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On a theorem of Weitzenböck in invariant theory

By

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Let V be an affine variety and G an algebraic group acting on V (on the left). Let R be the coordinate ring of V and $I_G(R)$ its subring of G-invariants. If the characteristic of the ground field is zero, and G is semi-simple, it is now a well-known result that $I_G(R)$ is finitely generated over the ground field (cf. [2] or [4]). The same result is also known to be true without any hypothesis on the characteristic of the ground field if G is a torus group (cf. [2]). In these cases, the essential part of the proof is that there is a canonical projection operator of the space of regular functions on the group G onto the space of constants, which can be extended to a projection operator of R onto $I_G(R)$ and then using the fact that the ring of polynomials in a finite number of variables over a field is Noetherian, we get the result. There is a less known result of Weitzenböck (cf. [3]), which says that if V is the affine space and G an algebraic 1-parameter group acting on V by linear transformations, $I_G(R)$ is again finitely generated, provided that the ground field is of characteristic zero; and it is realized easily that this result is equivalent to the assertion that $I_G(R)$ is finitely generated for the particular case $G=$the additive group G_a. This result is no longer true if V is an arbitrary affine variety (this is a consequence of the famous counter example of Nagata (cf. [1] or [2]) showing that $I_G(R)$ need not be, in general, finitely generated).

In this note, we prove a result (cf. Theorem 1) which is valid for any characteristic and of which Weitzenböck's result is an easy consequence. Even when the characteristic is zero, our method brings out clearly the underlying idea of Weitzenböck's proof.

§ **1.** Suppose that the varieties V and G are both defined over a field K, with points in a universal domain Ω. Suppose that there exists a dense subgroup H of K-rational points of G such that every element of H gives rise to a K-automorphism of V. Let R be the K-coordinate ring of V and $R_1=R\otimes_K\Omega$ its Ω-coordinate ring. Then we see easily (cf. [2]) that $I_H(R)\otimes_K\Omega=I_G(R_1)$, which shows that $I_H(R)$ is finitely generated over K if and only if $I_G(R_1)$ is over Ω. Since these conditions will be satisfied in the problems we consider, we assume hereafter that $K=\Omega$, or simply that K is algebraically closed.

Let V be a vector space of dimension 2 over K and e_1, e_2 a fixed chosen basis of V. We can then represent an element of V by a 2×1 matrix over K. If $A\in SL(2, K)$ and $\alpha=\alpha_1e_1+\alpha_2e_2\in V$, we define $A\circ\alpha$ as the element of V represented by the matrix $A\cdot\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$. Now if $W=V^m=\oplus V$ (m times), we extend the operation of $SL(2, K)$ to W by defining:

$$A(v_1, \cdots, v_m) = (Av_1, \cdots, Av_m)\,, \qquad A\in SL(2, K),\ \ v_i\in V\,.$$

Every element $w\in W$ can be represented by a $2\times m$ matrix:

$$w = \begin{pmatrix}\alpha_1, & \cdots, & \alpha_m\\ \beta_1, & \cdots, & \beta_m\end{pmatrix}, \qquad \alpha_i, \beta_i\in K\,.$$

Let R be the coordinate ring of W; then we can regard α_i, β_i as variable elements of R. Then $t_{ij}=det\begin{pmatrix}\alpha_i & \alpha_j\\ \beta_i & \beta_j\end{pmatrix}$ is an element invariant under $SL(2, K)$. *We denote by J the subring $K[t_{ij}]$ $(1\leqslant i\leqslant m,\ 1\leqslant j\leqslant m)$ of R. We write G for $SL(2, K)$.*

Proposition 1: *The ring $I_G(R)$ is finitely generated over K.*

The only non-trivial case we have to consider is when $m\geqslant 3$. Let W_{ij} be the G-invariant affine open subset of W defined by $t_{ij}\neq 0$ and let R_{ij} be its coordinate ring. Consider for example, the subvariety Z of W represented by points of the following form

$$\begin{pmatrix} \lambda & 0 & \alpha_3, \cdots, \alpha_m \\ 0 & 1 & \beta_3, \cdots, \beta_m \end{pmatrix}, \qquad \lambda \neq 0.$$

Then Z meets every G-orbit in W_{12} at a unique point, from which it is deduced easily that the *quotient variety**, W_{12}/G exists and can be canonically identified with the affine variety Z. This implies in particular that $I_G(R_{12})$ can be canonically identified with the coordinate ring of Z. Since $\lambda = t_{12}$, $\alpha_i = -t_{2i}$ and $\beta_i = \lambda^{-1} t_{1i}$, we see that $I_G(R_{12})$ is generated by t_{12}, t_{12}^{-1}, t_{1i} $(i \geqslant 3)$, t_{2i} $(i \geqslant 3)$ and that $I_G(R_{12}) = J_S$, where J_S represents the ring of quotients of J with respect to the multiplicatively closed subset S formed of positive powers of t_{12}. This shows that if V is the open subset of points P of W such that for at least one t_{ij}, $t_{ij}(P) \neq 0$, the quotient variety V/G exists and it can be identified with the open subset Y of the affine variety X whose coordinate ring is J, such that if $P \in Y$, at least for one t_{ij}, $t_{ij}(P) \neq 0$. Now it is trivial to see that the complement of Y in X reduces to a point, in particular it is of codimension $\geqslant 2$ in X. From this it follows easily that if $f \in I_G(R)$, it is integral over J, which shows that $I_G(R)$ is finitely generated over K since J is. q.e.d.

Remark: The statement analogous to Prop. 1 for the case $G = SL(n, K)$, $n > 2$, can be proved in a similar manner.

Let $V^{(m)}$ be the m-th symmetric power of the vector space V. A canonical basis for the vector space $V^{(m)}$ is given by homogeneous monomials of degree m in e_1, e_2 (the chosen basis of V). The operation of G can be canonically extended to $V^{(m)}$. Let $M = \bigoplus_{1 \leqslant i \leqslant n} V^{(m_i)}$; then the operation of G can be extended to M by defining

$$A \circ (x_1, \cdots, x_n) = (A \cdot x_1, \cdots, A \cdot x_n), \qquad x_i \in V^{(m_i)}, \ A \in G.$$

Let S be the coordinate ring of M.

Proposition 2: *The ring $I_G(S)$ is finitely generated over K.*

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* If G is an algebraic group acting on a variety X, we say that the quotient variety X/G exists if the canonical structure of a ringed space on the space of orbits in X is a variety.

We present an element q of V^{m_i} by the matrix

$$q=\begin{pmatrix}\alpha_1, \cdots, \alpha_{m_i}\\ \beta_1, \cdots, \beta_{m_i}\end{pmatrix}.$$

Let Q_i be the subset of V^{m_i} such that for every j, $1\leqslant j\leqslant m_i$, α_j or β_j is different from zero. The subset Q_i is open, G-invariant and its complement is of codimension $\geqslant 2$ in V^{m_i}. There is a canonical morphism $k_i: Q_i\to V^{(m_i)}$ defined by $k_i(q)=$ the element of $V^{(m_i)}$ represented by the homogeneous form $\prod_{1\leqslant j\leqslant m_i}(\alpha_j e_1+\beta_j e_2)$ in e_1, e_2. The morphism k_i is a G-morphism (i.e.) it commutes with the operation of G. The symmetric group Γ_i on m_i elements acts canonically on V^{m_i}, leaves Q_i invariant and also commutes with the operation of G. If T_i is the torus group of dimension (m_i-1) represented by $y=(y_1, \cdots, y_{m_i})$, $y_i\in K$, $\prod_{1\leqslant j\leqslant m_i} y_j=1$, T_i operates on V^{m_i} by defining $q\circ y$, $q\in V^{m_i}$ as the element

$$\begin{pmatrix}y_1\alpha_1, y_2\alpha_2, \cdots, y_{m_i}\alpha_{m_i}\\ y_1\beta_1, y_2\beta_2, \cdots, y_{m_i}\beta_{m_i}\end{pmatrix}$$

of V^{m_i}. We see that the operation of T_i also commutes with G.

Now if Q is the open subset of $W=\prod_{1\leqslant i\leqslant n} V^{m_i}$ defined by $Q=\prod_{1\leqslant i\leqslant n} Q_i$, Q is G-invariant and its complement is of codimension $\geqslant 2$ in M. If we now define the morphism $k: Q\to M$ by $k=k_1\times\cdots\times k_n$, k commutes with G. If $T=\prod_{1\leqslant i\leqslant n} T_i$ and $\Gamma=\prod_{1\leqslant i\leqslant n}\Gamma_i$, we see that $H=T\times\Gamma$ acts canonically on W, leaves Q invariant and commutes with the operation of G. Further we see easily that the quotient variety Q/H can be canonically identified with an open subset of M. If R is the coordinate ring of W, we see easily that $I_G(S)=I_H(I_G(R))$. By Prop. 1, $I_G(R)$ is finitely generated and since H is the product of a torus group and a finite group, $I_H(I_G(R))$ is also finitely generated over K. q.e.d.

§2. If X is an algebraic variety on which an algebraic group G operates (on the left), we call it a *transformation space* and denote this object by (X, G). A *morphism* between two such objects (X_1, H_1) and (X_2, H_2) is a morphism $f: X_1\to X_2$ of varieties together with a homomorphism $\rho: H_1 \to H_2$ of algebraic groups such that $f(g\cdot x)=\rho(g)f(x)$, $x\in X_1$, $g\in H_1$. We can then define an isomorphism between two transformation spaces. We call (W, G) a *fundamental space for the group* $G=SL(2, K)$, if (upto an isomorphism), W is a finite direct sum of vector spaces of the form $V^{(m)}$ and G operates canonically on W as we have seen before. The additive group G_a can be considered as a subgroup of G by means of the morphism $\lambda: G_a \to G$, $\lambda(\alpha)=\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$. Therefore if (W, G) is a fundamental space for G, G_a operates on W and the object (W, G_a) is called a *fundamental space for* G_a. If for example $W=V^{(m)}$, W can be identified with the vector space generated by the translates of a regular function on G_a.

Proposition 3: *Let* (X, G_a) *be a transformation space such that* X *is a finite dimensional vector space and every element of* G_a *operates by a linear transformation of* X. *Then* (X, G_a) *is a fundamental space for* G_a *if the characteristic of* K *is zero.*

Let $m=\dim. X$. Then if we identify the group of K-automorphisms of X with $GL(m, K)$ by choosing a basis of X, the operation of G_a on X is defined by a homomorphism $\chi: G_a \to GL(m, K)$ of algebraic groups. Since $\chi(G_a)$ is again isomorphic to G_a, we can assume that χ is *injective*. Then χ is uniquely determined by $(d\chi)(t)$, where t is a fixed chosen non-zero element of the tangent space at 0 of G_a. We note that every element of $\chi(G_a)$ is a unipotent matrix. Therefore $(d\chi)(t)$ can be identified with an $(m\times m)$ matrix A such that if

$$A=\begin{pmatrix} A_1 & 0 & 0\cdots 0\\ 0 & A_2 & 0\cdots 0\\ 0 & 0 & A_3\cdots 0\\ \cdots & \cdots & \cdots\\ 0 & \cdots\cdots & A_n \end{pmatrix} \qquad A_i=\begin{pmatrix} \lambda_i & 1 & 0 & \cdots\cdots 0\\ 0 & \lambda_i & 1 & \cdots\cdots 0\\ 0 & 0 & \lambda_i & 1\cdots 0\\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

is the Jordan canonical form of A, $\lambda_i=0$ for every i. Therefore to prove the proposition, we can assume that A itself has the form

$$\begin{pmatrix} 0 & 1 & 0\cdots\\ 0 & 0 & 1\cdots\\ \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots \end{pmatrix}.$$

As we have seen before, the group G operates canonically on $V^{(m-1)}$ and if we refer the operation of G to the canonical basis of $V^{(m-1)}$, we have a representation $\rho: G \rightarrow GL(m, K)$. Consider the representation $\varphi: G_a \rightarrow GL(m, K)$ defined by $\varphi = \rho \circ \lambda$. We check that the Jordan canonical form for $dX(t)$ is A, which proves the proposition. q.e.d.

Theorem 1: *Let* (X, G_a) *be a transformation space such that* X *is a finite dimensional vector space and every element of* G_a *operates by a linear transformation of* X. *Let* R *be the coordinate ring of* X. *Then* $I_{G_a}(R)$ *is finitely generated if* (i) (X, G_a) *is a fundamental space or* (ii) *the characteristic of* K *is zero.*

By Prop. 3, we can assume that (X, G_a) is fundamental for G_a so that the operation of G_a is induced by an operation of G on X and (X, G) is fundamental for G. Consider $L = X \oplus V$. Then G operates on L by defining, $g \circ (x, v) = (g \cdot x, g \cdot v)$ and (L, G) is fundamental for G. Consider the subvariety X_1 of L formed of elements (w, v) such that $w \in X$ and $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We identify G_a as a subgroup of G by the homomorphism λ of G_a into G. Then X_1 is left invariant by G_a and if an element g of G transforms an element of X_1 again into X_1, g in fact belongs to G_a; in particular the intersection of a G-orbit in L with X_1 is either empty or coincides with a G_a-orbit in X_1. We note that the G-invariant subset X_1^G generated by X_1 coincides with the complement of the closed subset of L formed of elements $(w, 0)$ where w is an arbitrary element of X and 0 is the 0-vector of V. Therefore X_1^G is open in L and its complement is of codimension $\geqslant 2$. Let S be the coordinate ring of L. Then every regular function of X_1^G is also regular on L; in particular the algebra of regular G-invariant functions of X_1^G can be identified with $I_G(S)$. By Prop. 2, $I_G(S)$ is finitely generated over K. We shall now show that every G_a-invariant regular function on X_1 is induced by a G-invariant regular function of X_1^G. This will complete the proof of the theorem since (X_1, G_a) is isomorphic to (X, G_a).

Consider the canonical mapping $f: G \times X_1 \rightarrow L$ defined by $f(g, x_1) = g \cdot x_1$. Then $f(G \times X_1) = X_1^G$. Now if x is G_a-invariant regular function on X_1, we extend it to a regular function y on $G\times X_1$ by defining $y(g, w)=x(w)$, $g\in G$, $w\in X_1$. Because of the condition that the intersection of a G-orbit of L with X_1 is either empty or a G_a-orbit in X_1, we see that y is constant on the fibre over every point of X_1^G. Therefore y goes down into a regular (G-invariant) function on X_1^G, if we assume that f is a separable morphism (i.e. the field of rational functions $F(G\times X_1)$ of $G\times X_1$ is a separable extension over the field of rational functions $F(L)$ of L), which is not difficult to check. (Actually, for the theorem, it is not necessary to use the fact that f is separable, for if Q is the quotient field of $I_G(S)$, we have the canonical inclusion $Q\subset F(L) \subset F(G\times X_1)$ and if N is the algebraic closure of Q in $F(G\times X_1)$, y can be identified with an element of N integral over $I_G(S)$. This fact is sufficient to conclude that $I_G(R)$ is finitely generated over K), q.e.d.

Remark: We do not know what happens to the above theorem if the characteristic of K is not zero and (X, G_a) is not fundamental.

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