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On the imbedding of the Schwarzschild space-time II.

By

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1. In the previous paper, we obtained the equation (4. 3):

$$(1)\qquad \boldsymbol{z} = \boldsymbol{y}(t, r) + r(\sin\theta \sin\varphi \boldsymbol{e}_4 + \sin\theta \cos\varphi \boldsymbol{e}_5 + \cos\theta \boldsymbol{e}_6),$$

which was the general imbedding function of the Schwarzschild space-time V^4 into a pseudo-euclidean E^6. The function $\boldsymbol{y}(t, r)$ in (1) should be chosen as satisfying (4. 8):

$$(2)\qquad \begin{aligned} &\left\langle \frac{\partial \boldsymbol{y}}{\partial t}, \frac{\partial \boldsymbol{y}}{\partial t} \right\rangle = \frac{r-2m}{r}, \quad \left\langle \frac{\partial \boldsymbol{y}}{\partial t}, \frac{\partial \boldsymbol{y}}{\partial r} \right\rangle = 0, \\ &\left\langle \frac{\partial \boldsymbol{y}}{\partial r}, \frac{\partial \boldsymbol{y}}{\partial r} \right\rangle = -\frac{2m}{r-2m}. \end{aligned}$$

Conversely, we can easily verify by a direct calculation that the four-dimensional subspace V^4 in E^6 defined by (1) has the induced metric which is the same as the one of the Schwarzschild space-time. The geometrical meaning of (2) is as follows. Since $\boldsymbol{y}$ is a three dimensional vector-valued function of two variables t and r, it defines a two-dimensional subspace V^2 in a pseudo-euclidean E^3, and the equation (2) shows that the induced metric of the V^2 is

$$(3)\qquad ds^2 = \frac{r-2m}{r} dt^2 - \frac{2m}{r-2m} dr^2.$$

Therefore the $\boldsymbol{y}$ may be regarded as the imbedding function of the two-dimensional Riemannian space V^2 with the indefinite metric (3) into the E^3. Thus we have

Theorem 1. *The imbedding problem of the Schwarzschild space-time V^4 into a pseudo-euclidean space E^6 is equivalent to the one of a two-dimensional Riemannian space V^2 with the indefinite metric* (3) *into a pseudo-euclidean space E^3. Namely, if $y(t, r)$ is any imbedding function of the V^2 into an E^3 with a signature $\varepsilon_1+\varepsilon_2+\varepsilon_3$, then the function* (1) *gives an imbedding of the Schwarzschild space-time V^4 into a pseudo-euclidean space E^6 with the signature $\varepsilon_1+\varepsilon_2+\varepsilon_3-3$, where ε_1, ε_2 and $\varepsilon_3=\pm 1$. And conversely, any imbedding function of the V^4 into an E^6 is obtained by this way.*

2. In this paper, we shall give another proof of the theorem. The following method of the proof is closely connected with the one of Kasner's paper [1] and clarifies the freedom of the imbedding.

We now introduce a three-dimensional Riemannian space $V^3(t)$ with the positive-definite metric

$$ds^2 = g_{ij}dx^i dx^j\,, \qquad i, j = 1, 2, 3,$$

$$g_{11} = \frac{r}{r-2m}\,, \quad g_{22} = r^2\,, \quad g_{33} = r^2\sin^2\theta\,,$$

$$g_{ij} = 0\,, \qquad i \neq j\,, \qquad r > 2m\,,$$

which is obtained from the Schwarzschild metric by putting $t=$ constant and changing the algebraic sign. We shall call the $V^3(t)$ the *stationary hypersurface* in the V^4. The Christoffel's symbols $\Gamma^i_{jk}(i, j, k=1, 2, 3)$ and the curvature tensor R_{ijkl} $(i, j, k, l=1, 2, 3)$ of the $V^3(t)$ are given as follows.

$$\Gamma^1_{11} = -\frac{m}{r(r-2m)}\,, \quad \Gamma^2_{21} = \Gamma^2_{12} = \Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r}\,,$$

$$\Gamma^3_{32} = \Gamma^3_{23} = \cot\theta\,, \quad \Gamma^1_{22} = -(r-2m)\,,$$

$\Gamma^1_{33}= -(r-2m)\sin^2\theta$, $\Gamma^2_{33}= -\sin\theta\cos\theta$, the other $\Gamma^i_{jk}=0$,

$$R_{1212} = -\frac{m}{r-2m}\,, \quad R_{2323} = 2mr\sin^2\theta\,, \quad R_{3131} = -\frac{m}{r-2m}\sin^2\theta\,,$$

the other $R_{ijkl} = 0$.

3. We first consider the stationary $V^3(t)$ and shall prove

Theorem 2. *The stationary hypersurface $V^3(t)$ in the V^4 can be imbedded in the euclidean space E^4 with the positive-definite metric, and then the $V^3(t)$ is rigid*.*

Proof. The Gauss and Weingarten formulas of $V^3(t)$ in an pseudo-enclidean E^4 (I-(1.5)) are

$$\nabla_j \boldsymbol{u}_i = e b_{ij} \boldsymbol{n}\,, \quad e = \pm 1\,, \tag{4}$$

$$\nabla_j \boldsymbol{n} = -g^{ik} b_{ij} \boldsymbol{u}_k\,, \tag{5}$$

and the Gauss and Codazzi equations (I-(1. 7), (1. 8)) are

$$e R_{ijkl} = b_{ik} b_{jl} - b_{il} b_{jk}\,, \tag{6}$$

$$\nabla_k b_{ij} - \nabla_j b_{ik} = 0\,. \tag{7}$$

Following the general theory developed by T. Y. Thomas [2], we shall solve the algebraic equation (6). First we get from (6)

$$\begin{aligned}[\det(b_{ij})]^2 &= \begin{vmatrix} eR_{2323} & eR_{2331} & eR_{2312} \\ eR_{3123} & eR_{3131} & eR_{3112} \\ eR_{1223} & eR_{1231} & eR_{1212} \end{vmatrix} \\ &= eR_{2323} R_{3131} R_{1212} = e\,\frac{2m^3 r \sin^4\theta}{(r-2m)^2} > 0\,.\end{aligned}$$

Hence we must take the sign $e = +1$, and the b_{ij} are given by**

$$b_{ii} = \frac{R_{ijij} R_{ikik}}{(R_{2323} R_{3131} R_{1212})^{1/2}}\,, \quad b_{ij} = 0, \quad i, j, k \neq\,.$$

Thus we have the unique system of solutions of (6) as follows.

$$\begin{aligned} b_{11} &= \frac{m}{(r-2m)(2mr)^{1/2}}\,, \quad b_{22} = -(2mr)^{1/2}\,, \\ b_{33} &= -(2mr)^{1/2} \sin^2\theta\,, \quad b_{ij} = 0\,, \quad i \neq j\,. \end{aligned} \tag{8}$$

By substitution from (8) into (7), we see easily that these (8) satisfy (7) automatically. Therefore the $V^3(t)$ under consideration can be imbedded in the euclidean E^4. It follows from $\det(b_{ij}) \neq 0$ that the hypersurface $V^3(t)$ in E^4 is of type three in the sense of T. Y. Thomas, and hence we have the rigidity of the $V^3(t)$ regarded as a hypersurface of the E^4.

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* See [5] of [2]. ** See (9.4) of [2].

4. Next, we shall find the imbedding function $\boldsymbol{u}$ of the $V^3(t)$ into E^4. The Gauss formula (4) is now rewritten in the forms

$$
(9)\quad
\begin{aligned}
&\frac{\partial^2 \boldsymbol{u}}{\partial r^2} = -\frac{m}{r(r-2m)}\frac{\partial \boldsymbol{u}}{\partial r} + \frac{m}{(r-2m)(2mr)^{1/2}}\boldsymbol{n}\,,\\
&\frac{\partial^2 \boldsymbol{u}}{\partial \theta^2} = -(r-2m)\frac{\partial \boldsymbol{u}}{\partial r} - (2mr)^{1/2}\boldsymbol{n}\,,\\
&\frac{\partial^2 \boldsymbol{u}}{\partial \varphi^2} = -(r-2m)\sin^2\theta\frac{\partial \boldsymbol{u}}{\partial r} - \sin\theta\cos\theta\frac{\partial \boldsymbol{u}}{\partial \theta} - (2mr)^{1/2}\sin^2\theta\,\boldsymbol{n}\,,\\
&\frac{\partial^2 \boldsymbol{u}}{\partial r\partial\theta} = \frac{1}{r}\frac{\partial \boldsymbol{u}}{\partial \theta}\,,\quad \frac{\partial^2 \boldsymbol{u}}{\partial \theta\partial\varphi} = \cot\theta\frac{\partial \boldsymbol{u}}{\partial \varphi}\,,\quad \frac{\partial^2 \boldsymbol{u}}{\partial r\partial\varphi} = \frac{1}{r}\frac{\partial \boldsymbol{u}}{\partial \varphi}\,,
\end{aligned}
$$

and the Weingarten formula (5) is expressed in the forms

$$
(10)\quad
\begin{aligned}
&\frac{\partial \boldsymbol{n}}{\partial r} = -\frac{(2mr)^{1/2}}{2r^2}\frac{\partial \boldsymbol{u}}{\partial r}\,,\\
&\frac{\partial \boldsymbol{n}}{\partial \theta} = \frac{(2mr)^{1/2}}{r^2}\frac{\partial \boldsymbol{u}}{\partial \theta}\,,\\
&\frac{\partial \boldsymbol{n}}{\partial \varphi} = \frac{(2mr)^{1/2}}{r^2}\frac{\partial \boldsymbol{u}}{\partial \varphi}\,.
\end{aligned}
$$

Furthermore, the imbedding function $\boldsymbol{u}$ should satisfy the isometry conditions:

$$
(11)\quad
\begin{aligned}
&\left\langle\frac{\partial \boldsymbol{u}}{\partial r}, \frac{\partial \boldsymbol{u}}{\partial r}\right\rangle = \frac{r}{r-2m}\,,\quad \left\langle\frac{\partial \boldsymbol{u}}{\partial \theta}, \frac{\partial \boldsymbol{u}}{\partial \theta}\right\rangle = r^2\\
&\left\langle\frac{\partial \boldsymbol{u}}{\partial \varphi}, \frac{\partial \boldsymbol{u}}{\partial \varphi}\right\rangle = r^2\sin^2\theta\,,\\
&\left\langle\frac{\partial \boldsymbol{u}}{\partial r}, \frac{\partial \boldsymbol{u}}{\partial \theta}\right\rangle = \left\langle\frac{\partial \boldsymbol{u}}{\partial r}, \frac{\partial \boldsymbol{u}}{\partial \varphi}\right\rangle = \left\langle\frac{\partial \boldsymbol{u}}{\partial \theta}, \frac{\partial \boldsymbol{u}}{\partial \varphi}\right\rangle = 0\,,
\end{aligned}
$$

and the normal $\boldsymbol{n}$ should accept the normality conditions:

$$
(12)\quad
\begin{aligned}
&\left\langle\frac{\partial \boldsymbol{u}}{\partial r}, \boldsymbol{n}\right\rangle = \left\langle\frac{\partial \boldsymbol{u}}{\partial \theta}, \boldsymbol{n}\right\rangle = \left\langle\frac{\partial \boldsymbol{u}}{\partial \varphi}, \boldsymbol{n}\right\rangle = 0\,,\\
&\langle \boldsymbol{n}, \boldsymbol{n}\rangle = 1\,.
\end{aligned}
$$

The solutions $\boldsymbol{u}$ and $\boldsymbol{n}$ of the equations (9) and (10) are easily obtained by

$$
(13)\quad \boldsymbol{u} = -2[2m(r-2m)]^{1/2}\boldsymbol{e}_1 + r\sin\theta(\sin\varphi\boldsymbol{e}_4 + \cos\varphi\boldsymbol{e}_5) + r\cos\theta\boldsymbol{e}_6\,,
$$

$$(14) \quad \boldsymbol{n} = \left(\frac{r-2m}{r}\right)^{1/2} \boldsymbol{e}_1 + \frac{(2mr)^{1/2}}{r} \sin\theta(\sin\varphi \boldsymbol{e}_4 + \cos\varphi \boldsymbol{e}_5) + \cos\theta \boldsymbol{e}_6 .$$

Then the equations (11) and (12) imply that the $\boldsymbol{e}_1$, $\boldsymbol{e}_4$, $\boldsymbol{e}_5$ and $\boldsymbol{e}_6$ are orthonormal unit vectors.

5. We now return to the Schwarzschild space-time V^4. Its metric is expressed by

$$ds^2 = \frac{r-2m}{r} dt^2 - (ds^2 \text{ of the } V^3(t)) ,$$

and so the equation (13) shows that the Schwarzschild metric ds^2 is written in the form

$$(15) \qquad ds^2 = \frac{r-2m}{r} dt^2 - (dy)^2 - (dz_4)^2 - (dz_5)^2 - (dz_6)^2 ,$$

where we put

$$(16) \qquad \begin{aligned} &y = -2[2m(r-2m)]^{1/2} , \quad z_4 = r \sin\theta \sin\varphi , \\ &z_5 = r \sin\theta \cos\varphi , \quad z_6 = r\cos\theta . \end{aligned}$$

The equation (15) is equivalent to the equation (4) of Kasner's paper [1]. The y is the function of r only and we have

$$\frac{r-2m}{r} dt^2 - (dy)^2 = \frac{r-2m}{r} dt^2 - \frac{2m}{r-2m} dr^2 .$$

which is the same as (3). Therefore we have proved Theorem 1.

6. Consequently we may say that our imbedding problem of the Schwarzschild space-time V^4 is reduced to the one of the V^2 with the metric (3) into an E^3. The Christoffel's symbols $\Gamma^i_{jk}(i, j, k=0, 1)$ of the V^2 are given by

$$(17) \qquad \begin{aligned} &\Gamma^0_{00} = \Gamma^1_{01} = \Gamma^0_{11} = 0 , \quad \Gamma^1_{00} = \frac{r-2m}{2r^2} , \\ &\Gamma^0_{01} = \frac{m}{r(r-2m)} , \quad \Gamma^1_{11} = -\frac{1}{2(r-2m)} , \end{aligned}$$

and the curvature tensor of the V^2 is

$$(18) \qquad R_{0101} = \frac{3m}{2r^3} .$$

Hence the Gauss and Codazzi equations of the V^2 in the E^3 are written in the forms

$$b_{00}b_{11} - (b_{01})^2 = \frac{3m}{2r^3}\varepsilon, \quad \varepsilon = \pm 1, \tag{19}$$

$$\begin{cases} \dfrac{\partial b_{00}}{\partial r} - \dfrac{\partial b_{01}}{\partial t} - \dfrac{m}{r(r-2m)} b_{00} + \dfrac{r-2m}{2r^2} b_{11} = 0, \\ \dfrac{\partial b_{01}}{\partial r} - \dfrac{\partial b_{11}}{\partial t} + \dfrac{r+2m}{2r(r-2m)} b_{01} = 0. \end{cases} \tag{20}$$

The imbedding function $\boldsymbol{y}$ and the normal $\boldsymbol{m}$ are the solutions of the Gauss and Weingarten formulas as follows:

$$\begin{cases} \dfrac{\partial^2 \boldsymbol{y}}{\partial t^2} = \dfrac{r-2m}{2r^2}\dfrac{\partial \boldsymbol{y}}{\partial r} + b_{00}\boldsymbol{m}, \\ \dfrac{\partial^2 \boldsymbol{y}}{\partial t \partial r} = \dfrac{m}{r(r-2m)}\dfrac{\partial \boldsymbol{y}}{\partial t} + b_{01}\boldsymbol{m}, \\ \dfrac{\partial^2 \boldsymbol{y}}{\partial r^2} = -\dfrac{1}{2(r-2m)}\dfrac{\partial \boldsymbol{y}}{\partial r} + b_{11}\boldsymbol{m}, \end{cases} \tag{21}$$

$$\begin{cases} \dfrac{\partial \boldsymbol{m}}{\partial t} = -\dfrac{r}{r-2m} b_{00}\dfrac{\partial \boldsymbol{y}}{\partial t} + \dfrac{r-2m}{2m} b_{01}\dfrac{\partial \boldsymbol{y}}{\partial r}, \\ \dfrac{\partial \boldsymbol{m}}{\partial r} = -\dfrac{r}{r-2m} b_{01}\dfrac{\partial \boldsymbol{y}}{\partial t} + \dfrac{r-2m}{2m} b_{11}\dfrac{\partial \boldsymbol{y}}{\partial r}. \end{cases} \tag{22}$$

The three-dimensional vectors $\boldsymbol{y}$ and $\boldsymbol{m}$ satisfy the well-known algebraic conditions:

$$\begin{cases} \left\langle \dfrac{\partial \boldsymbol{y}}{\partial r}, \dfrac{\partial \boldsymbol{y}}{\partial r} \right\rangle = -\dfrac{2m}{r-2m}, \quad \left\langle \dfrac{\partial \boldsymbol{y}}{\partial r}, \dfrac{\partial \boldsymbol{y}}{\partial t} \right\rangle = 0, \\ \left\langle \dfrac{\partial \boldsymbol{y}}{\partial t}, \dfrac{\partial \boldsymbol{y}}{\partial t} \right\rangle = \dfrac{r-2m}{r}, \end{cases} \tag{23}$$

$$\begin{cases} \left\langle \dfrac{\partial \boldsymbol{y}}{\partial r}, \boldsymbol{m} \right\rangle = \left\langle \dfrac{\partial \boldsymbol{y}}{\partial t}, \boldsymbol{m} \right\rangle = 0, \\ \langle \boldsymbol{m}, \boldsymbol{m} \rangle = 1, \end{cases} \tag{24}$$

where the symbol $\langle , \rangle$ is the inner product in the E^3. As is well-known, *a system of solutions* b_{ij} *of* (19) *and* (20) *corresponds one-to-one* to an imbedding function $\boldsymbol{y}$ of the V^2 (within a motion in the E^3) and hence *to an imbedding function* $\boldsymbol{z}$ *of the Schwarzschild space time* V^4.

7. For example, we consider the case where b_{00}, b_{01} and b_{11} in (19) and (20) *do not depend upon t*. These b_{ij} are immediately given by

$$
(25) \qquad \begin{aligned}
(b_{00})^2 &= \frac{m(r-2m)}{2r^4}+\frac{c^2}{r}+\frac{c'(r-2m)}{r}, \\
b_{01} &= \frac{c(r)^{1/2}}{r-2m}, \\
b_{11} &= \frac{1}{b_{00}}\left(\frac{3m}{2r^3}+\frac{c^2 r}{(r-2m)^2}\right),
\end{aligned}
$$

where c and c' are integral constants. On the other hand, we have obtained the *stationary solutions* at the end of the previous paper. If we deal with the stationary solution (i), then we have

$$
(dy_1)^2+(dy_2)^2-(dy_3)^2 = \frac{r-2m}{r}dt^2-\frac{2m}{r-2m}dr^2,
$$

from which it follows that

$$
\begin{aligned}
\frac{\partial \boldsymbol{y}}{\partial t} &= \left(\gamma^{1/2}\cos\frac{t}{k},\ -\gamma^{1/2}\sin\frac{t}{k},\ 0\right), \\
\frac{\partial^2 \boldsymbol{y}}{\partial t^2} &= \left(-\frac{\gamma^{1/2}}{k}\sin\frac{t}{k},\ -\frac{\gamma^{1/2}}{k}\cos\frac{t}{k},\ 0\right), \\
\frac{\partial^2 y}{\partial t\partial r} &= \left(\frac{m}{r^2\gamma^{1/2}}\cos\frac{t}{k},\ -\frac{m}{r^2\gamma^{1/2}}\sin\frac{t}{k},\ 0\right), \\
\boldsymbol{m} &= \left(p\sin\frac{t}{k},\ p\cos\frac{t}{k},\ p\frac{km}{r^2\gamma^{1/2}f'}\right),
\end{aligned}
$$

where we put

$$
\gamma = \frac{r-2m}{r}, \quad p^2 = 1+\frac{mk^2}{2r^3}.
$$

Hence the equation (21) gives

$$
b_{01} = 0, \quad (b_{00})^2 = \frac{r-2m}{r}\left(\frac{m}{2r^3}+\frac{1}{k^2}\right).
$$

These are obtained from (25), if we take $c=0$ and $c'=\frac{1}{k^2}$. Similarly we can verify that the other stationary solutions (ii) and (iii) are special cases of (25), the former being $c=0$, $c'=-\frac{1}{k^2}$, and the the later being $c=0$, $c'=0$. Thus we see that *the stationary solutions in the previous paper are the special cases of the solutions* (25), *where we put* $c=0$.

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