\mathfrak{t};\mathfrak{k})} \frac{\langle \lambda + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} \right).$$

Now note that the factors $\langle \lambda + \rho_K, \alpha \rangle / \langle \alpha, \rho_k \rangle$ either reduce to factors of 1 (when $\alpha \perp \lambda$) or contribute factors of the form $\langle \lambda, \alpha \rangle / \langle \alpha, \rho_K \rangle$ to the leading term of $p_{\overline{\mathcal{O}_i}}$. We thus need only account for the roots that have components along $\gamma_1, \ldots, \gamma_i$. Let

$$\Delta_i^+ = \left\{ \alpha \in \Delta^+(\mathfrak{t};\mathfrak{k}) \mid \langle \alpha, \gamma_j \rangle \neq 0 \text{ for some } j \in \{1, \ldots, i\} \right\}$$

The leading term of the Hilbert polynomial for $p_{\overline{\mathcal{O}}_i}(t)$ is thus

$$LT\left(p_{\overline{\mathcal{O}}_i}\right) = \left(\prod_{\alpha \in \Delta_i^+} \frac{1}{\langle \rho_K, \alpha \rangle} \right) LT\left(\sum_{\lambda \in \Lambda_t} \left(\prod_{\alpha \in \Delta_i^+} \langle \lambda, \alpha \rangle \right) \right)$$

To compute the products $\prod_{\alpha \in \Delta_i^+} \langle \lambda, \alpha \rangle$, we just need to know Σ, the restricted root systems with multiplicities associated with $\{\gamma_1, \ldots, \gamma_n\}$. Since each restricted root will be of one of the types $a_n,\ A_n, \ldots, d_n$,and since the roots of each type share a common multiplicity, we can write

$$\prod_{\alpha \in \Delta_i^+} \langle \lambda, \alpha \rangle = \left(\prod_{\alpha \in (a_n)_i^+} \langle \lambda, \alpha \rangle \right)^{m_a} \left(\prod_{\alpha \in (A_n)_i^+} \langle \lambda, \alpha \rangle \right)^{m_A} \cdots\cdots \left(\prod_{\alpha \in (d_n)_i^+} \langle \lambda, \alpha \rangle \right)^{m_d}$$

where

$$\begin{aligned}
(a_n)_i^+ &= \left\{ \frac{1}{2}\gamma_j - \frac{1}{2}\gamma_k \mid 1 \leq j < k \leq i \right\} \cup \left\{ \frac{1}{2}\gamma_j - \frac{1}{2}\gamma_k \mid 1 \leq j \leq i < k \leq n \right\}, \\
(A_n)_i^+ &= \{\gamma_j - \gamma_k \mid 1 \leq j < k \leq i\} \cup \{\gamma_j - \gamma_k \mid 1 \leq j \leq i < k \leq n\}\,, \\
(b_n)_i^+ &= \left\{ \frac{1}{2}\gamma_j \mid 1 \leq j \leq i \right\}, \\
(C_n)_i^+ &= \{\gamma_j \mid 1 \leq j \leq i\}\,, \\
(a_n)_i^+ &= \left\{ \frac{1}{2}\gamma_j \pm \frac{1}{2}\gamma_k \mid 1 \leq j < k \leq i \right\} \cup \left\{ \frac{1}{2}\gamma_j \pm \frac{1}{2}\gamma_k \mid 1 \leq j \leq i < k \leq n \right\}.
\end{aligned}$$

We thus have

$$\left(\prod_{\alpha\in(a_n)_i^+}\langle\lambda,\alpha\rangle\right)^{m_a} = \left(\frac{1}{2}\right)^{m_a i(2n-i-1)/2}\left(\prod_{1\le j<k\le i}(a_j-a_k)^{m_a}\right)\left(\prod_{1\le j\le i}(a_j)^{m_a(n-i)}\right),$$

$$\left(\prod_{\alpha\in(A_n)_i^+}\langle\lambda,\alpha\rangle\right)^{m_A} = \left(\prod_{1\le j\le i}(a_j)^{m_A(n-i)}\right)\left(\prod_{1\le j<k\le i}(a_j-a_k)^{m_A}\right),$$

$$\left(\prod_{\alpha\in(b_n)_i^+}\langle\lambda,\alpha\rangle\right)^{m_b} = \left(\frac{1}{2}\right)^{im_b}\left(\prod_{1\le j\le i}(a_j)^{m_b}\right),$$

$$\left(\prod_{\alpha\in(C_n)_i^+}\langle\lambda,\alpha\rangle\right)^{m_C} = \left(\prod_{1\le j\le i}(a_j)^{m_C}\right),$$

$$\left(\prod_{\alpha\in(d_n)_i^+}\langle\lambda,\alpha\rangle\right)^{m_d} = \left(\frac{1}{2}\right)^{m_d i(2n-i-1)}\left(\prod_{1\le j<k\le i}(a_j+a_k)^{m_d}\right) \times\left(\prod_{1\le j<k\le i}(a_j-a_k)^{m_d}\right)\left(\prod_{1\le j\le i}(a_j)^{2(n-i)m_d}\right).$$

And so we get

$$\left(\prod_{\alpha\in\Delta_i^+}\langle\lambda,\alpha\rangle\right) = \left(\frac{1}{2}\right)^{\frac{1}{2}i(2n-i-1)(m_a+2m_d)+im_b}\left(\prod_{1\le j\le i}a_j\right)^{m_b+m_C+(n-i)(m_a+m_A+2m_d)} \times\left(\prod_{1\le j<k\le i}(a_j-a_k)\right)^{m_a+m_A}\left(\prod_{1\le j<k\le i}\left(a_j^2-a_k^2\right)\right)^{m_d}.$$

In summary,

Lemma 4.1.

$$LT\left(p_{\overline{\mathcal{O}}_i}\right) = c_i\ LT\left(\sum_{\lambda\in\Lambda_t}\left(\prod_{1\le j\le i}a_j\right)^q\left(\prod_{1\le j<k\le i}(a_j-a_k)\right)^r\left(\prod_{1\le j<k\le i}\left(a_j^2-a_k^2\right)\right)^s\right)$$

where

$$\begin{aligned} c_i &= \frac{1}{\prod_{\alpha\in\Delta_i^+}\langle\rho_K,\alpha\rangle}\left(\frac{1}{2}\right)^{\frac{1}{2}i(2n-i-1)(m_a+2m_d)+im_b}, \\ q &= (n-i)\,(m_a+m_A+2m_d)+(m_b+m_C)\,, \\ r &= m_a+m_A, \\ s &= m_d. \end{aligned}$$

Next we observe that the sum over Λ_t can be carried out as an iterated sum of the form

$$\sum_{\lambda\in\Lambda_t}(\cdots)=\sum_{a_1=0}^{t}\ \sum_{a_2=0}^{\min(a_1,t-a_1)}\cdots\sum_{a_i=0}^{\min(a_{i-1},t-a_1-\cdots-a_{i-i})}(\cdots)$$

and that the quantity to be summed,

$$F(\mathbf{a})\equiv\left(\prod_{j=1}^{i}(a_j)^q\right)\left(\prod_{1\le j<k\le i}(a_j-a_k)^r\left(a_j^2-a_k^2\right)^s\right)$$

is homogeneous of degree

$$deg(F)=qi+i(i-1)(r+2s)/2$$

in the variables $a_1,\ldots,a_i$.

Lemma 4.2. *Suppose $\Omega_t\subset\mathbb{N}^n$ is a region of the form*

$$\Omega_t=\{\mathbf{a}\in\mathbb{N}^n\mid 0\le a_1\le t\ ,\ 0\le a_2\le\phi_2(t,a_1)\ ,\ \cdots,0\le a_n\le\phi_n(t,a_1,\ldots,a_{n-1})\}\,,$$

where each ϕ_i is a homogeneous linear function of its arguments. Then for large t

$$\sum_{\mathbf{a}\in\Omega_t}a_1^{m_1}\cdots a_n^{m_n}=$$
$$\int_0^t\int_0^{\phi_2(t,x_1)}\cdots\int_0^{\phi_n(t,x_1,\ldots,x_{n-1})}x_1^{m_1}\cdots x_n^{m_n}\,dx_n\cdots dx_1+\text{ lower order terms.}$$

Proof. From Faulhaber's formula [F]

$$\sum_{i=0}^{t}t^p=\frac{1}{p+1}\sum_{k=1}^{p+1}(-1)^{\delta_{k,p}}\binom{p+1}{k}B_{p+1-k}t^k$$

(where $\delta_{k,p}$ is the Kronecker delta symbol, $\left(\begin{array}{c} a \\ b \end{array}\right)$ is the usual binomial coefficients, and B_q is the q^{th} Bernoulli number) one sees that

$$\begin{aligned} S_p(t) \equiv \sum_{i=0}^{t} i^p &= \frac{1}{p+1} t^{p+1} + \frac{1}{2} t^p - \frac{p}{12} t^{p-1} + \cdots \\ &= \int_0^t x^p dx + \text{ lower order terms.} \end{aligned}$$

The result now follows from an easy computation and inductive argument. $\square$

Remark 4.3. Note that

$$P(t) = \int_0^t \int_0^{\phi_2(t,x_1)} \cdots \int_0^{\phi_n(t,x_1,\ldots,x_{n-1})} x_1^{m_1} \cdots x_n^{m_n} dx_n \cdots dx_1$$

is a monomial of degree $m_1 + \cdots + m_n + n$ in t. Its leading coefficient is thus

$$P(1) = \int_0^1 \int_0^{\phi_2(1,x_1)} \cdots \int_0^{\phi_n(1,x_1,\ldots,x_{n-1})} x_1^{m_1} \cdots x_n^{m_n} dx_n \cdots dx_1.$$

Thus if we set

$$\begin{aligned} R_t = \{\mathbf{x} \in \mathbb{R}^n \mid 0 \le x_1 \le 1, 0 \le x_2 \le \phi_2(1, x_1), \\ \cdots, 0 \le x_n \le \phi_n(1, x_1, \ldots, x_{n-1})\} \end{aligned}$$

we have

$$\sum_{\mathbf{a} \in \Omega_t} a_1^{m_1} \cdots a_n^{m_n} \approx \left(\int_{R_1} x_1^{m_1} \cdots x_n^{m_n} d^n x \right) t^{m_1 + \cdots + m_n + n}.$$

Proposition 4.4. *Let*

$$\mathcal{S}_{n,t} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge 0 \ , \ \sum_{i=1}^{n} x_i \le t \right\}$$

and

$$\Lambda_t = \mathcal{S}_{n,t} \cap \mathbb{N}^n$$

If $F(x_1, \ldots, x_n)$ is homogeneous of degree d, then

$$\sum_{\mathbf{a} \in \Lambda_t} F(\mathbf{a}) = \left(\int_{\mathcal{S}_{n,1}} F(\mathbf{x}) \, dx_n \cdots dx_1 \right) t^{d+n} + \textit{lower order terms.}$$

Proof. We can decompose the sum over Λ_t into a sum of sums

$$\sum_{\Lambda_t} F(a_1, \cdots, a_n) = \sum_{i=1}^{N} \sum_{\Lambda_{t,i}} F(a_1, \cdots, a_n)$$

where each region $\Lambda_{t,i}$ is a region of the elementary form considered in Lemma (4.1). One can then apply the lemma and the subsequent remark to get

$$\sum_{\Lambda_t} F(a_1, \cdots, a_n) \approx \sum_{i=1}^{N} \left(\int_{R_{1,i}} F(x_1, \cdots, x_n)\, d^n x \right) t^{d+n} + \text{ lower order terms}$$

and then reassemble the region $\mathcal{S}_{n,1}$ from the regions $R_{1,i}$ to obtain the desired result. □

Applying the preceding proposition to our formula for the Hilbert polynomial $p_{\overline{\mathcal{O}_i}}(t)$, we can conclude:

Theorem 4.5. *Let G be a semisimple Lie group, $\{\gamma_1, \ldots, \gamma_n\}$ a maximal sequence of strongly orthogonal noncompact weights, Σ the corresponding restricted root system (specified as in (2.1)), and let $\mathcal{O}_i$ be the multiplicity-free $K_{\mathcal{C}}$-orbit associated to a subsequence $\{\gamma_1, \ldots, \gamma_i\}$. The dimension of $\overline{\mathcal{O}_i}$ is given by*

$$\dim(\mathcal{O}_i) = i(q+1) + i(i-1)(r+2s)/2$$

and its degree is given by

$$Deg\left(\overline{\mathcal{O}_i}\right) = \frac{c_i}{\dim(\mathcal{O}_i)!} \left(\prod_{\alpha \in \Delta^+(\mathfrak{t};\mathfrak{k})} \frac{1}{\langle \rho_K, \alpha \rangle} \right)$$
$$\times \int_{\mathcal{S}_i} \left(\prod_{j=1}^{i} x_i \right)^q \left(\prod_{1 \le j < k \le i} (x_j - x_k) \right)^r \left(\prod_{1 \le j < k \le i} \left(x_j^2 - x_k^2\right) \right)^s d^i x$$

where $\mathcal{S}_i$ is the domain

$$\mathcal{S}_i = \left\{ x \in \mathbb{R}^i \mid x_1 \ge x_2 \ge \cdots \ge x_i \ge 0 \quad , \quad \sum_{j=1}^{i} x_j \le 1 \right\}$$

and the constants c_i, q, r and s are as in Lemma 4.1 *and Table* 2 *below.*

Remark 4.6. From the data tabulated above, one finds that the formula for the degree of $\overline{\mathcal{O}_i}$ is either of the form

$$\int_{S_i} \left(\prod_{j=1} x_j \right)^q \left(\prod_{1 \le j < k \le i} (x_j - x_k) \right)^r d^i x \tag{4.1}$$

(which happens only when G/K is Hermitian symmetric), or

$$\int_{S_i} \left(\prod_{j=1} x_j \right)^q \left(\prod_{1 \le j < k \le i} \left(x_j^2 - x_k^2\right) \right)^s d^i x. \tag{4.2}$$

Table 2.

G	$\dim \mathcal{O}_i$	q	r	s
$SL(n,\mathbb{R})$	$i\left(2\left[\frac{n}{2}\right]-i\right)$	$2\left(\left[\frac{n}{2}\right]-i\right)$	0	1
$SL(n,\mathbb{H})$	$4i\left(2\left[\frac{n}{2}\right]-i\right)$	$8\left(\left[\frac{n}{2}\right]-i\right)+3$	0	4
$SU(p,q)$	$i(p+q-i)$	$p+q-2i$	2	0
$SO(2,q)$, $q>2$	$i(i(2-q)+q-4)/2$	$(q-2)(2-i)$	$q-2$	0
$SO(p,q)$ I , $2<p\leq q$	$i(p+q-2i-1)$	$p+q-4i$	0	2
$SO^*(2n)$	$i\left(8\left[\frac{n}{2}\right]-4i-1\right)$	$8\left(\left[\frac{n}{2}\right]-i\right)+2$	0	4
$Sp(n,\mathbb{R})$	$i(2n-i+1)/2$	$n-i$	1	0
$Sp(p,q)$	$2i(p+q-i+1)$	$2(p+q-2i+1)$	0	2

The first form is very much akin to the famous Selberg integral [Se]

$$S_{n,r,s,t}=\frac{1}{n!}\int_{[0,1]^n}\left(\prod_{i=1}^{n}x_i\right)^s\left(\prod_{i=1}^{n}(1-x_i)\right)^t\left(\prod_{1\leq i<j\leq n}|x_i-x_j|\right)^r d^nx.$$

Indeed, by setting $t=0$ and making a change of variables ([M], pg. 286) (4.1) can be explicitly evaluated using Selberg's formula. The more generic case (4.2), however, seems to be lacking an explicit evaluation. We do note, however, that Nishiyama, Ochiai and Zhu [NOZ] encountered and evaluated certain integrals of the form (4.2) in their study of theta liftings of nilpotent orbits. In [B] we provide several other methods of evaluating integrals of the general form (4.2).

Appendix A. Closure relations for spherical nilpotent orbits of classical real linear Groups

To indicate exactly which spherical orbits are constructible by our sequences of strongly orthogonal noncompact weights, we display below the closure relations ([O], [D]) for the spherical orbits of classical real linear Lie groups([Ki]); or rather those cases for which we've identified a nice pattern (the closure diagrams of $SU(p,q)$ and $SO(p,q)$ get rather complicated as p and q increase). The double lines in the diagram indicate the simple chains of spherical nilpotent orbits closures corresponding to sequences of strongly orthogonal noncompact weights (cf. Remark 2.1.3.). In the Hermitian symmetric cases we indicate both the chains lying in $\mathfrak{p}_+$ and those lying in $\mathfrak{p}_-$. Our notation for the orbits is somewhere between that of [Ki] and [D]. Briefly, as in [Ki] we indicate particular orbits by expressions of the form $(\pm n_1)^{m_1}(\pm n_2)^{m_2}\cdots(\pm n_k)^{m_k}$, where a factor of the form $(\pm n_i)^{m_i}$ indicates the occurrence of a signed a row of alternating '+' and '−' signs, of length n_i, beginning with a $\pm$ sign, and occurring with multiplicity m_i. However, Djokovic's algorithm makes use of unsigned rows (actually, unsigned "genes") rather than rows that are more commonly represented as even signed rows; we indicate such an unsigned row of length n occurring with multiplicity m by a factor of the form $(n)^m$. Thus, for example,

$$(+3)^2(2)(+1)^2(-1) \quad \sim \quad \begin{ytableau} + & - & + \\ + & - & + \\ \; & \; \\ + \\ + \\ - \end{ytableau} \; .$$

Figure 1. $SL\,(n,\mathbb{R})$

$$\begin{tikzcd}[column sep=small]
\mathcal{O}^{I}_{(2)^{[n/2]}} \arrow[dr, Rightarrow] & & \mathcal{O}^{II}_{(2)^{[n/2]}} \arrow[dl, Rightarrow] \\
& \mathcal{O}_{(2)^{[n/2]-1}(1)^2} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(2)^{[n/2]-2}(1)^4} \arrow[d, Rightarrow] & \\
& \vdots \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(1)^n} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{O}_{(2)^{[n/2]}} \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)^{[n/2]-1}(1)^2} \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)^{[n/2]-2}(1)^4} \arrow[d, Rightarrow] \\
\vdots \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] \\
\mathcal{O}_{(1)^n}
\end{tikzcd}$$

n even $\qquad\qquad$ n odd

Figure 2. $SU\,(2,q)$

$$\begin{tikzcd}[column sep=tiny]
& & & \mathcal{O}_{(-3)^2(-1)^{q-4}} \arrow[dlll] \arrow[drrr] & & & \\
\mathcal{O}_{(-3)(-2)(-1)^{q-3}} \arrow[dd] \arrow[drrr] & & & & & & \mathcal{O}_{(-3)(+2)(-1)^{q-3}} \arrow[dlll] \arrow[dd] \\
& & & \mathcal{O}_{(-3)(+1)(-1)^{q-1}} \arrow[d] & \mathcal{O}_{(+3)(-1)^{q-1}} \arrow[dl] & & \\
\mathcal{O}_{(-2)^2(-1)^{q-2}} \arrow[drr, Rightarrow] & & & \mathcal{O}_{(+2)(-2)(-1)^{q-2}} \arrow[dl] \arrow[dr] & & & \mathcal{O}_{(+2)^2(-1)^{q-2}} \arrow[dll, Rightarrow] \\
& & \mathcal{O}_{(-2)(+1)(-1)^{q-1}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(+2)(+1)(-1)^{q-1}} \arrow[dl, Rightarrow] & & \\
& & & \mathcal{O}_{(+1)^2(-1)^q} & & &
\end{tikzcd}$$

Figure 3. $SL\,(n,\mathbb{H})$

$$
\begin{array}{c}
\mathcal{O}_{(2)^{[n/2]}} \\
\Downarrow \\
\mathcal{O}_{(2)^{[n/2]-1}(1)^2} \\
\Downarrow \\
\vdots \\
\Downarrow \\
\mathcal{O}_{(2)(1)^{n-2}} \\
\Downarrow \\
\mathcal{O}_{(1)^n} \\
\\
n \text{ even}
\end{array}
\qquad\qquad
\begin{array}{c}
\mathcal{O}_{(2)^{[n/2]}(1)} \\
\Downarrow \\
\mathcal{O}_{(2)^{[n/2]-1}(1)^3} \\
\Downarrow \\
\vdots \\
\Downarrow \\
\mathcal{O}_{(2)(1)^{n-2}} \\
\Downarrow \\
\mathcal{O}_{(1)^n} \\
\\
n \text{ odd}
\end{array}
$$

Figure 4. $SO\,(2,p) \quad ; \quad p > 4$

$$
\begin{tikzcd}
 & & \mathcal{O}_{(-3)^2(-1)^{p-4}} \arrow[d] & & \\
\mathcal{O}^{I}_{(+3)(-1)^{p-1}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(-3)(+1)(-1)^{p-2}} \arrow[dl] \arrow[dr] & & \mathcal{O}^{II}_{(+3)(-1)^{p-1}} \arrow[dl, Rightarrow] \\
 & \mathcal{O}^{I}_{(+2)(-2)(-1)^{p-2}} \arrow[dr, Rightarrow] & & \mathcal{O}^{II}_{(+2)(-2)(-1)^{p-2}} \arrow[dl, Rightarrow] & \\
 & & \mathcal{O}_{(+1)^2(-1)^{p}} & &
\end{tikzcd}
$$

Figure 5. $SO^*\,(2n)$

$$
\begin{tikzcd}
\mathcal{O}_{[\frac{n}{2}],0} \arrow[dr, Rightarrow] & & \mathcal{O}_{[\frac{n}{2}]-1,1} \arrow[dl] \arrow[dr] & & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & & \mathcal{O}_{1,[\frac{n}{2}]-1} \arrow[dl] \arrow[dr] & & \mathcal{O}_{0,[\frac{n}{2}]} \arrow[dl, Rightarrow] \\
 & \mathcal{O}_{[\frac{n}{2}]-1,0} \arrow[ddr, dotted, Rightarrow] & \cdots & {} & & {} & & \mathcal{O}_{0,[\frac{n}{2}]-1} \arrow[ddl, dotted, Rightarrow] & \\
 & & & & \mathcal{O}_{(3)(1)^{n-3}} \arrow[d] & & & & \\
 & & \mathcal{O}_{2,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{1,1} \arrow[dl] \arrow[dr] & & \mathcal{O}_{0,2} \arrow[dl, Rightarrow] & & \\
 & & & \mathcal{O}_{1,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{0,1} \arrow[dl, Rightarrow] & & & \\
 & & & & \mathcal{O}_{0,0} & & & &
\end{tikzcd}
$$

where $\mathcal{O}_{r,s} = \mathcal{O}_{(+2)^r(-2)^s(1)^{n-2r-2s}}$

Figure 6. $Sp\,(n, \mathbb{R})$

$$\begin{tikzcd}[column sep=small]
\mathcal{O}_{n,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{n-1,1} \arrow[dl] \arrow[ddrr, dotted] & \cdots & \cdots & \cdots & \mathcal{O}_{1,n-1} \arrow[ddll, dotted] \arrow[dr] & & \mathcal{O}_{0,n} \arrow[dl, Rightarrow] \\
& \mathcal{O}_{n-1,0} \arrow[ddrr, Rightarrow, dotted] & & \cdots & \cdots & & & \mathcal{O}_{0,n-1} \arrow[ddll, Rightarrow, dotted] & \\
& & & & \mathcal{O}_{1,1} \arrow[dl] \arrow[dr] & & & & \\
& & & \mathcal{O}_{1,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{0,1} \arrow[dl, Rightarrow] & & & \\
& & & & \mathcal{O}_{0,0} & & & &
\end{tikzcd}$$

where $\mathcal{O}_{r,s} = \mathcal{O}_{(+2)^r(-2)^s(+1)^{n-r-s}(-1)^{n-r-s}}$

Figure 7. $Sp(p, q) \quad p \leq q$

$$\begin{tikzcd}[column sep=small]
& \mathcal{O}_{(+2)^p} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(+2)^{p-1}(+1)(-1)} \arrow[d, Rightarrow, dotted] & \\
\mathcal{O}_{(+3)(+2)(+1)^{p-3}(-1)^{q-2}} \arrow[d] \arrow[dr] & \mathcal{O}_{(+2)^3(+1)^{p-3}(-1)^{q-3}} \arrow[d, Rightarrow] & \mathcal{O}_{(-3)(+2)(+1)^{p-2}(-1)^{q-3}} \arrow[d] \arrow[dl] \\
\mathcal{O}_{(+3)(+1)^{p-2}(-1)^{q-1}} \arrow[dr, Rightarrow] & \mathcal{O}_{(+2)^2(+1)^{p-2}(-1)^{q-2}} \arrow[d, Rightarrow] & \mathcal{O}_{(-3)(+1)^{p-1}(-1)^{q-2}} \arrow[dl, Rightarrow] \\
& \mathcal{O}_{(+2)(+1)^{p-1}(-1)^{q-1}} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(+1)^p(-1)^q} &
\end{tikzcd}$$

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